МАТЕМАТИЧКИ ВЕСНИК
55 (2003), 83–96

UDK 517.546
оригинални научни рад
research paper

A CLASS OF UNIVALENT FUNCTIONS DEFINED BY USING HADAMARD PRODUCT

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Abstract. In this paper we introduce the class $L^*_\alpha(\lambda,\beta)$ of functions defined by $f * S_\alpha(z)$ of $f(z)$ and $S_\alpha = \dfrac{z}{(1-z)^{2(1-\alpha)}}$. We determine coefficient estimates, closure theorems, distortion theorems and radii of close-to-convexity, starlikeness and convexity. Also we find integral operators and some results for Hadamard products of functions in the class $L^*_\alpha(\lambda,\beta)$. Finally, in terms of the operators of fractional calculus, we derive several sharp results depicting the growth and distortion properties of functions belonging to the class $L^*_\alpha(\lambda,\beta)$.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $U = \{\, z : |z| < 1 \,\}$. And let S denote the subclass of A consisting of analytic and univalent functions $f(z)$ in U.

A function $f(z)$ from S is said to be starlike of order α if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in U)$$

for some α, $0 \leqslant \alpha < 1$. We denote the class of all starlike functions of order α by $S^*(\alpha)$. Further, a function $f(z)$ from S is said to be convex of order α if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad (z \in U)$$

for some α, $0 \leqslant \alpha < 1$. And we denote the class of all convex functions of order α by $K(\alpha)$. We note that $f(z) \in K(\alpha)$ if and only if $zf'(z) \in S^*(\alpha)$. The classes $S^*(\alpha)$ and $K(\alpha)$ were first introduced by Robertson [7], and later were studied by Schild [9], MacGregor [2] and Pinchuk [6].

AMS Subject Classification: 30 C 45
Keywords and phrases: Univalent functions, Hadamard product, fractional calculus.

Now, the function

$$S_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \qquad (0 \leqslant \alpha < 1)$$

is the well-known extremal function for the class $S^*(\alpha)$. Setting

$$C(\alpha, n) = \frac{1}{(n-1)!}\prod_{k=2}^{n}(k-2\alpha) \qquad (n \geqslant 2),$$

$S_\alpha(z)$ can be written in the form $S_\alpha(z) = z + \sum_{n=2}^{\infty} C(\alpha, n)z^n$. Then we can see that $C(\alpha, n)$ is a decreasing function in α and satisfies

$$\lim_{n\to\infty} C(\alpha, n) = \begin{cases} \infty, & \alpha < 1/2, \\ 0, & \alpha > 1/2, \\ 1, & \alpha = 1/2. \end{cases}$$

Let $f * g(z)$ denote the Hadamard product (convolution) of two functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, then

$$f * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Let T denote the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqslant 0). \tag{1.2}$$

We say that a function $f(z)$ defined by (1.1) belongs to the class $L_\alpha(\lambda, \beta)$ if $f(z)$ satisfies the following condition

$$\operatorname{Re}\left\{\frac{(f * S_\alpha(z))'}{\lambda(f * S_\alpha(z))' + (1-\lambda)}\right\} > \beta \tag{1.3}$$

for some α, $0 \leqslant \alpha < 1$, λ, $0 \leqslant \lambda < 1$, β, $0 \leqslant \beta < 1$ and for all $z \in U$.

Further we denote by $L^*_\alpha(\lambda, \beta)$ the class obtained by taking intersection of the class $L_\alpha(\lambda, \beta)$ with T, that is $L^*_\alpha(\lambda, \beta) = L_\alpha(\lambda, \beta) \cap T$. We note that:

(i) $L^*_{1/2}(0, \beta) = T^{**}(\beta)$ (Sarangi and Uralegaddi [8] and Al-Amiri [1]);

(ii) $L^*_{1/2}(\lambda, \beta)$ represents the class of functions $f(z) \in T$ satisfying the condition

$$\operatorname{Re}\left\{\frac{f'(z)}{\lambda f'(z) + (1-\lambda)}\right\} > \beta,$$

where $0 \leqslant \lambda < 1$ and $0 \leqslant \beta < 1$;

(iii) $L_\alpha(0, \beta)$ represents the class of functions $f(z) \in T$ satisfying the condition $\operatorname{Re}\{(f * S_\alpha(z))'\} > \beta$.

2. Coefficient estimates

THEOREM 1. *Let the function $f(z)$ be defined by (1.2). Then $f(z)$ is in the class $L^*_\alpha(\lambda, \beta)$ if and only if*

$$\sum_{n=2}^{\infty} n(1-\lambda\beta)C(\alpha, n)a_n \leqslant 1 - \beta. \tag{2.1}$$

The result is sharp.

Proof. Assume that inequality (2.1) holds and let $|z| < 1$. Then we have

$$\left|\frac{(f * S_\alpha(z))'}{\lambda(f * S_\alpha(z))' + (1-\lambda)} - 1\right| = \left|\frac{-(1-\lambda)\sum_{n=2}^{\infty} nC(\alpha,n)a_n z^{n-1}}{1-\lambda\sum_{n=2}^{\infty} nC(\alpha,n)a_n z^{n-1}}\right|$$

$$< \frac{(1-\lambda)\sum_{n=2}^{\infty} nC(\alpha,n)a_n}{1-\lambda\sum_{n=2}^{\infty} nC(\alpha,n)a_n} \leqslant 1-\beta.$$

This shows that the values of $\dfrac{(f * S_\alpha(z))'}{\lambda(f * S_\alpha(z))' + (1-\lambda)}$ lie in the circle centered at $w = 1$ whose radius is $1-\beta$. Hence $f(z)$ satisfies condition (1.3).

Conversely, assume the function $f(z)$ defined by (1.2) is in the class $L_\alpha^*(\lambda,\beta)$. Then

$$\operatorname{Re}\left\{\frac{(f * S_\alpha(z))'}{\lambda(f * S_\alpha(z))' + (1-\lambda)}\right\} = \operatorname{Re}\left\{\frac{1-\sum_{n=2}^{\infty} nC(\alpha,n)a_n z^{n-1}}{1-\lambda\sum_{n=2}^{\infty} nC(\alpha,n)a_n z^{n-1}}\right\} > \beta \qquad (2.2)$$

for $z \in U$. Choose values of z on the real axis so that $\dfrac{(f * S_\alpha(z))'}{\lambda(f * S_\alpha(z))' + (1-\lambda)}$ is real. Upon clearing the denominator in (2.2) and letting $z \to 1^-$ through real values, we obtain

$$1 - \sum_{n=2}^{\infty} nC(\alpha,n)a_n \geqslant \beta\left\{1-\lambda\sum_{n=2}^{\infty} nC(\alpha,n)a_n\right\}$$

which gives (2.1). Finally, the result is sharp with the extremal function $f(z)$ given by

$$f(z) = z - \frac{1-\beta}{n(1-\lambda\beta)C(\alpha,n)}z^n \qquad (n \geqslant 2). \quad \blacksquare \qquad (2.3)$$

COROLLARY 1. *Let the function $f(z)$ defined by (1.2) be in the class $L_\alpha^*(\lambda,\beta)$. Then we have*

$$a_n \leqslant \frac{1-\beta}{n(1-\lambda\beta)C(\alpha,n)} \qquad (n \geqslant 2). \qquad (2.4)$$

The equality in (2.4) is attained for the function $f(z)$ given by (2.3).

3. Some properties of the class $L_\alpha^*(\lambda,\beta)$

THEOREM 2. *Let $0 \leqslant \alpha < 1$, $0 \leqslant \lambda_1 \leqslant \lambda_2 < 1$ and $0 \leqslant \beta < 1$. Then $L_\alpha^*(\lambda_1,\beta) \subset L_\alpha^*(\lambda_2,\beta)$.*

Proof. It follows from Theorem 1 that

$$\sum_{n=2}^{\infty} n(1-\lambda_2\beta)C(\alpha,n)a_n \leqslant \sum_{n=2}^{\infty} n(1-\lambda_1\beta)C(\alpha,n)a_n \leqslant 1-\beta$$

for $f(z) \in L_\alpha^*(\lambda_1,\beta)$. Hence $f(z)$ is in $L_\alpha^*(\lambda_2,\beta)$. $\blacksquare$

THEOREM 3. *Let $0 \leqslant \alpha_1 \leqslant \alpha_2 < 1$, $0 \leqslant \lambda < 1$ and $0 \leqslant \beta < 1$. Then we have $L_{\alpha_1}^*(\lambda,\beta) \subset L_{\alpha_2}^*(\lambda,\beta)$.*

Proof. Since $C(\alpha,n)$ is a decreasing function in α, it follows from Theorem 1 that

$$\sum_{n=2}^{\infty} n(1-\lambda\beta)C(\alpha_2,n)a_n \leqslant \sum_{n=2}^{\infty} n(1-\lambda\beta)C(\alpha_1,n)a_n \leqslant 1-\beta$$

for $f(z) \in L_{\alpha_1}^*(\lambda,\beta)$. Hence $f(z)$ is in $L_{\alpha_2}^*(\lambda,\beta)$. $\blacksquare$

4. Closure theorems

We shall prove the following results for the closure of functions in the class $L_\alpha^*(\lambda,\beta)$.

THEOREM 4. *Let the functions* $f_j(z)$, $j=1,2,\dots,m$, *defined by*

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n \qquad (a_{n,j} \geqslant 0) \tag{4.1}$$

for $z \in U$, *be in the class* $L_\alpha^*(\lambda,\beta)$. *Then the function* $h(z)$ *defined by*

$$h(z) = z - \sum_{n=2}^{\infty} b_n z^n$$

also belongs to the class $L_\alpha^*(\lambda,\beta)$, *where* $b_n = \frac{1}{m}\sum_{j=1}^{m} a_{n,j}$.

Proof. Since $f_j(z) \in L_\alpha^*(\lambda,\beta)$, it follows from Theorem 1 that

$$\sum_{n=2}^{\infty} n(1-\lambda\beta)C(\alpha,n)a_{n,j} \leqslant 1-\beta \qquad (j=1,2,\dots,m).$$

Therefore

$$\sum_{n=2}^{\infty} n(1-\lambda\beta)C(\alpha,n)b_n = \sum_{n=2}^{\infty} n(1-\lambda\beta)C(\alpha,n)\Big(\frac{1}{m}\sum_{j=1}^{m} a_{n,j}\Big)$$
$$= \frac{1}{m}\sum_{j=1}^{m}\Big\{\sum_{n=2}^{\infty} n(1-\lambda\beta)C(\alpha,n)a_{n,j}\Big\} \leqslant 1-\beta.$$

Hence by Theorem 1, $h(z) \in L_\alpha^*(\lambda,\beta)$. Thus we have the theorem. ∎

Employing the techniques used earlier by Silverman [11], and with the aid of Theorem 1, we can prove the following

THEOREM 5. *The class* $L_\alpha^*(\lambda,\beta)$ *is closed under convex linear combinations.*

As a consequence of Theorem 5, there exist extreme points of the class $L_\alpha^*(\lambda,\beta)$.

THEOREM 6. *Let* $f_1(z) = z$ *and*

$$f_n(z) = z - \frac{1-\beta}{n(1-\lambda\beta)C(\alpha,n)} z^n \qquad (n \geqslant 2) \tag{4.2}$$

for $0 \leqslant \alpha < 1$, $0 \leqslant \lambda < 1$ *and* $0 \leqslant \beta < 1$. *Then* $f(z)$ *is in the class* $L_\alpha^*(\lambda,\beta)$ *if and only if it can be expressed in the form* $f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z)$, *where* $\mu_n \geqslant 0$ $(n \geqslant 1)$ *and* $\sum_{n=1}^{\infty} \mu_n = 1$.

COROLLARY 2. *The extreme points of the class* $L_\alpha^*(\lambda,\beta)$ *are the functions* $f_n(z)$ $(n \geqslant 1)$ *given by Theorem 6.*

5. Distortion theorems

With the aid of Theorem 1, we may now find bounds of the modulus of $f(z)$ and $f'(z)$ for $f(z) \in L_\alpha^*(\lambda,\beta)$.

THEOREM 7. *If the function* $f(z)$ *defined by (1.2) is in the class* $L_\alpha^*(\lambda,\beta)$, $0 \leqslant \lambda < 1$, $0 \leqslant \beta < 1$, *and either* $0 \leqslant \alpha \leqslant 5/6$ *or* $|z| \leqslant 3/4$, *then*

$$|f(z)| \geqslant \max\left\{0,\ |z| - \frac{1-\beta}{4(1-\lambda\beta)(1-\alpha)}\,|z|^2\right\},$$

and $|f(z)| \leqslant |z| + \dfrac{1-\beta}{4(1-\lambda\beta)(1-\alpha)}\,|z|^2$. *The bounds are sharp.*

Proof. By virtue of Theorem 1, we note that

$$|f(z)| \geqslant \max\left\{0,\ |z| - \max_{n\in\mathbf{N}\setminus\{1\}} \frac{1-\beta}{n(1-\lambda\beta)C(\alpha,n)}\,|z|^n\right\},$$

$$|f(z)| \leqslant |z| + \max_{n\in\mathbf{N}\setminus\{1\}} \frac{1-\beta}{n(1-\lambda\beta)C(\alpha,n)}\,|z|^n$$

for $z \in U$. Hence it suffices to deduce that

$$G(\alpha,\lambda,\beta,|z|,n) = \frac{1-\beta}{n(1-\lambda\beta)C(\alpha,n)}\,|z|^n$$

is a decreasing function of n $(n \geqslant 2)$. Since $C(\alpha,n+1) = \dfrac{n+1-2\alpha}{n}C(\alpha,n)$, we can see that, for $|z| \neq 0$, $G(\alpha,\lambda,\beta,|z|,n) \geqslant G(\alpha,\lambda,\beta,|z|,n+1)$ if and only if

$$H(\alpha,|z|,n) = (n+1)(n+1-2\alpha) - n^2|z| \geqslant 0.$$

It is easy to see that $H(\alpha,|z|,n)$ is a decreasing function of α for fixed $|z|$. Consequently it follows that

$$H(\alpha,|z|,n) \geqslant H(5/6,|z|,n) = n^2(1-|z|) + \frac{1}{3}(n-2) \geqslant 0$$

for $0 \leqslant \alpha \leqslant 5/6$, $z \in U$ and $n \geqslant 2$.

Further, since $H(\alpha,|z|,n)$ is decreasing in $|z|$ and increasing in n, we obtain that $H(\alpha,|z|,n) > H(1,|z|,n) \geqslant H(1,3/4,2) = 0$ for $0 \leqslant \alpha \leqslant 1$, $|z| \leqslant 3/4$ and $n \geqslant 2$. Thus $\max_{n\in\mathbf{N}\setminus\{1\}} G(\alpha,\lambda,\beta,|z|,n)$ is attained at $n = 2$.

Finally, since the functions $f_n(z)$ $(n \geqslant 2)$ defined in Theorem 6 are extreme points of the class $L_\alpha^*(\lambda,\beta)$, we can see that the bounds of Theorem 7 are attained by the function $f_2(z)$, that is

$$f_2(z) = z - \frac{1-\beta}{4(1-\lambda\beta)(1-\alpha)}\,z^2. \quad \blacksquare$$

COROLLARY 3. *Let the function* $f(z)$ *defined by (1.2) be in the class* $L_\alpha^*(\lambda,\beta)$, $0 \leqslant \alpha \leqslant 5/6$, $0 \leqslant \lambda < 1$ *and* $0 \leqslant \beta < 1$. *Then* $f(z)$ *is included in the disc with the center at the origin and radius* r *given by* $r = 1 + \dfrac{1-\beta}{4(1-\lambda\beta)(1-\alpha)}$.

THEOREM 8. *If the function* $f(z)$ *defined by (1.2) is in the class* $L_\alpha^*(\lambda,\beta)$, $0 \leqslant \lambda < 1$, $0 \leqslant \beta < 1$, *and either* $0 \leqslant \alpha \leqslant 1/2$ *or* $|z| \leqslant 1/2$, *then*

$$1 - \frac{1-\beta}{2(1-\lambda\beta)(1-\alpha)}\,|z| \leqslant |f'(z)| \leqslant 1 + \frac{1-\beta}{2(1-\lambda\beta)(1-\alpha)}\,|z|.$$

The bounds are sharp.

Proof. It is similar to the proof of Theorem 7. $\blacksquare$

6. Radii of close-to-convexity, starlikeness and convexity

THEOREM 9. *$L^*_\alpha(\lambda,\beta)$ is a subclass of S if and only if $0 \leqslant \alpha \leqslant 1/2$.*

Proof. Naote that the function $f(z)$ defined by (1.2) is in the class S if $\sum_{n=2}^{\infty} n|a_n| \leqslant 1$ (cf. [11]). Hence it suffices to prove that $(1-\lambda\beta)C(\alpha,n) \geqslant 1-\beta$ for $0 \leqslant \alpha \leqslant 1/2$, $0 \leqslant \lambda < 1$, $0 \leqslant \beta < 1$ and $n \geqslant 2$ by means of Theorem 1. Since $C(\alpha,n) \geqslant C(1/2,n) = 1$ for $0 \leqslant \alpha \leqslant 1/2$, we can see that, for $0 \leqslant \alpha \leqslant 1/2$, $0 \leqslant \lambda < 1$ and $0 \leqslant \beta < 1$,

$$(1-\lambda\beta)C(\alpha,n) - (1-\beta) \geqslant (1-\lambda\beta) - (1-\beta) \geqslant 0.$$

Conversely, if we assume $\alpha > 1/2$, then $\lim_{n\to\infty} C(\alpha,n) = 0$. Taking the function $f_n(z)$ given by (4.2), we have

$$f_n'(z) = 1 - \frac{1-\beta}{(1-\lambda\beta)C(\alpha,n)}\, z^{n-1} = 0$$

for $z^{n-1} = \dfrac{(1-\lambda\beta)C(\alpha,n)}{1-\beta}$ which is less than one for n sufficiently large. Thus $f_n(z)$ is not univalent for $\alpha > 1/2$ and $n = n(\alpha)$ sufficiently large. ■

By using Theorem 1, we can prove the following

THEOREM 10. *Let the function $f(z)$ defined by (1.2) be in the class $L^*_\alpha(\lambda,\beta)$, $0 \leqslant \alpha \leqslant 1/2$, $0 \leqslant \lambda < 1$ and $0 \leqslant \beta < 1$. Then $f(z)$ is close-to-convex of order ρ $(0 \leqslant \rho < 1)$ in $|z| \leqslant R_1$, where*

$$R_1 = \inf_n \left\{ \frac{(1-\rho)(1-\lambda\beta)C(\alpha,n)}{1-\beta} \right\}^{1/(n-1)} \qquad (n \geqslant 2).$$

The result is sharp, with extremal function $f(z)$ given by (2.3).

THEOREM 11. *Let the function $f(z)$ defined by (1.2) be in the class $L^*_\alpha(\lambda,\beta)$, $0 \leqslant \alpha \leqslant 1/2$, $0 \leqslant \lambda < 1$ and $0 \leqslant \beta < 1$. Then $f(z)$ is starlike of order ρ $(0 \leqslant \rho < 1)$ in $|z| \leqslant R_2$, where*

$$R_2 = \inf_n \left\{ \frac{n(1-\rho)(1-\lambda\beta)C(\alpha,n)}{(n-\rho)(1-\beta)} \right\}^{1/(n-1)} \qquad (n \geqslant 2).$$

The result is sharp, with extremal function $f(z)$ given by (2.3).

COROLLARY 4. *Let the function $f(z)$ defined by (1.2) be in the class $L^*_\alpha(\lambda,\beta)$, $0 \leqslant \alpha \leqslant 1/2$, $0 \leqslant \lambda < 1$ and $0 \leqslant \beta < 1$. Then $f(z)$ is convex of order ρ $(0 \leqslant \rho < 1)$ in $|z| \leqslant R_3$, where*

$$R_3 = \inf_n \left\{ \frac{(1-\rho)(1-\lambda\beta)C(\alpha,n)}{(n-\rho)(1-\beta)} \right\}^{1/(n-1)} \qquad (n \geqslant 2).$$

The result is sharp, with extremal function $f(z)$ given by (2.3).

7. Integral operators

THEOREM 12. *Let the function $f(z)$ defined by (1.2) be in the class $L_\alpha^*(\lambda,\beta)$, and let d be a real number such that $d>-1$. Then the function $F(z)$ defined by*

$$F(z)=\frac{d+1}{z^d}\int_0^z t^{d-1}f(t)\,dt \tag{7.1}$$

also belongs to the class $L_\alpha^*(\lambda,\beta)$.

Proof. From the representation of $F(z)$, it follows that $F(z)=z-\sum_{n=2}^{\infty}b_nz^n$, where $b_n=(\frac{d+1}{d+n})a_n$. Therefore

$$\begin{aligned}\sum_{n=2}^{\infty}n(1-\lambda\beta)C(\alpha,n)b_n&=\sum_{n=2}^{\infty}n(1-\lambda\beta)C(\alpha,n)\left(\frac{d+1}{d+n}\right)a_n\\ &\leqslant\sum_{n=2}^{\infty}n(1-\lambda\beta)C(\alpha,n)a_n\leqslant 1-\beta,\end{aligned}$$

since $f(z)\in L_\alpha^*(\lambda,\beta)$. Hence by Theorem 1, $F(z)\in L_\alpha^*(\lambda,\beta)$. ∎

THEOREM 13. *Let the function $F(z)=z-\sum_{n=2}^{\infty}a_nz^n$ $(a_n\geqslant 0)$ be in the class $L_\alpha^*(\lambda,\beta)$, and let d be a real number such that $d>-1$. Then the function $f(z)$ defined by (7.1) is univalent in $|z|<R^*$, where*

$$R^*=\inf_n\left\{\frac{(1-\lambda\beta)C(\alpha,n)(d+1)}{(1-\beta)(d+n)}\right\}^{1/(n-1)}\qquad (n\geqslant 2.)$$

The result is sharp.

Proof. From (7.1) we have

$$f(z)=\frac{z^{1-d}(z^dF(z))'}{d+1}=z-\sum_{n=2}^{\infty}\left(\frac{d+n}{d+1}\right)a_nz^n.$$

In order to obtain the required result it suffices to show that $|f'(z)-1|<1$ in $|z|<R^*$. Now

$$|f'(z)-1|=\left|-\sum_{n=2}^{\infty}n\left(\frac{d+n}{d+1}\right)a_nz^{n-1}\right|\leqslant\sum_{n=2}^{\infty}n\left(\frac{d+n}{d+1}\right)a_n|z|^{n-1}.$$

Thus $|f'(z)-1|<1$ if

$$\sum_{n=2}^{\infty}n\left(\frac{d+n}{d+1}\right)a_n|z|^{n-1}\leqslant 1. \tag{7.2}$$

But Theorem 1 confirms that $\sum_{n=2}^{\infty}\frac{n(1-\lambda\beta)C(\alpha,n)}{1-\beta}a_n\leqslant 1$. Hence (7.2) will be satisfied if

$$\frac{n(d+n)}{d+1}|z|^{n-1}\leqslant\frac{n(1-\lambda\beta)C(\alpha,n)}{1-\beta}\qquad (n\geqslant 2)$$

or if

$$|z|\leqslant\left\{\frac{(1-\lambda\beta)C(\alpha,n)(d+1)}{(1-\beta)(d+n)}\right\}^{1/(n-1)}\qquad (n\geqslant 2). \tag{7.3}$$

The required result follows now from (7.3). The result is sharp for the function

$$f(z)=z-\frac{(1-\beta)(d+n)}{n(1-\lambda\beta)C(\alpha,n)(d+1)}z^n\qquad (n\geqslant 2).\quad ∎$$

8. Modified Hadamard products

Let the functions $f_j(z)$ $(j = 1, 2)$ be defined by (4.1). The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$f_1 * f_2(z) = z - \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n.$$

THEOREM 14. *Let the functions $f_j(z)$ $(j = 1, 2)$ defined by (4.1) be in the class $L^*_\alpha(\lambda, \beta)$ with $0 \leqslant \alpha \leqslant 1/2$, $0 \leqslant \lambda < 1$ and $0 \leqslant \beta < 1$. Then $f_1 * f_2(z) \in L^*_\alpha(\lambda, \gamma(\alpha, \lambda, \beta))$ where*

$$\gamma(\alpha, \lambda, \beta) = 1 - \frac{(1-\lambda)(1-\beta)^2}{4(1-\lambda\beta)^2(1-\alpha) - \lambda(1-\beta)^2}.$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [10], we need to find the largest $\gamma(\alpha, \lambda, \beta)$ such that

$$\sum_{n=2}^{\infty} \frac{n(1-\lambda\gamma)C(\alpha, n)}{1-\gamma} a_{n,1} a_{n,2} \leqslant 1.$$

Since $\sum_{n=2}^{\infty} \frac{n(1-\lambda\beta)C(\alpha, n)}{1-\beta} a_{n,1} \leqslant 1$ and $\sum_{n=2}^{\infty} \frac{n(1-\lambda\beta)C(\alpha, n)}{1-\beta} a_{n,2} \leqslant 1$, by the Cauchy- Schwarz inequality we have

$$\sum_{n=2}^{\infty} \frac{n(1-\lambda\beta)C(\alpha, n)}{1-\beta} \sqrt{a_{n,1} a_{n,2}} \leqslant 1.$$

Thus it is sufficient to show that

$$\frac{n(1-\lambda\gamma)C(\alpha, n)}{1-\gamma} a_{n,1} a_{n,2} \leqslant \frac{n(1-\lambda\beta)C(\alpha, n)}{1-\beta} \sqrt{a_{n,1} a_{n,2}} \quad (n \geqslant 2),$$

that is that $\sqrt{a_{n,1} a_{n,2}} \leqslant \frac{(1-\lambda\beta)(1-\gamma)}{(1-\lambda\gamma)(1-\beta)}$. Note that $\sqrt{a_{n,1} a_{n,2}} \leqslant \frac{1-\beta}{n(1-\lambda\beta)C(\alpha, n)}$ $(n \geqslant 2)$. Consequently, we need only to prove that

$$\frac{1-\beta}{n(1-\lambda\beta)C(\alpha, n)} \leqslant \frac{(1-\lambda\beta)(1-\gamma)}{(1-\lambda\gamma)(1-\beta)} \quad (n \geqslant 2),$$

or, equivalently, that $\gamma \leqslant 1 - \frac{(1-\lambda)(1-\beta)^2}{n(1-\lambda\beta)^2 C(\alpha, n) - \lambda(1-\beta)^2}$ $(n \geqslant 2)$. Since

$$A(n) = 1 - \frac{(1-\lambda)(1-\beta)^2}{n(1-\lambda\beta)^2 C(\alpha, n) - \lambda(1-\beta)^2} \tag{8.1}$$

is an increasing function of n $(n \geqslant 2)$, for $0 \leqslant \alpha \leqslant 1/2$, $0 \leqslant \lambda < 1$ and $0 \leqslant \beta < 1$, letting $n = 2$ in (8.1), we obtain

$$\gamma \leqslant A(2) = 1 - \frac{(1-\lambda)(1-\beta)^2}{4(1-\lambda\beta)^2 C(\alpha, n) - \lambda(1-\beta)^2},$$

which completes the proof of Theorem 14.

Finally, by taking the functions $f_j(z)$ given by

$$f_j(z) = z - \frac{1-\beta}{4(1-\lambda\beta)(1-\alpha)}\, z^2 \qquad (j = 1, 2), \tag{8.2}$$

we can see that the result is sharp. ■

THEOREM 15. *Let the function* $f_1(z)$ *defined by (4.1) be in the class* $L^*_\alpha(\lambda, \beta)$ *with* $0 \leqslant \alpha \leqslant 1/2$, $0 \leqslant \lambda < 1$ *and* $0 \leqslant \beta < 1$, *and the function* $f_2(z)$ *defined by (4.1) be in the class* $L^*_\alpha(\lambda, \tau)$ *with* $0 \leqslant \alpha \leqslant 1/2$, $0 \leqslant \lambda < 1$ *and* $0 \leqslant \tau < 1$. *Then* $f_1 * f_2(z) \in L^*_\alpha(\lambda, \zeta(\alpha, \lambda, \beta, \tau))$, *where*

$$\zeta(\alpha, \lambda, \beta, \tau) = 1 - \frac{(1-\lambda)(1-\beta)(1-\tau)}{4(1-\lambda\beta)(1-\lambda\tau)(1-\alpha) - \lambda(1-\beta)(1-\tau)}.$$

The result is sharp.

Proof. Proceeding as in the proof of Theorem 14, we get

$$\zeta \leqslant B(n) = 1 - \frac{(1-\lambda)(1-\beta)(1-\tau)}{n(1-\lambda\beta)(1-\lambda\tau)C(\alpha, n) - \lambda(1-\beta)(1-\tau)} \quad (n \geqslant 2). \tag{8.3}$$

Since the function $B(n)$ is an increasing function of n ($n \geqslant 2$), for $0 \leqslant \alpha \leqslant 1/2$, $0 \leqslant \lambda < 1$ and $0 \leqslant \tau < 1$, letting $n = 2$ in (8.3), we obtain

$$\zeta \leqslant B(2) = 1 - \frac{(1-\lambda)(1-\beta)(1-\tau)}{4(1-\lambda\beta)(1-\lambda\tau)(1-\alpha) - \lambda(1-\beta)(1-\tau)},$$

which evidently proves Theorem 15.

Finally, the result is best possible for the functions

$$f_1(z) = z - \frac{1-\beta}{4(1-\lambda\beta)(1-\alpha)}\, z^2 \quad \text{and} \quad f_2(z) = z - \frac{1-\tau}{4(1-\lambda\tau)(1-\alpha)}\, z^2. \quad ■$$

COROLLARY 4. *Let the functions* $f_j(z)$ $(j = 1, 2, 3)$ *defined by (4.1) be in the class* $L^*_\alpha(\lambda, \beta)$ *with* $0 \leqslant \alpha \leqslant 1/2$, $0 \leqslant \lambda < 1$ *and* $0 \leqslant \beta < 1$. *Then* $f_1 * f_2 * f_3(z) \in L^*_\alpha(\lambda, \eta(\alpha, \lambda, \beta))$, *where*

$$\eta(\alpha, \lambda, \beta) = 1 - \frac{(1-\lambda)(1-\beta)^3}{16(1-\lambda\beta)^3(1-\alpha)^2 - \lambda(1-\beta)^3}.$$

The result is best possible for the functions $f_j(z) = z - \dfrac{1-\beta}{4(1-\lambda\beta)(1-\alpha)}\, z^2$ $(j = 1, 2, 3)$.

Proof. From Theorem 14, we have $f_1 * f_2(z) \in L^*_\alpha(\lambda, \gamma(\alpha, \lambda, \beta))$. We use now Theorem 15, and we get $f_1 * f_2 * f_3(z) \in L^*_\alpha(\lambda, \eta(\alpha, \lambda, \beta, \gamma))$, where

$$\begin{aligned} \eta(\alpha, \lambda, \beta, \gamma) &= 1 - \frac{(1-\lambda)(1-\beta)(1-\gamma)}{4(1-\lambda\beta)(1-\lambda\gamma)(1-\alpha) - \lambda(1-\beta)(1-\gamma)} \\ &= 1 - \frac{(1-\lambda)(1-\beta)^3}{16(1-\lambda\beta)^3(1-\alpha)^2 - \lambda(1-\beta)^3}. \end{aligned}$$

This completes the proof of Corollary 4. ■

THEOREM 16. *Let the functions $f_j(z)$ $(j = 1,2)$ defined by (4.1) be in the class $L^*_\alpha(\lambda,\beta)$ with $0 \leqslant \alpha \leqslant 1/2$, $0 \leqslant \lambda < 1$ and $0 \leqslant \beta < 1$. Then the function*

$$h(z) = z - \sum_{n=2}^{\infty}(a_{n,1}^2 + a_{n,2}^2)z^n$$

belongs to the class $L^*_\alpha(\lambda,\phi(\alpha,\lambda,\beta))$, where

$$\phi(\alpha,\lambda,\beta) = 1 - \frac{(1-\lambda)(1-\beta)^2}{2(1-\lambda\beta)^2(1-\alpha) - \lambda(1-\beta)^2}.$$

The result is sharp for the functions $f_j(z)$ $(j = 1,2)$ defined by (8.2).

Proof. By virtue of Theorem 1, we obtain

$$\sum_{n=2}^{\infty}\left[\frac{n(1-\lambda\beta)C(\alpha,n)}{1-\beta}\right]^2 a_{n,1}^2 \leqslant \left[\sum_{n=2}^{\infty}\frac{n(1-\lambda\beta)C(\alpha,n)}{1-\beta}a_{n,1}\right]^2 \leqslant 1 \qquad (8.4)$$

and

$$\sum_{n=2}^{\infty}\left[\frac{n(1-\lambda\beta)C(\alpha,n)}{1-\beta}\right]^2 a_{n,2}^2 \leqslant \left[\sum_{n=2}^{\infty}\frac{n(1-\lambda\beta)C(\alpha,n)}{1-\beta}a_{n,2}\right]^2 \leqslant 1. \qquad (8.5)$$

It follows from (8.4) and (8.5) that

$$\sum_{n=2}^{\infty}\frac{1}{2}\left[\frac{n(1-\lambda\beta)C(\alpha,n)}{1-\beta}\right]^2\left[a_{n,1}^2 + a_{n,2}^2\right] \leqslant 1.$$

Therefore, we need to find the largest $\phi = \phi(\alpha,\lambda,\beta)$ such that

$$\frac{n(1-\lambda\phi)C(\alpha,n)}{1-\phi} \leqslant \frac{1}{2}\left[\frac{n(1-\lambda\beta)C(\alpha,n)}{1-\beta}\right]^2 \qquad (n \geqslant 2),$$

that is $\phi \leqslant 1 - \dfrac{2(1-\lambda)(1-\beta)^2}{n(1-\lambda\beta)^2C(\alpha,n) - 2\lambda(1-\beta)^2}$ $(n \geqslant 2)$. Since

$$D(n) = 1 - \frac{2(1-\lambda)(1-\beta)^2}{n(1-\lambda\beta)^2C(\alpha,n) - 2\lambda(1-\beta)^2}$$

is an increasing function of n $(n \geqslant 2)$, for $0 \leqslant \alpha \leqslant 1/2$, $0 \leqslant \lambda < 1$ and $0 \leqslant \beta < 1$, we readily have

$$\phi \leqslant D(2) = 1 - \frac{(1-\lambda)(1-\beta)^2}{2(1-\lambda\beta)^2(1-\alpha) - \lambda(1-\beta)^2},$$

and Theorem 15 follows at once. ■

9. Fractional calculus operators

The object of this section is to obtain several growth and distortion properties of functions in the class $L^*_\alpha(\lambda,\beta)$ involving a family of operators of fractional calculus (that is, fractional integral and fractional derivative).

First of all, in terms of Gauss hypergeometric function

$$ {}_2F_1(\delta,\tau;\gamma;z)=\sum_{k=0}^{\infty}\frac{(\delta)_k(\tau)_k}{(\gamma)_k}\frac{z^k}{k!}\quad (z\in U;\ \delta,\tau,\gamma\in\mathbf{C};\ \gamma\neq 0,-1,-2,\dots), $$

where $(m)_k=\dfrac{\Gamma(m+k)}{\Gamma(m)}$ denotes the Pochhammer symbol, we recall the definitions of fractional integral operator $I_{0,z}^{\mu,\nu,\eta}$ and the fractional derivative operator $J_{0,z}^{\mu,\nu,\eta}$ as follows (cf., e.g., [4] and [14], see also [13]).

DEFINITION 1. The fractional integral of order μ is defined, for a function $f(z)$, by

$$ I_{0,z}^{\mu,\nu,\eta}f(z)=\frac{z^{-\mu-\nu}}{\Gamma(\mu)}\int_0^z(z-\zeta)^{\mu-1}{}_2F_1\left(\mu+\nu,-\eta;\mu;1-\frac{\zeta}{z}\right)f(\zeta)\,d\zeta\quad(\mu>0), $$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\mu-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta>0$, provided further that

$$ f(z)=O(|z|^{\varepsilon})\qquad(z\to 0;\ \varepsilon>\max\{0,\nu-\eta\}-1). \tag{9.1} $$

DEFINITION 2. The fractional derivative of order μ is defined, for a function $f(z)$, by

$$ J_{0,z}^{\mu,\nu,\eta}f(z)= $$

$$ =\begin{cases}\dfrac{1}{\Gamma(1-\mu)}\dfrac{d}{dz}\left\{z^{\mu-\nu}\int_0^z(z-\zeta)^{-\mu}{}_2F_1(\nu-\mu,1-\eta;1-\mu;1-(\zeta/z))f(\zeta)\,d\zeta\right\}\\ \hfill(0\leqslant\mu<1)\\ \dfrac{d^n}{dz^n}J_{0,z}^{\mu-n,\nu,\eta}f(z)\qquad\qquad(n\leqslant\mu<n+1;\ n\in\mathbf{N}),\end{cases} $$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\mu}$ is removed, as in Defintion 1, and ε is given by the order estimate (9.1).

It follows from Definitions 1 and 2 that

$$ I_{0,z}^{\mu,-\mu,\eta}f(z)=D_z^{-\mu}f(z)\qquad(\mu>0) \tag{9.2} $$

and

$$ J_{0,z}^{\mu,\mu,\eta}f(z)=D_z^{\mu}f(z)\qquad(0\leqslant\mu<1), \tag{9.3} $$

where $D_z^{\mu}f(z)$ $(\mu\in\mathbf{R})$ is the fractional calculus operator considered by Owa [3] and subsequently by Owa and Srivastava [5] and in many other works (cf., e.g., [12] and [13]). Furthermore, in terms of Gamma functions Definitions 1 and 2 readily yield

LEMMA 1. (cf. Srivastava et al. [14]) *The (generalized) fractional integral and the (generalized) fractional derivative of a power function are given by*

$$ I_{0,z}^{\mu,\nu,\eta}z^{\rho}=\frac{\Gamma(\rho+1)\Gamma(\rho-\nu+\eta+1)}{\Gamma(\rho-\nu+1)\Gamma(\rho+\mu+\eta+1)}\,z^{\rho-\nu}\quad(\mu>0;\ \rho>\max\{0,\nu-\eta\}-1) \tag{9.4} $$

and

$$J_{0,z}^{\mu,\nu,\eta} z^{\rho} = \frac{\Gamma(\rho+1)\Gamma(\rho-\nu+\eta+1)}{\Gamma(\rho-\nu+1)\Gamma(\rho-\mu+\eta+1)}\, z^{\rho-\nu} \quad (0 \leqslant \mu < 1;\ \rho > \max\{0, \nu-\eta\}-1). \tag{9.5}$$

THEOREM 17. *Let the function $f(z)$ defined by (1.2) be in the class $L_{\alpha}^{*}(\lambda,\beta)$, with $0 \leqslant \alpha \leqslant 1/2$, $0 \leqslant \lambda < 1$ and $0 \leqslant \beta < 1$. Then*

$$\begin{aligned} &\frac{\Gamma(2-\nu+\eta)}{\Gamma(2-\nu)\Gamma(2+\mu+\eta)}|z|^{1-\nu}\left\{1-\frac{(1-\beta)(2-\nu+\eta)}{2(2-\nu)(2+\mu+\eta)(1-\lambda\beta)(1-\alpha)}|z|\right\} \\ &\leqslant |I_{0,z}^{\mu,\nu,\eta} f(z)| \leqslant \\ &\leqslant \frac{\Gamma(2-\nu+\eta)}{\Gamma(2-\nu)\Gamma(2+\mu+\eta)}|z|^{1-\nu}\left\{1+\frac{(1-\beta)(2-\nu+\eta)}{2(2-\nu)(2+\mu+\eta)(1-\lambda\beta)(1-\alpha)}|z|\right\} \end{aligned} \tag{9.6}$$

$(z \in U_0;\ \mu > 0,\ \max\{\nu, \nu-\eta, -\mu-\eta\} < 2;\ \nu(\mu+\eta) \leqslant 3\mu)$, and

$$\begin{aligned} &\frac{\Gamma(2-\nu+\eta)}{\Gamma(2-\nu)\Gamma(2-\mu+\eta)}|z|^{1-\nu}\left\{1-\frac{(1-\beta)(2-\nu+\eta)}{2(2-\nu)(2-\mu+\eta)(1-\lambda\beta)(1-\alpha)}|z|\right\} \leqslant \\ &\leqslant |J_{0,z}^{\mu,\nu,\eta} f(z)| \\ &\leqslant \frac{\Gamma(2-\nu+\eta)}{\Gamma(2-\nu)\Gamma(2-\mu+\eta)}|z|^{1-\nu}\left\{1+\frac{(1-\beta)(2-\nu+\eta)}{2(2-\nu)(2-\mu+\eta)(1-\lambda\beta)(1-\alpha)}|z|\right\} \end{aligned} \tag{9.7}$$

$(z \in U_0;\ 0 \leqslant \mu < 1,\ \max\{\nu, \nu-\eta, \mu-\eta\} < 2;\ \nu(\mu-\eta) \geqslant 3\mu)$, where $U_0 = \begin{cases} U, & (\nu \leqslant 1), \\ U\setminus\{0\}, & (\nu > 1). \end{cases}$ Each of these results is sharp for the function $f(z)$ given by

$$f(z) = z - \frac{1-\beta}{4(1-\lambda\beta)(1-\alpha)}\, z^2. \tag{9.8}$$

Proof. First of all, since the function $f(z)$ defined by (1.2) is in the class $L_{\alpha}^{*}(\lambda,\beta)$, $0 \leqslant \alpha \leqslant 1/2$, $0 \leqslant \lambda < 1$ and $0 \leqslant \beta < 1$, we can apply Theorem 1 to deduce that

$$\sum_{n=2}^{\infty} a_n \leqslant \frac{1-\beta}{4(1-\lambda\beta)(1-\alpha)}. \tag{9.9}$$

Next, making use of the assertion 9.4 of Lemma 1, we find from (1.2) that

$$F(z) = \frac{\Gamma(2-\nu)\Gamma(2+\mu+\eta)}{\Gamma(2-\nu+\eta)}\, z^{\nu} I_{0,z}^{\mu,\nu,\eta} f(z) = z - \sum_{n=2}^{\infty} \Phi(n) a_n z^n, \tag{9.10}$$

where, for convenience,

$$\Phi(n) = \frac{(1)_n (2-\nu+\eta)_{n-1}}{(2-\nu)_{n-1}(2+\mu+\eta)_{n-1}} \quad (n \in \mathbf{N}\setminus\{1\}). \tag{9.11}$$

The function $\Phi(n)$ defined by (9.11) can easily be seen to be nonincreasing under the parametric constraints stated already after (9.6), and thus we have

$$0 < \Phi(n) \leqslant \Phi(2) = \frac{2(2-\nu+\eta)}{(2-\nu)(2+\mu+\eta)} \quad (n \in \mathbf{N}\setminus\{1\}). \tag{9.12}$$

Now the assertion (9.6) of the theorem follows readily from (9.9), (9.10) and (9.12).

The assertion (9.7) of the theorem can be proven similarly by noting from (9.5) that

$$G(z)=\frac{\Gamma(2-\nu)\Gamma(2-\mu+\eta)}{\Gamma(2-\nu+\eta)}\,z^{\nu}J_{0,z}^{\mu,\nu,\eta}f(z)=z-\sum_{n=2}^{\infty}\Psi(n)a_nz^n,$$

where

$$0<\Psi(n)=\frac{(1)_n(2-\nu+\eta)_{n-1}}{(2-\nu)_{n-1}(2-\mu+\eta)_{n-1}}\leqslant\Psi(2)=\frac{2(2-\nu+\eta)}{(2-\nu)(2-\mu+\eta)},$$

$(n\in\mathbf{N}\setminus\{1\})$ under the parametric constraints stated already after (9.7).

Finally, by observing that the equalities in each of the assertions (9.6) and (9.7) are attained by the function $f(z)$ given by (9.8), we complete the proof of the theorem. ■

In view of the relationships (9.2) and (9.3), by setting $\nu=-\mu$ and $\nu=\mu$ in our assertions (9.6) and (9.7), respectively, we obtain

COROLLARY 5. *Let the function $f(z)$ defined by (1.2) be in the class $L_\alpha^*(\lambda,\beta)$, $0\leqslant\alpha\leqslant 1/2$, $0\leqslant\lambda<1$ and $0\leqslant\beta<1$. Then*

$$\frac{|z|^{1+\mu}}{\Gamma(2+\mu)}\left\{1-\frac{1-\beta}{2(2+\mu)(1-\lambda\beta)(1-\alpha)}\,|z|\right\}\leqslant|D_z^{-\mu}f(z)|\leqslant$$
$$\leqslant\frac{|z|^{1+\mu}}{\Gamma(2+\mu)}\left\{1+\frac{1-\beta}{2(2+\mu)(1-\lambda\beta)(1-\alpha)}\,|z|\right\}\quad(z\in U;\ \mu>0)\quad(9.13)$$

and

$$\frac{|z|^{1-\mu}}{\Gamma(2-\mu)}\left\{1-\frac{1-\beta}{2(2-\mu)(1-\lambda\beta)(1-\alpha)}\,|z|\right\}\leqslant|D_z^{\mu}f(z)|\leqslant$$
$$\leqslant\frac{|z|^{1-\mu}}{\Gamma(2-\mu)}\left\{1+\frac{1-\beta}{2(2-\mu)(1-\lambda\beta)(1-\alpha)}\,|z|\right\}\quad(z\in U;\ 0\leqslant\mu<1).\quad(9.14)$$

Each of these results is sharp for the function $f(z)$ given by (9.8).

The assertions (9.13) and (9.14) of Corollary 5 can indeed be applied further in order to deduce the following interesting results for functions in the class $L_\alpha^*(\lambda,\beta)$.

COROLLARY 6. *Under the hypotheses of Corollary 5, $D_z^{-\mu}f(z)$ $(\mu>0)$ is included in the disc with its center at the origin and radius r_1 given by*

$$r_1=\frac{1}{\Gamma(2+\mu)}\left\{1+\frac{1-\beta}{2(2+\mu)(1-\lambda\beta)(1-\alpha)}\right\}.$$

COROLLARY 7. *Under the hypotheses of Corollary 5, $D_z^{\mu}f(z)$ $(0\leqslant\mu<1)$ is included in the disc with its center at the origin and radius r_2 given by*

$$r_2=\frac{1}{\Gamma(2-\mu)}\left\{1+\frac{1-\beta}{2(2-\mu)(1-\lambda\beta)(1-\alpha)}\right\}.$$

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(received 03.08.2002)

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