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SUFFICIENT CONDITIONS FOR ELLIPTIC PROBLEM OF OPTIMAL CONTROL IN $\mathbf{R}^n$, WHERE $n > 2$

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Abstract. This paper is concerned with the local minimization problem for a variety of non Frechet-differentiable Gâteaux functional $J(f) \equiv \int_Q v(x,u,f)\,dx$ in the Sobolev space $(W_0^{1,2}(Q), \|\cdot\|_p)$, where u is the solution of the Dirichlet problem for a linear uniformly elliptic operator with nonhomogenous term f and $\|\cdot\|_p$ is the norm generated by the metric space $L^p(Q)$, $(p > 1)$. We use a recent extension of Frechet-differentiability (approach of Taylor mappings, see [5]), and we give various assumptions on v to guarantee a critical point to be a strict local minimum. Finally, we give an example of a control problem where classical Frechet differentiability cannot be used and their approach of Taylor mappings works.

1. Preliminaries

1.1. Description of the optimization problem

Let A be an elliptic operator of second order

$$Au \equiv \sum_{|l|\le 1, |s|\le 1} (-1)^l \mathcal{D}^l(a_{ls}(x)\mathcal{D}^s u),$$

where $a_{ls}(x) \in \mathcal{D}(\overline{Q})$. Suppose that Q is a sufficiently smooth and bounded domain in$\mathbf{R}^n$. Let us consider the problem

$$Au = f, \tag{1.1}$$

$$u/_{\partial Q} = 0. \tag{1.2}$$

For this problem, let us state Agmon's-Douglis-Niremberg's theorem.

THEOREM 1.1. *If $1 < q < \infty$, then we have that $\forall f \in L^q(Q)$, there exists a unique solution $u \in W^{2,q}(Q) \cap W_0^{1,q}(Q)$ of problem (1.1), (1.2). Moreover, $\forall m \ge 0$ if $f \in W^{m,q}(Q)$, then $u \in W^{m+2,q}(Q)$ and $\|u\|_{W^{m+2,q}(Q)} \le c\|f\|_{W^{m,q}(Q)}$.*

Let $f \in F \subset W_0^{1,2}(Q)$ be a control and let u be the solution of problem (1.1), (1.2) in $W_0^{1,2}(Q) \cap W^{2,2}(Q)$ associated to f. Let us consider $J_k(f) = \int_Q v_k(x,u,f)\,dx + c_k\|f\|^2_{W^{1,2}(Q)}$, $(k = 0,1,2,\dots,s_1)$ and $J_k(f) = \int_Q v_k(x,u,f)\,dx$,

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($k = s_1+1, s_1+2, \dots, s_1+s_2$), where the sequence of functions $v_k\colon Q\times\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ is measurable on $Q\times\mathbf{R}\times\mathbf{R}$ and has second derivative with respect to (u,f) on $\mathbf{R}\times\mathbf{R}$ for almost all $x\in Q$.

We consider three problems of minimizing the functional $J_0(f)$:

$$i)\ J_0(f)\to\min, \tag{1.3}$$

$$ii)\ J_0(f)\to\min,\ J(f)=0,\ \text{where } J=(J_{s_1+1},\dots,J_{s_1+s_2}), \tag{1.4}$$

$$iii)\ J_0(f)\to\min,\ J(f)=0,\ J_k(f)\le 0,\ (k=1,2,\dots,s_1). \tag{1.5}$$

We must choose a control f^0 in order that the solution u^0 of the problem (1.1), (1.2) with $f=f^0$ satisfies the inequality type $J_k(f)\le 0$, $(1\le k\le s_1)$ and the equality type $J_k(f)=0$, $(s_1+1\le k\le s_1+s_2)$ and the functional $J_0(f)$ takes the minimum value. This control f^0 will be called optimal.

1.2. Taylor mappings and lower semi-Taylor mappings

Let $\|\cdot\|_{W^{1,2}(Q)}$ be the usual norm in $W_0^{1,2}(Q)$, F a subset of $W_0^{1,2}(Q)$, τ a topology in F, Y a normed space, and $\|\cdot\|_Y$ the norm in Y. According to [5], a mapping $r\colon F\to Y$ (respectively, $r\colon F\to\mathbf{R}$) is said to be infinitesimally $(\tau,\|\cdot\|_{W^{1,2}(Q)})$-small (respectively, infinitesimally lower $(\tau,\|\cdot\|_{W^{1,2}(Q)})$-semismall) of order p_1 at $f\in F$ if: $\forall\varepsilon>0$, $\exists\ O_f\in\tau$, $\forall h\in W_0^{1,2}(Q)$ we have

$$f+h\in O_f \implies \|r(f+h)\|_Y\le\varepsilon\|h\|^{p_1}_{W^{1,2}(Q)},$$

(respectively $\forall\varepsilon>0$, $\exists O_f\in\tau$, $\forall h\in W_0^{1,2}(Q)$ we have

$$f+h\in O_f \implies r(f+h)\ge-\varepsilon\|h\|^{p_1}_{W^{1,2}(Q)});$$

here and below, O_f is a neighborhood of f in (F,τ).

A mapping $J\colon F\to Y$ (respectively, $J\colon F\to\mathbf{R}$) is called a $(\tau,\|\cdot\|_{W^{1,2}(Q)})$-Taylor (respectively, lower $(\tau,\|\cdot\|_{W^{1,2}(Q)})$-semi-Taylor) mapping of order p_1 at $f\in F$ if there exist k linear symmetric (not necessarily continuous) mappings $J^{(k)}(f)\colon (W_0^{1,2}(Q))^k\to Y$ (respectively, $J^{(k)}(f)\colon (W_0^{1,2}(Q))^k\to\mathbf{R}$), $k=1,\dots,p_1$, such that

$$\begin{aligned}&J(f+h)-J(f)=\\&\quad=J^{(1)}(f)h+2^{-1}J^{(2)}(f)(h,h)+\dots+(p_1)!^{-1}J^{(p_1)}(f)(h,\dots,h)+r(f+h),\end{aligned}$$

where $r\colon F\to Y$ (respectively, $r\colon F\to\mathbf{R}$) is an infinitesimally $(\tau,\|\cdot\|_{W^{1,2}(Q)})$-small (respectively, infinitesimally lower $(\tau,\|\cdot\|_{W^{1,2}(Q)})$-semismall) mapping of order p_1 at $f\in F$.

We note that $J^{(1)}(f),\dots,J^{(p_1)}(f)$ are not in general single-valued. The set of tuples $(J^{(1)}(f),\dots,J^{(p_1)}(f))$ is denoted by $S_n(J,f)$.

Let us solve the problems (1.3), (1.4) and (1.5).

For the problem (1.5) let us introduce the Lagrange functions:

$$\mathcal{L}(f, y^*, \lambda, \lambda_0) = \sum_{k=0}^{s_1} \lambda_k J_k(f) + \langle y^*, J(f)\rangle, \tag{1.6}$$

$$\mathcal{L}_f(f, y^*, \lambda, \lambda_0) = \sum_{k=0}^{s_1} \lambda_k J_k^{(1)}(f) + \langle y^*, J^{(1)}(f)\rangle, \tag{1.7}$$

$$\mathcal{L}_{ff}(f, y^*, \lambda, \lambda_0) = \sum_{k=0}^{s_1} \lambda_k J_k^{(2)}(f) + \langle y^*, J^{(2)}(f)\rangle, \tag{1.8}$$

where $\lambda_0 \in \mathbf{R}$, $y^* \in (\mathbf{R}^{s_2})^*$, $\lambda \in (\mathbf{R}^{s_1})^*$.

Similarly, for the problem (1.4), let us introduce the Lagrange functions:

$$\mathcal{L}(f, y^*, \lambda_0) = \lambda_0 J_0(f) + \langle y^*, J(f)\rangle, \tag{1.9}$$

$$\mathcal{L}_f(f, y^*, \lambda_0) = \lambda_0 J_0^{(1)}(f) + \langle y^*, J^{(1)}(f)\rangle \tag{1.10}$$

$$\mathcal{L}_{ff}(f, y^*, \lambda_0) = \lambda_0 J_0^{(2)}(f) + \langle y^*, J^{(2)}(f)\rangle, \tag{1.11}$$

where $\lambda_0 \in \mathbf{R}$, $y^* \in (\mathbf{R}^{s_2})^*$.

Let us give the following lemma where the proof can be traced back to [5].

LEMMA 1.1. *Let (Ω, Σ, μ) be a measure space with σ-finite measure, and let X be a complete linear metric space continuously imbedded in the metric space $M(\Omega)$ of equivalence classes of measurable almost everywhere finite functions $x\colon \Omega \to \mathbf{R}$, with the metrizable topology $\tau(meas)$ of convergence in measure on each set of σ-finite measure.*

Suppose that X contains with each element $x(s)$ the function $|x(s)|$, the metric in X is translation-invariant, and $\rho(x, 0) = \rho(|x|, 0)$ for each $x \in X$. Then for each sequence $x_n \to 0$ in X there exist a subsequence x_{n_k} and an element $z \in X$ such that: $|x_{n_k}(s)| \le z(s)$, $k = 1, 2, \ldots$ in the sense of the natural order on classes of functions.

2. Sufficient conditions of local minimum for Gâteaux functional of second order Dirichlet problem

Suppose that Q is a sufficiently smooth and bounded domain in $\mathbf{R}^n$, where $n > 2$. Let F be a subset of $W_0^{1,2}(Q)$. Let G be the functional defined on F by $G(f) = \int_Q v(x, u(x), f(x))\, dx$, where $u(x)$ is the solution of problem (1.1), (1.2) in $W_0^{1,2}(Q) \cap W^{2,2}(Q)$ and the function $v\colon Q \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is measurable on $Q \times \mathbf{R} \times \mathbf{R}$ and has second derivative with respect to (u, f) on $\mathbf{R} \times \mathbf{R}$ for almost all $x \in Q$. Suppose also that $v, v_{uf}^{(2)}, v_{fu}^{(2)}$ are continuous in $Q \times \mathbf{R} \times \mathbf{R}$.

Let τ_p be the topology generated by the metric space $L^p(Q)$, where $p > 1$. In the rest of this section $a = const$.

THEOREM 2.1. *Suppose that the following conditions are added to the conditions of paragraph (1) and (2):*

$$|v(x,u,f)| \le a(|u|^{\nu} + |f|^{\nu}) + b_0(x),$$
$$|v_u^{(1)}(x,u,f)| + |v_f^{(1)}(x,u,f)| \le a(|u|^{\nu-1} + |f|^{\nu-1}) + b_1(x),$$
$$|v_{uu}^{(2)}(x,u,f)| + 2|v_{uf}^{(2)}(x,u,f)| + |v_{ff}^{(2)}(x,u,f)| \le a(|u|^{\frac{2p}{n}} + |f|^{\frac{2p}{n}}) + b_2(x),$$

where $\nu = \frac{2n}{n-2}$, $b_0(x) \in L^1(Q)$, $b_1(x) \in L^{\frac{2n}{n+2}}(Q)$, $b_2(x) \in L^{\frac{n}{2}}(Q)$, *and* $1 < p \le \frac{2n}{n-2}$. *Then* G *is a* $(\tau_p, \|.\|_{W^{1,2}(Q)})$*-Taylor mapping of second order at each point* $f \in F$. *Moreover,1* $G^{(2)}(f) \in \mathcal{B}((W_0^{1,2}(Q), \|\cdot\|_{W^{1,2}(Q)}), \mathbf{R})$ *and* $G^{(1)}(f) \in \mathcal{L}((W_0^{1,2}(Q), \|\cdot\|_{W^{1,2}(Q)}), \mathbf{R})$.

Proof. Let us prove first that the functional G is finite. We have

$$\begin{aligned} |G(f)| &= \left|\int_Q v(x,u,f)\,dx\right| \le \int_Q |v(x,u,f)|\,dx \\ &\le a\Big(\int_Q |u(x)|^{\nu}\,dx + \int_Q |f(x)|^{\nu}\,dx\Big) + \int_Q b_0(x)\,dx \\ &\le a\big(\|u(x)\|^{\nu}_{L^{\nu}(Q)} + \|f(x)\|^{\nu}_{L^{\nu}(Q)}\big) + \|b_0(x)\|_{L^1(Q)} \\ &\le c_1\big(\|u(x)\|^{\nu}_{W^{1,2}(Q)} + \|f(x)\|^{\nu}_{W^{1,2}(Q)}\big) + \|b_0(x)\|_{L^1(Q)} < \infty. \end{aligned}$$

Thus the functional G is finite.

Let $R\colon W_0^{1,2}(Q) \to W_0^{1,2}(Q)$, where $(R(h))(x)$ is a solution of the problem

$$Au = h, \tag{2.1}$$
$$u/_{\partial Q} = 0. \tag{2.2}$$

Such a solution exists $\forall h \in W_0^{1,2}(Q)$.

Let $G^{(1)}(f)$ and $G^{(2)}(f)$ be defined by:

$$\begin{aligned} G^{(1)}(f)h &= \lim_{\lambda\to 0} \lambda^{-1}(G(f+\lambda h) - G(f)) \\ &= \lim_{\lambda\to 0} \lambda^{-1} \int_Q [v(x, u+\lambda R(h), f+\lambda h) - v(x,u,f)]\,dx \\ &= \lim_{\lambda\to 0} \lambda^{-1} \int_Q [v(x, u+\lambda R(h), f+\lambda h) - v(x,u,f+\lambda h) \\ &\qquad + v(x,u,f+\lambda h) - v(x,u,f)]\,dx \\ &= \lim_{\lambda\to 0} \int_Q \Big[\int_0^1 v_u^{(1)}(x, u+\theta\lambda R(h), f+\lambda h) R(h)\,d\theta \\ &\qquad + \int_0^1 v_f^{(1)}(x,u,f+\rho\lambda h) h\,d\rho\Big]\,dx \\ &= \lim_{\lambda\to 0} \int_Q \Big[\int_0^1 [v_u^{(1)}(x, u+\theta\lambda R(h), f+\lambda h) - v_u^{(1)}(x,u,f)] R(h)\,d\theta \end{aligned}$$

$$+\int_0^1 v_u^{(1)}(x,u,f)R(h)\,d\theta + \int_0^1 [v_f^{(1)}(x,u,f+\rho\lambda h) - v_f^{(1)}(x,u,f)]h\,d\rho$$
$$+ h\int_0^1 v_f^{(1)}(x,u,f)\,d\rho\Big]\,dx$$
$$= \int_Q v_u^{(1)}(x,u,f)R(h)\,dx + \int_Q h v_f^{(1)}(x,u,f)\,dx$$

and

$$G^{(2)}(f)(h_1,h_2) = \lim_{\lambda\to 0}\lambda^{-1}[G^{(1)}(f+\lambda h_2) - G^{(1)}(f)]h_1$$
$$= \lim_{\lambda\to 0}\lambda^{-1}\Big[\int_Q \big[v_u^{(1)}(x,u+\lambda R(h_2),f+\lambda h_2) - v_u^{(1)}(x,u,f)\big]R(h_1)\,dx$$
$$+ \int_Q \big[v_f^{(1)}(x,u+\lambda R(h_2),f+\lambda h_2) - v_f^{(1)}(x,u,f)\big]h_1\,dx\Big]$$
$$= \lim_{\lambda\to 0}\lambda^{-1}\Big[\int_Q \big[v_u^{(1)}(x,u+\lambda R(h_2),f+\lambda h_2) - v_u^{(1)}(x,u,f+\lambda h_2)$$
$$+ v_u^{(1)}(x,u,f+\lambda h_2) - v_u^{(1)}(x,u,f)\big]R(h_1)\,dx$$
$$+ \int_Q \big[v_f^{(1)}(x,u+\lambda R(h_2),f+\lambda h_2) - v_f^{(1)}(x,u,f+\lambda h_2)$$
$$+ v_f^{(1)}(x,u,f+\lambda h_2) - v_f^{(1)}(x,u,f)\big]h_1\,dx\Big]$$
$$= \lim_{\lambda\to 0}\lambda^{-1}\Big[\int_Q\Big[\int_0^1 v_{uu}^{(2)}(x,u+\theta\lambda R(h_2),f+\lambda h_2)\lambda R(h_2)\,d\theta$$
$$+ \int_0^1 v_{fu}^{(2)}(x,u,f+\rho\lambda h_2)\lambda h_2\,d\rho\big]R(h_1)\,dx$$
$$+ \int_Q\Big[\int_0^1 v_{uf}^{(2)}(x,u+\theta\lambda R(h_2),f+\lambda h_2)\lambda R(h_2)\,d\theta$$
$$+ \int_0^1 v_{ff}^{(2)}(x,u,f+\rho\lambda h_2)\lambda h_2\,d\rho\Big]h_1 dx\Big]$$
$$= \int_Q v_{uu}^{(2)}(x,u,f)R(h_1)R(h_2)\,dx + \int_Q v_{uf}^{(2)}(x,u,f)R(h_1)h_2\,dx$$
$$+ \int_Q v_{fu}^{(2)}(x,u,f)h_1R(h_2)\,dx + \int_Q v_{ff}^{(2)}(x,u,f)h_1h_2\,dx.$$

Therefore $G^{(1)}(f) = \int_Q v_u^{(1)}(x,u,f)R(h)\,dx + \int_Q v_f^{(1)}(x,u,f)h\,dx$ and

$$G^{(2)}(f)(h_1,h_2) = \int_Q v_{uu}^{(2)}(x,u,f)R(h_1)R(h_2)\,dx + \int_Q v_{uf}^{(2)}(x,u,f)R(h_1)h_2\,dx$$
$$+ \int_Q v_{fu}^{(2)}(x,u,f)h_1R(h_2)\,dx + \int_Q v_{ff}^{(2)}(x,u,f)h_1h_2\,dx.$$

The linearity and bilinearity of $G^{(1)}(f)$ and $G^{(2)}(f)$ are obvious. Let us prove now that they are bounded.

We have

$$
\begin{aligned}
|G^{(1)}(f)h| &\le \int_Q |v_u^{(1)}(x,u,f)R(h)|\,dx + \int_Q |v_f^{(1)}(x,u,f)h|\,dx \\
&\le \int_Q \Big[a\big(|u(x)|^{\nu-1} + |f(x)|^{\nu-1}\big) + |b_1(x)|\Big]\big[|R(h)| + |h|\big]\,dx \\
&\le \Bigg[a\bigg[\int_Q \big(|u(x)|^{\nu-1}\big)^{\frac{2n}{n+2}}\,dx\bigg]^{\frac{n+2}{2n}} + a\bigg[\int_Q \big(|f(x)|^{\nu-1}\big)^{\frac{2n}{n+2}}dx\bigg]^{\frac{n+2}{2n}} \\
&\quad + \bigg[\int_Q |b_1(x)|^{\frac{2n}{n+2}}\,dx\bigg]^{\frac{n+2}{2n}}\Bigg]\Bigg[\bigg[\int_Q |R(h)|^{\frac{2n}{n-2}}\,dx\bigg]^{\frac{n-2}{2n}} + \bigg[\int_Q |h|^{\frac{2n}{n-2}}\,dx\bigg]^{\frac{n-2}{2n}}\Bigg] \\
&= \Big[a\Big(\|u(x)\|_{L^p(Q)}^{\frac{n+2}{2n}} + \|f(x)\|_{L^p(Q)}^{\frac{n+2}{2n}}\Big) + \|b_1(x)\|_{L^{\frac{2n}{n+2}}(Q)}\Big]\times \\
&\quad \times \Big[\|R(h)\|_{L^{\frac{2n}{n-2}}(Q)} + \|h\|_{L^{\frac{2n}{n-2}}(Q)}\Big] \\
&\le c_0\Big[\|u(x)\|_{W^{1,2}(Q)}^{\frac{n+2}{2n}} + \|f(x)\|_{W^{1,2}(Q)}^{\frac{n+2}{2n}} + \|b_1(x)\|_{L^{\frac{2n}{n+2}}(Q)}\Big]\times \\
&\quad \times \Big[\|R(h)\|_{W^{1,2}(Q)} + \|h\|_{W^{1,2}(Q)}\Big].
\end{aligned}
$$

Thus $\exists\ c_2 > 0$ such that

$$|G^{(1)}(f)h| \le c_2\big(\|R(h)\|_{W^{1,2}(Q)} + \|h\|_{W^{1,2}(Q)}\big).$$

On the other hand, $R(h)$ depends continually on h, thus $|G^{(1)}(f)h| \le c_3\|h\|_{W^{1,2}(Q)}$, where $c_3 > 0$. Consequently, $G^{(1)}(f) \in \mathcal{L}((W_0^{1,2}(Q), \|.\|_{W^{1,2}(Q)}), \mathbf{R})$.

Let us prove now that $G^{(2)}(f)$ is also bounded. We have

$$
\begin{aligned}
|G^{(2)}(f)(h_1,h_2)| &\le \int_Q |v_{uu}^{(2)}(x,u,f)R(h_1)R(h_2)|\,dx + \int_Q |v_{uf}^{(2)}(x,u,f)R(h_1)h_2|\,dx \\
&\quad + \int_Q |v_{fu}^{(2)}(x,u,f)h_1R(h_2)|\,dx + \int_Q |v_{ff}^{(2)}(x,u,f)h_1h_2|\,dx \\
&\le \int_Q |v_{uu}^{(2)}(x,u,f)R(h_1)R(h_2)|\,dx \\
&\quad + \int_Q 2|v_{uf}^{(2)}(x,u,f)|\big[|R(h_1)||h_2| + |R(h_2)||h_1|\big]\,dx + \int_Q |v_{ff}^{(2)}(x,u,f)h_1h_2|\,dx \\
&\le \int_Q \big[|v_{uu}^{(2)}(x,u,f)| + 2|v_{uf}^{(2)}(x,u,f)| + |v_{ff}^{(2)}(x,u,f)|\big] \\
&\quad \times \big[|R(h_1)R(h_2)| + |R(h_1)h_2| + |h_1R(h_2)| + |h_1h_2|\big]\,dx \\
&\le \int_Q \Big[a\Big(|u|^{\frac{2p}{n}} + |f|^{\frac{2p}{n}}\Big) + |b_2(x)|\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\times \left[|R(h_1)R(h_2)| + |R(h_1)h_2| + |h_1R(h_2)| + |h_1h_2|\right] dx \\
&\leq \left[a\left[\int_Q \left(|u(x)|^{\frac{2p}{n}}\right)^{\frac{n}{2}} dx\right]^{\frac{2}{n}} + a\left[\int_Q \left(|f(x)|^{\frac{2p}{n}}\right)^{\frac{n}{2}} dx\right]^{\frac{2}{n}} + \left[\int_Q |b_2(x)|^{\frac{n}{2}} dx\right]^{\frac{2}{n}}\right] \\
&\times \left[\left[\int_Q \left(|R(h_1)||R(h_2)|\right)^{\frac{n}{n-2}} dx\right]^{\frac{n-2}{n}} + \left[\int_Q \left(|R(h_1)||h_2|\right)^{\frac{n}{n-2}} dx\right]^{\frac{n-2}{n}}\right. \\
&+ \left.\left[\int_Q \left(|h_1||R(h_2)|\right)^{\frac{n}{n-2}} dx\right]^{\frac{n-2}{n}} + \left[\int_Q \left(|h_1||h_2|\right)^{\frac{n}{n-2}} dx\right]^{\frac{n-2}{n}}\right] \\
&\leq \Big[\|R(h_1)\|_{L^{\frac{2n}{n-2}}(Q)}\|R(h_2)\|_{L^{\frac{2n}{n-2}}(Q)} + \|R(h_1)\|_{L^{\frac{2n}{n-2}}(Q)}\|h_2\|_{L^{\frac{2n}{n-2}}(Q)} \\
&+ \|h_1\|_{L^{\frac{2n}{n-2}}(Q)}\|R(h_2)\|_{L^{\frac{2n}{n-2}}(Q)} + \|h_1\|_{L^{\frac{2n}{n-2}}(Q)}\|h_2\|_{L^{\frac{2n}{n-2}}(Q)}\Big] \\
&\times \left[a(\|u(x)\|^{\frac{2p}{n}}_{L^p(Q)} + \|f(x)\|^{\frac{2p}{n}}_{L^p(Q)}) + \|b_2(x)\|_{L^{\frac{n}{2}}(Q)}\right] \\
&\leq c_4\Big[\|R(h_1)\|_{W^{1,2}(Q)}\|R(h_2)\|_{W^{1,2}(Q)} + \|R(h_1)\|_{W^{1,2}(Q)}\|h_2\|_{W^{1,2}(Q)} \\
&+ \|h_1\|_{W^{1,2}(Q)}\|R(h_2)\|_{W^{1,2}(Q)} + \|h_1\|_{W^{1,2}(Q)}\|h_2\|_{W^{1,2}(Q)}\Big] \\
&\leq c_5\|h_1\|_{W^{1,2}(Q)}\|h_2\|_{W^{1,2}(Q)}.
\end{aligned}
$$

Thus $G^{(2)}(f) \in \mathcal{B}((W_0^{1,2}(Q), \|\cdot\|_{W^{1,2}(Q)}), \mathbf{R})$.

Let us prove now that G is a $(\tau_p, \|.\|_{W^{1,2}(Q)})$-Taylor mapping, where τ_p is the topology generated by $L^p(Q)$.

Let $f \in F$ and let us prove that $r(h) = G(f+h) - G(f) - G^{(1)}(f)h - 2^{-1}G^{(2)}(f)(h,h)$ is infinitesimally $(\tau_p, \|.\|_{W^{1,2}(Q)})$-small of second order at zero. Assume the contrary. Then $\exists (\tilde{h}_m)_{m\in\mathbf{N}} \in F$ and $\varepsilon > 0$ such that $\tilde{h}_m \to 0$ in $L^p(Q)$ and $r(\tilde{h}_m) \geq \varepsilon \|\tilde{h}_m\|^2_{W^{1,2}(Q)}$.

Using the Agmon's-Douglis-Niremberg's theorem, we obtain $R(\tilde{h}_m) \to 0$ in $L^p(Q)$ and using Lemma 1.1, we deduce that $\exists \tilde{Z}(x) \in L^p(Q)$ such that $|(R(\tilde{h}_m))(x)| \leq \tilde{z}(x)$.

Let $\tilde{Z}_0(x) = \tilde{z}(x) + |u(x)|$, then $|u(x)| + |(R(\tilde{h}_m))(x)| \leq \tilde{Z}_0(x)$, where $\tilde{Z}_0(x) \in L^p(Q)$. Analogously for $f \in W^{1,2}(Q)$, we obtain $|f(x)| + |\tilde{h}_m| \leq \tilde{Z}_1$, where $\tilde{Z}_1 \in L^p(Q)$. We have

$$
\begin{aligned}
r(h) = \int_Q &\left[v(x, u+R(h), f+h) - v(x,u,f) - v_u^{(1)}(x,u,f)R(h) - v_f^{(1)}(x,u,f)h\right. \\
&\left.- 2^{-1}\left[v_{uu}^{(2)}(x,u,f)R^2(h) + 2v_{uf}^{(2)}(x,u,f)R(h)h + v_{ff}^{(2)}(x,u,f)h^2\right]\right] dx.
\end{aligned}
$$

Indeed,

$$
\begin{aligned}
&v(x, u+R(h), f+h) - v(x,u,f) \\
&\quad = v(x, u+R(h), f+h) - v(x,u,f+h) + v(x,u,f+h) - v(x,u,f)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^1 v_u^{(1)}(x, u+\theta R(h), f+h) R(h)\, d\theta + \int_0^1 v_f^{(1)}(x, u, f+\lambda h) h\, d\lambda \\
&= \int_0^1 R(h) \big[v_u^{(1)}(x, u+\theta R(h), f+h) - v_u^{(1)}(x, u+\theta R(h), f) \\
&\qquad + v_u^{(1)}(x, u+\theta R(h), f) \big]\, d\theta + \int_0^1 v_f^{(1)}(x, u, f+\lambda h) h\, d\lambda \\
&= \int_0^1 R(h) v_u^{(1)}(x, u+\theta R(h), f)\, d\theta + \int_0^1 v_f^{(1)}(x, u, f+\lambda h) h\, d\lambda \\
&\qquad + \int_0^1 \int_0^1 v_{fu}^{(2)}(x, u+\theta R(h), f+\lambda h) h R(h)\, d\lambda\, d\theta.
\end{aligned}
$$

So,

$$
\begin{aligned}
r(h) = &\int_Q \int_0^1 \Big[v_u^{(1)}(x, u+\theta R(h), f+h) R(h) - v_u^{(1)}(x, u, f) R(h) \\
&\quad - 2^{-1}(v_{uu}^{(2)}(x, u, f) R^2(h) \Big]\, d\theta\, dx \\
&+ \int_Q \int_0^1 \big[v_f^{(1)}(x, u, f+\lambda h) h - v_f^{(1)}(x, u, f) h - 2^{-1} v_{ff}^{(2)}(x, u, f) h^2 \big]\, d\lambda\, dx \\
&- \int_Q \int_0^1 v_{uf}^{(2)}(x, u, f) R(h) h\, d\lambda\, dx \\
&+ \int_Q \int_0^1 \int_0^1 v_{uf}^{(2)}(x, u+\theta R(h), f+\lambda h) h R(h)\, d\lambda\, d\theta\, dx \\
= &\int_Q \int_0^1 \big[v_u^{(1)}(x, u+\theta R(h), f) - v_u^{(1)}(x, u, f) - 2^{-1}(v_{uu}^{(2)}(x, u, f) R(h) \big] R(h)\, d\theta\, dx \\
&+ \int_Q \int_0^1 \big[v_f^{(1)}(x, u, f+\lambda h) - v_f^{(1)}(x, u, f) - 2^{-1} v_{ff}^{(1)}(x, u, f) h \big] h\, d\lambda\, dx \\
&- \int_Q \int_0^1 v_{uf}^{(2)}(x, u, f) R(h) h\, d\lambda\, dx \\
&+ \int_Q \int_0^1 \int_0^1 v_{uf}^{(2)}(x, u+\theta R(h), f+\lambda h) h R(h)\, d\lambda\, d\theta\, dx.
\end{aligned}
$$

Let A_m, B_m be two functions defined by:

$$
A_m(x, \theta) = \begin{cases} \dfrac{v_u^{(1)}(x, u+\theta R(\tilde{h}_m), f) - v_u^{(1)}(x, u, f)}{R(\tilde{h}_m)} - \theta v_{uu}^{(2)}(x, u, f), & R(\tilde{h}_m) \neq 0, \\ 0, & R(\tilde{h}_m) = 0, \end{cases}
$$

$$
B_m(x, \lambda) = \begin{cases} \dfrac{v_f^{(1)}(x, u, f+\lambda \tilde{h}_m) - v_f^{(1)}(x, u, f)}{\tilde{h}_m} - \lambda v_{ff}^{(2)}(x, u, f), & \tilde{h}_m \neq 0, \\ 0, & \tilde{h}_m = 0. \end{cases}
$$

Let F_m be defined by $F_m(x,\theta,\lambda) = v^{(2)}_{uf}(x, u(x)+\theta R(\tilde h_m), f+\lambda\tilde h_m) - v^{(2)}_{uf}(x, u(x), f)$. So,

$$|r(\tilde h_m)| = \left|\int_Q\int_0^1 A_m(x,\theta)R^2(\tilde h_m)\,d\theta\,dx + \int_Q\int_0^1 B_m(x,\lambda)\tilde h_m^2\,d\lambda\,dx \right.$$
$$\left. + \int_Q\int_0^1\int_0^1 F_m(x,\theta,\lambda)R(\tilde h_m)\tilde h_m\,d\lambda\,d\theta\,dx\right|.$$

Thus

$$|r(\tilde h_m)| \le \int_0^1\int_Q \left|A_m(x,\theta)R^2(\tilde h_m)\right|\,dx\,d\theta + \int_0^1\int_Q\left|B_m(x,\lambda)\tilde h_m^2\right|\,d\lambda\,dx$$
$$+ \int_0^1\int_0^1\int_Q |F_m(x,\theta,\lambda)|\left|R(\tilde h_m)\right|\left|\tilde h_m\right|\,dx\,d\theta\,d\lambda$$
$$\le \int_0^1\left[\int_Q |A_m(x,\theta)|^{\frac n2}\,dx\right]^{\frac 2n}\left[\left(\int_Q |R(\tilde h_m)|^{\frac{2n}{n-2}}dx\right)^{\frac{n-2}{2n}}\right]^2 d\theta$$
$$+ \int_0^1\left[\int_Q |B_m(x,\lambda)|^{\frac n2}\,dx\right]^{\frac 2n}\left[\left(\int_Q |\tilde h_m|^{\frac{2n}{n-2}}dx\right)^{\frac{n-2}{2n}}\right]^2 d\lambda$$
$$+ \int_0^1\int_0^1\left[\int_Q |F_m(x,\theta,\lambda)|^{\frac n2}\,dx\right]^{\frac 2n}\left[\int_Q |R(\tilde h_m)\tilde h_m|^{\frac{n}{n-2}}dx\right]^{\frac{n-2}{n}} d\theta\,d\lambda$$
$$= \int_0^1\left[\int_Q |A_m(x,\theta)|^{\frac n2}dx\right]^{\frac 2n} d\theta\,\|R(\tilde h_m)\|^2_{L^{\frac{2n}{n-2}}(Q)}$$
$$+ \int_0^1\left[\int_Q |B_m(x,\lambda)|^{\frac n2}dx\right]^{\frac 2n} d\lambda\,\|\tilde h_m\|^2_{L^{\frac{2n}{n-2}}(Q)}$$
$$+ \int_0^1\int_0^1\left[\int_Q |F_m(x,\theta,\lambda)|^{\frac n2}\,dx\right]^{\frac 2n} d\theta\,d\lambda\times$$
$$\times\left[\left(\int_Q |R(\tilde h_m)|^{\frac{2n}{n-2}}\,dx\right)^{\frac12}\left(\int_Q |\tilde h_m|^{\frac{2n}{n-2}}\,dx\right)^{\frac12}\right]^{\frac{n-2}{n}}$$
$$\le c_6\int_0^1\left[\int_Q |A_m(x,\theta)|^{\frac n2}dx\right]^{\frac 2n} d\theta\,\|R(\tilde h_m)\|^2_{W^{1,2}(Q)}$$
$$+ c_7\|\tilde h_m\|^2_{W^{1,2}(Q)}\int_0^1\int_0^1\left[\int_Q |F_m(x,\theta,\lambda)|^{\frac n2}\,dx\right]^{\frac 2n} d\theta\,d\lambda$$
$$\le c_8\left[\int_0^1\left[\int_Q |A_m(x,\theta)|^{\frac n2}\,dx\right]^{\frac 2n} d\theta + \int_0^1\left[\int_Q |B_m(x,\lambda)|^{\frac n2}\,dx\right]^{\frac 2n} d\lambda\right.$$
$$\left. + \int_0^1\int_0^1\left[\int_Q |F_m(x,\theta,\lambda|^{\frac n2}\,dx\right]^{\frac 2n} d\theta\,d\lambda\right]|\tilde h_m\|^2_{W^{1,2}(Q)} \tag{2.3}$$

Let us remark that $A_m(x,\theta), B_m(x,\lambda), F_m(x,\theta,\lambda) \to 0$ almost everywhere.

On the other hand, using the mean value theorem, we deduce that there exists a sequence $k_m(x)$ such that $0 \le k_m(x) \le 1$ and

$$\begin{aligned}
|A_m(x,\theta)|^{\frac{n}{2}} &= \big[\big|v^{(2)}_{uu}(x,u(x)+k_m(x)\ \theta[R(\tilde h_m)](x),f) - v^{(2)}_{uu}(x,u(x),f)\big|\theta\big]^{\frac{n}{2}} \\
&\le \big[\big|v^{(2)}_{uu}(x,u(x)+k_m(x)\ \theta\ [R(\tilde h_m)](x),f)\big| + \big|v^{(2)}_{uu}(x,u(x),f)\big|\big]^{\frac{n}{2}} \\
&\le \big[a\big(|u(x)+k_m(x)\theta[R(\tilde h_m)](x)|^{\frac{2p}{n}} + |f(x)|^{\frac{2p}{n}}\big) + |b_2(x)| \\
&\quad + a\big(|u(x)|^{\frac{2p}{n}} + |f(x)|^{\frac{2p}{n}}\big) + |b_2(x)|\big]^{\frac{n}{2}} \\
&\le \big[a\big[(|u(x)| + |R(\tilde h_m)(x)|)^{\frac{2p}{n}} + |f(x)|^{\frac{2p}{n}}\big] \\
&\quad + a\big[|u(x)|^{\frac{2p}{n}} + |f(x)|^{\frac{2p}{n}}\big] + 2|b_2(x)|\big]^{\frac{n}{2}} \\
&\le \big[2a(|\tilde Z_0(x)|^{\frac{2p}{n}} + |f(x)|^{\frac{2p}{n}}) + 2|b_2(x)|\big]^{\frac{n}{2}} \in L^1(Q).
\end{aligned}$$

Analogously, for B_m we deduce that there exists $S_m(x)$: $0 \le S_m(x) \le 1$ and

$$\begin{aligned}
|B_m(x,\lambda)|^{\frac{n}{2}} &= \big|v^{(2)}_{ff}(x,u(x),f(x)+\xi_m\lambda\tilde h_m) - v^{(2)}_{ff}(x,u(x),f(x))\lambda\big|^{\frac{n}{2}} \\
&\le \big[a(|u(x)|^{\frac{2p}{n}} + |f(x)+\xi_m\lambda\tilde h_m|^{\frac{2p}{n}}) + |b_2(x)| \\
&\quad + a(|u(x)|^{\frac{2p}{n}} + |f(x)|^{\frac{2p}{n}}) + |b_2(x)|\big]^{\frac{n}{2}} \\
&\le \big[2a(|u(x)|^{\frac{2p}{n}} + 2|\tilde Z_1(x)|^{\frac{2p}{n}}) + 2|b_2(x)|\big]^{\frac{n}{2}} \in L^1(Q).
\end{aligned}$$

Analogously, for F_m we obtain

$$\begin{aligned}
|F_m(x,\theta,\lambda)|^{\frac{n}{2}} &= |v^{(2)}_{uf}(x,u(x)+\theta R(\tilde h_m), f+\lambda\tilde h_m) - v^{(2)}_{uf}(x,u(x),f)|^{\frac{n}{2}} \\
&\le \big[a(|u(x)+\theta R(\tilde h_m)|^{\frac{2p}{n}} + |f+\lambda\tilde h_m|^{\frac{2p}{n}} \\
&\quad + |u(x)|^{\frac{2p}{n}} + |f(x)|^{\frac{2p}{n}}) + 2|b_2(x)|]^{\frac{n}{2}} \\
&\le \big[2a(|\tilde Z_0(x)|^{\frac{2p}{n}} + |\tilde Z_1(x)|^{\frac{2p}{n}}) + 2|b_2(x)|\big]^{\frac{n}{2}} \in L^1(Q).
\end{aligned}$$

Let us remark that $A_m(x,\theta) \to 0$, $B_m(x,\lambda) \to 0$, $F_m(x,\theta,\lambda) \to 0$ almost everywhere. Thus, using the dominated convergence theorem, we conclude that

$$\int_0^1 \Big[\int_Q |A_m(x,\theta)|^{\frac{n}{2}}\,dx\Big]^{\frac{2}{n}} d\theta \to 0,$$

$$\int_0^1 \Big[\int_Q |B_m(x,\lambda)|^{\frac{n}{2}}\,dx\Big]^{\frac{2}{n}} d\lambda \to 0,$$

$$\int_0^1 \int_0^1 \Big[\int_Q |F_m(x,\theta,\lambda)|^{\frac{n}{2}}\,dx\Big]^{\frac{2}{n}} d\lambda\,d\theta \to 0,$$

but this contradicts (2.3). ∎

THEOREM 2.2. *Let the following condition be added to the conditions of Theorem 2.1:*

$$|v^{(1)}_u(x,u,f)| + |v^{(1)}_f(x,u,f)| \le a\big(|u|^{\frac{2p}{n}} + |f|^{\frac{2p}{n}}\big) + |\widehat b_1(x)|.$$

Then the functional G is a $(\tau_p, \|.\|_{W^{1,2}(Q)})$-Taylor mapping of first and second order at each point $f \in F$.

Proof. We must estimate $r(h) \equiv G(f+h) - G(f) - G^{(1)}(f)h$ as in the proof of Theorem 2.1, representing $r(h_m)$ in the form:

$$r(h_m) = \int_Q A_m(x) R(\tilde{h}_m)\, dx + \int_Q B_m(x) \tilde{h}_m \, dx,$$

where

$$A_m(x) = \begin{cases} \dfrac{v(x, u + R(\tilde{h}_m), f + \tilde{h}_m) - v(x, u, f + \tilde{h}_m)}{R(\tilde{h}_m)} - v_u^{(1)}(x, u, f), & R(\tilde{h}_m) \neq 0, \\ 0, & R(\tilde{h}_m) = 0, \end{cases}$$

$$B_m(x) = \begin{cases} \dfrac{v(x, u, f + \tilde{h}_m) - v(x, u, f)}{\tilde{h}_m} - v_f^{(1)}(x, u, f), & \tilde{h}_m \neq 0, \\ 0, & \tilde{h}_m = 0, \end{cases}$$

while h_m is the same as in the proof of Theorem 2.1. These estimates are omitted. ■

Now let us give sufficient conditions of optimality for the problems (1.3), (1.4) and (1.5).

THEOREM 2.3. *Suppose that in the problem (1.4), v_k satisfies the conditions of Theorems 2.1 and 2.2. Then the functionals*

$$J_k(f) \equiv \int_Q v_k(x, u, f)\, dx \quad , (k = s_1 + 1, \ldots, s_1 + s_2)$$

are $(\tau_p, \|\cdot\|_{W^{1,2}(Q)})$-Taylor mappings of first and second order at each point $f \in F$ and $J_k(f) = \int_Q v_k(x, u, f)\, dx + c_k \|f\|^2_{W^{1,2}(Q)}$, $(k = 0, \ldots, s_1)$ are lower $(\tau_p, \|\cdot\|_{W^{1,2}(Q)})$-semi-Taylor mappings of first and second order at each point $f \in F$. Consequently, $\exists J_k^{(1)}(f)$ and $\exists J_k^{(2)}(f)$, $(k = 0, \ldots, s_1 + s_2)$.

Let us suppose also that $J(\widehat{f}) = 0$, $\exists \widehat{y}^* \in (\mathbf{R}^{s_2})^*$, $\exists \alpha > 0$ $\mathcal{L}_f(\widehat{f}, \widehat{y}^*, 1) = 0$ and $\forall h \in ker J^{(1)}(\widehat{f})$ $\mathcal{L}_{ff}(\widehat{f}, \widehat{y}^*, 1)(h, h) \geq 2\alpha \|h\|^2_{W^{1,2}(Q)}$, where $\mathcal{L}_f(\widehat{f}, \widehat{y}^*, 1)$ and $\mathcal{L}_{ff}(\widehat{f}, \widehat{y}^*, 1)$ are given by formulas (1.10), (1.11). Then $\widehat{f}$ is a strict τ_p-local minimum point.

Proof. All conditions of Theorem 1.5 in [5] are satisfied, so $\widehat{f}$ is a strict τ_p-local minimum point. ■

THEOREM 2.4. *Suppose that in the problem (1.5), v_k satisfies the conditions of Theorems 2.1 and 2.2. Then the functionals*

$$J_k(f) \equiv \int_Q v_k(x, u, f)\, dx, \quad (k = s_1 + 1, \ldots, s_1 + s_2)$$

are $(\tau_p, \|\cdot\|_{W^{1,2}(Q)})$-Taylor mappings of first and second order at each point $f \in F$ and $J_k(f) = \int_Q v_k(x, u, f)\, dx + c_k \|f\|^2_{W^{1,2}(Q)}$, $(k = 0, \ldots, s_1)$ are lower

$(\tau_p, \|\cdot\|_{W^{1,2}(Q)})$-semi-Taylor mappings of first and second order at each point $f \in F$. Consequently, $\exists J_k^{(1)}(f)$ and $\exists J_k^{(2)}(f)$, $(k = 0, \dots, s_1 + s_2)$.

Let us suppose also that $\widehat{f} \in F$, $J(\widehat{f}) = 0$, $J_k(\widehat{f}) = 0$, $(k = 0, \dots, s_1)$. Let us put $L = \{h \in W_0^{1,2}(Q) / J_k^{(1)}(\widehat{f})h = 0, k = 1, \dots, s_1, J^{(1)}(\widehat{f})h = 0\}$. Suppose that $\exists \widehat{\lambda} \in (\mathbf{R}^{s_1})^*$, $\exists \widehat{y}^* \in (\mathbf{R}^{s_2})^*$, $\exists \gamma \geq 0$, $\exists \widehat{\lambda}_k > 0$, $(k = 1, .., s_1)$: $\mathcal{L}_f(\widehat{f}, \widehat{y}^*, \widehat{\lambda}, 1) = 0$ and $\forall h \in L$ $\mathcal{L}_{ff}(\widehat{f}, \widehat{y}^*, \widehat{\lambda}, 1)(h, h) \geq 2\gamma \|h\|^2_{W^{1,2}(Q)}$, where $\mathcal{L}_f(\widehat{f}, \widehat{y}^*, \widehat{\lambda}, 1)$ and $\mathcal{L}_{ff}(\widehat{f}, \widehat{y}^*, \widehat{\lambda}, 1)$ are defined by formulas 1.7 and 1.8. Then $\widehat{f}$ is a strict τ_p-local minimum point.

Proof. All conditions of Theorem 1.6 in [5] are satisfied, so $\widehat{f}$ is a strict τ_p-local minimum point. ■

THEOREM 2.5. *Suppose that in the problem 1.3, v_k satisfies the conditions of Theorems 2.1 and 2.2. Then the functionals*

$$J_k(f) \equiv \int_Q v_k(x, u, f)\, dx, \quad (k = s_1 + 1, \dots, s_1 + s_2)$$

are $(\tau_p, \|\cdot\|_{W^{1,2}(Q)})$-Taylor mappings of first and second order at each point $f \in F$ and $J_k(f) = \int_Q v_k(x, u, f)\, dx + c_k \|f\|^2_{W^{1,2}(Q)}$, $(k = 0, \dots, s_1)$ are lower $(\tau_p, \|\cdot\|_{W^{1,2}(Q)})$-semi-Taylor mappings of first and second order at each point $f \in F$. Consequently, $\exists J_k^{(1)}(f)$ and $\exists J_k^{(2)}(f)$, $(k = 0, \dots, s_1 + s_2)$.

Let us suppose also that $J_0^{(1)}(\widehat{f}) = 0$ and $\exists \alpha > 0$, $\forall h \in W_0^{1,2}(Q)$ $J_0^{(2)}(\widehat{f})(h, h) \geq 2\alpha \|h\|^2_{W^{1,2}(Q)}$. Then $\widehat{f}$ is a strict τ_p-local minimum point.

Proof. All conditions of Theorem 1.4 in [5] are satisfied, so $\widehat{f}$ is a strict τ_p-local minimum point. ■

REMARK 2.1. Let us remark that in Theorems 2.1 and 2.2, the increase conditions satisfied by v are not sufficient to certify the Frechet differentiability of functional $G : (W_0^{1,2}(Q), \|\cdot\|_{L^p(Q)}) \to \mathbf{R}^{s_2}$.

Indeed, suppose we have $n = 3$ and $\frac{3}{2} < p < 1$. Let us define $v \colon Q \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ by: $v(x, u, f) = a\big[|u|^{\frac{5}{2}} + |f|^{\frac{5}{2}}\big] + |b_0(x)|$, where $b_0(x) \in C(\overline{Q}), a \in \mathbf{R}, a > 0$.

Let $d_m \to +\infty$ and put $\alpha_m = |d_m|^{\frac{1}{2}}$, so $\alpha_m \to +\infty$ and $\forall x \in Q$ $\forall u \in \mathbf{R}$ $\forall m \in \mathbf{N}$

$$|v(x, u, d_m)| \geq a|d_m|^{\frac{5}{2}} = a|d_m|^{\frac{1}{2}}|d_m|^2 \geq a|d_m|^{\frac{1}{2}}|d_m|^p = a\alpha_m |d_m|^p.$$

Let $\tilde{f} \in W_0^{1,2}(Q)$. By the countable additivity of Lebesgue measure, $\exists c > 0$ $\exists Q' \subset Q$: $\mu(Q') > 0$ and $\rho(Q', \partial Q) > 0$ and $\forall x \in Q'$ $|\tilde{f}(x)| \leq c$.

In this case put: $\mathcal{D} \equiv \max\{|v(x, u, f)| / |u| \leq c, |f| \leq c, x \in \overline{Q}\} < \infty$. Let us choose $Q_m \subset Q'$ such that $\mu(Q_m) = |d_m|^{-p}\alpha_m^{-\frac{1}{2}}$. Let $\tilde{h}_m$ defined by:

$$\tilde{h}_m(x) = \begin{cases} d_m - \tilde{f}(x), & \text{when } x \in Q_m, \\ 0, & \text{when } x \in Q \setminus Q_m. \end{cases}$$

We have:

$$\begin{aligned}\|\tilde{h}_m(x)\|_{L^p(Q)} &\le \|d_m\|_{L^p(Q_m)} + |\tilde{f}(x)\|_{L^p(Q_m)}\\ &\le d_m\big(\mu(Q_m)\big)^{\frac{1}{p}} + c\big(\mu(Q_m)\big)^{\frac{1}{p}} \le \alpha_m^{-\frac{1}{2p}} + c\big(\mu(Q_m)\big)^{\frac{1}{p}}.\end{aligned}$$

Consequently, $\|\tilde{h}_m(x)\|_{L^p(Q)} \to 0$, i.e., $\tilde{h}_m(x) \to 0$ in $L^p(Q)$.

On the other hand, we have

$$\begin{aligned}&|G(\tilde{f}+\tilde{h}_m) - G(\tilde{f})| =\\ &\qquad = \left|\int_{Q_m}\big[v(x,[R(\tilde{f}+\tilde{h}_m)](x),\tilde{f}(x)+\tilde{h}_m(x)) - v(x,[R(\tilde{f})](x),\tilde{f}(x))\big]\,dx\right|\\ &\qquad \ge \left|\int_{Q_m} v(x,[R(\tilde{f}+\tilde{h}_m)](x),d_m)\,dx\right| - \left|\int_{Q_m} v(x,[R(\tilde{f})](x),\tilde{f}(x))\,dx\right|\\ &\qquad \ge a\alpha_m|d_m|^p - \mathcal{D}\mu(Q_m) = a\alpha_m^{\frac{1}{2}} \to +\infty.\end{aligned}$$

Therefore, $|G(\tilde{f}+\tilde{h}_m) - G(\tilde{f})| \to +\infty$. Thus G is not Frechet differentiable at each point $f \in W_0^{1,2}(Q)$. ■

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