SEMIREGULAR PROPERTIES AND GENERALIZED LINDELÖF SPACES

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Abstract. Let (X,τ) be a topological space and (X,τ^*) its semiregularization. Then a topological property $\mathcal P$ is semiregular provided that τ has property $\mathcal P$ if and only if τ^* has the same property. In this work we study semiregular property of almost Lindelöf, weakly Lindelöf, nearly regular-Lindelöf, almost regular-Lindelöf and weakly regular-Lindelöf spaces. We prove that all these topological properties, on the contrary of Lindelöf property, are semiregular properties.

1. Introduction

Among various covering properties of topological spaces a lot of attention has been made to those covers which involve regularly open sets and regularly closed sets. In 1959 Frolik [5] introduced the notion of weakly Lindelöf space that was afterwards studied by several authors. In 1982 Balasubramanian [1] introduced and studied the notion of nearly Lindelöf spaces as a generalization of the nearly compact spaces defined by Singal and Mathur [8]. Then in 1986 Mršević et al. [7] gave some characterizations of the classes of nearly Lindelöf spaces. In 1984 Willard and Dissanayake [9] gave the notion of almost Lindelöf spaces and in 1996 Cammaroto and Santoro [4] introduced the notion of nearly regular-Lindelöf, almost regular-Lindelöf and weakly regular-Lindelöf spaces by using the regular covers which were introduced by Cammaroto and Lo Faro [3] in 1981. The purpose of this paper is to prove that all these generalizations of Lindelöf spaces, which are topological properties, are semiregular properties.

Throughout this paper, a space X means a topological space (X, τ) on which no separation axioms are assumed unless explicitly stated otherwise. The interior and the closure of any subset A of (X, τ) will be denoted by $\operatorname{Int}(A)$ or $\operatorname{Int}_{\tau}(A)$ and $\operatorname{Cl}(A)$ or $\operatorname{Cl}_{\tau}(A)$ respectively. Recall that a subset $A \subseteq X$ is called regularly open (regularly closed) if $A = \operatorname{Int}(\operatorname{Cl}(A))$ ($A = \operatorname{Cl}(\operatorname{Int}(A))$). The class of all regular open sets in a space (X, τ) is a base for a coarser topology than τ on X, called the

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semiregularization topology of X which is denoted by (X, τ^*) . Semiregularization topologies were studied by Mršević et al. [6] in 1985. If $\tau = \tau^*$ then X is said to be semiregular. The following results, to be used in the sequel, are known and easily can be proved.

Lemma 1.1. Let (X, τ) be a topological space and (X, τ^*) its semiregularization. Then

- (a) regularly open sets of (X, τ) are the same as regularly open sets of (X, τ^*) .
- (b) regularly closed sets of (X, τ) are the same as regularly closed sets of (X, τ^*) .
- (c) $\operatorname{Int}_{\tau}(C) = \operatorname{Int}_{\tau^*}(C)$ for any regularly closed set C.
- (d) $\operatorname{Cl}_{\tau}(A) = \operatorname{Cl}_{\tau^*}(A)$ for every $A \in \tau$.

DEFINITION 1.1. (see [2]) Let (X, τ) be a topological space and (X, τ^*) its semiregularization. A topological property \mathcal{P} is called semiregular provided that τ has the property \mathcal{P} if and only if τ^* has the property \mathcal{P} .

2. Semiregularization of Lindelöf spaces

DEFINITION 2.1. (see [1], [9] and [5]). A topological space X is said to be nearly Lindelöf, almost Lindelöf and weakly Lindelöf if, for every open cover $\{U_{\alpha} : \alpha \in \Delta\}$ of X, there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\} \subseteq \Delta$ such that

$$X = \bigcup_{n \in \mathbf{N}} \mathrm{Int}(\mathrm{Cl}(U_{\alpha_n})), \quad X = \bigcup_{n \in \mathbf{N}} \mathrm{Cl}(U_{\alpha_n}) \quad \text{and} \quad X = \mathrm{Cl}\Big(\bigcup_{n \in \mathbf{N}} U_{\alpha_n}\Big),$$

respectively.

Note first that a space (X,τ) is nearly Lindelöf if and only if (X,τ^*) is Lindelöf (see [7], Theorem 1). Moreover, the Lindelöf property is not a semiregular property (see [7], Example 1). Note also that, in distinction to the Lindelöf property, the nearly Lindelöf property is a semiregular property (see [7], Corollary 4). Next we prove that almost Lindelöf and weakly Lindelöf properties are semiregular properties as we conclude from the following theorems.

Theorem 2.1. Let (X, τ) be a topological space. Then (X, τ) is almost Lindelöf if and only if (X, τ^*) is almost Lindelöf.

Proof. Suppose that (X,τ) is almost Lindelöf and let $\{U_{\alpha}: \alpha \in \Delta\}$ be an open cover of (X,τ^*) , thus an open cover of the almost Lindelöf space (X,τ) since $\tau^* \subseteq \tau$. So there exists a countable subset $\{\alpha_n: n \in \mathbf{N}\} \subseteq \Delta$ such that $X = \bigcup_{n \in \mathbf{N}} \operatorname{Cl}_{\tau}(U_{\alpha_n})$. By Lemma 1.1(d), $X = \bigcup_{n \in \mathbf{N}} \operatorname{Cl}_{\tau^*}(U_{\alpha_n})$. This implies that (X,τ^*) is almost Lindelöf.

Conversely, suppose that (X, τ^*) is almost Lindelöf and let $\{V_\alpha : \alpha \in \Delta\}$ be an open cover of (X, τ) . Then $\{\operatorname{Int}_\tau(\operatorname{Cl}_\tau(V_\alpha)) : \alpha \in \Delta\}$ is an open cover of the almost Lindelöf space (X, τ^*) . Thus it has a countable subfamily $\{\operatorname{Int}_\tau(\operatorname{Cl}_\tau(V_{\alpha_n})) : n \in \mathbf{N}\}$ such that

$$X = \bigcup_{n \in \mathbf{N}} \operatorname{Cl}_{\tau^*}(\operatorname{Int}_{\tau}(\operatorname{Cl}_{\tau}(V_{\alpha_n}))) = \bigcup_{n \in \mathbf{N}} \operatorname{Cl}_{\tau}(\operatorname{Int}_{\tau}(\operatorname{Cl}_{\tau}(V_{\alpha_n}))) = \bigcup_{n \in \mathbf{N}} \operatorname{Cl}_{\tau}(V_{\alpha_n}).$$

This shows that (X,τ) is almost Lindelöf and completes the proof.

COROLLARY 2.1. Almost Lindelöf property is a semiregular property.

Theorem 2.2. Let (X, τ) be a topological space. Then (X, τ) is weakly Lindelöf if and only if (X, τ^*) is weakly Lindelöf.

 ${\it Proof.}$ The proof is quite similar to the proof of Theorem 2.1 above on using the fact that

$$\mathrm{Cl}_{\tau^*}\Big(\bigcup_{n\in\mathbf{N}}\mathrm{Int}_\tau(Cl_\tau(V_{\alpha_n}))\Big)\subseteq\mathrm{Cl}_{\tau^*}\Big(\bigcup_{n\in\mathbf{N}}V_{\alpha_n}\Big)=\mathrm{Cl}_\tau\Big(\bigcup_{n\in\mathbf{N}}V_{\alpha_n}\Big).$$

Thus we choose to omit the details. \blacksquare

COROLLARY 2.2. Weakly Lindelöf property is a semiregular property.

3. Semiregularization of regular-Lindelöf Spaces

Recall that an open cover $\{U_{\alpha}: \alpha \in \Delta\}$ of a topological space X is called regular [3] if, for every $\alpha \in \Delta$, there exists a nonempty regularly closed subset C_{α} of X such that $C_{\alpha} \subseteq U_{\alpha}$ and $X = \bigcup_{\alpha \in \Delta} \operatorname{Int}(C_{\alpha})$. The following definition was given in [4] and is using regular covers to give some generalizations of Lindelöf spaces.

Definition 3.1. A topological space X is called weakly regular-Lindelöf (resp. almost regular-Lindelöf, nearly regular-Lindelöf) if every regular cover $\{U_\alpha:\alpha\in\Delta\}$ of X admits a countable subfamily $\{U_{\alpha_n}:n\in\mathbf{N}\}$ such that

$$X = \operatorname{Cl}\Big(\bigcup_{n \in \mathbf{N}} U_{\alpha_n}\Big) \quad \Big(\text{ resp. } X = \bigcup_{n \in \mathbf{N}} \operatorname{Cl}(U_{\alpha_n}), \quad X = \bigcup_{n \in \mathbf{N}} \operatorname{Int}(\operatorname{Cl}(U_{\alpha_n}))\Big).$$

The following implications are obvious from the definitions.

Now we prove that the nearly regular-Lindelöf property is a semiregular property depending on Lemma 1.1 above.

Theorem 3.1. Let (X, τ) be a topological space. Then (X, τ) is nearly regular-Lindelöf if and only if (X, τ^*) is nearly regular-Lindelöf.

Proof. Suppose that (X,τ) is nearly regular-Lindelöf and let $\{U_\alpha:\alpha\in\Delta\}$ be a regular cover of (X,τ^*) . Since $\tau^*\subseteq\tau$ and by Lemma 1.1 (a, b and d) we have, $\{U_\alpha:\alpha\in\Delta\}$ is a regular cover of the nearly regular-Lindelöf space (X,τ) . So it has a countable subset $\{U_{\alpha_n}:n\in\mathbf{N}\}$ such that $X=\bigcup_{n\in\mathbf{N}}\operatorname{Int}_{\tau}(\operatorname{Cl}_{\tau}(U_{\alpha_n}))=\bigcup_{n\in\mathbf{N}}\operatorname{Int}_{\tau^*}(\operatorname{Cl}_{\tau^*}(U_{\alpha_n}))$ (by Lemma 1.1(c,d)). This implies that (X,τ^*) is nearly regular-Lindelöf.

Conversely, suppose that (X, τ^*) is nearly regular-Lindelöf and let $\{U_\alpha : \alpha \in \Delta\}$ be a regular cover of (X, τ) . Then since $U_\alpha \subseteq \operatorname{Int}_\tau(\operatorname{Cl}_\tau(U_\alpha))$, Lemma 1.1 above

implies that $\{\operatorname{Int}_{\tau}(\operatorname{Cl}_{\tau}(U_{\alpha})): \alpha \in \Delta\}$ is a regular cover of the nearly regular-Lindelöf space (X, τ^*) . Thus there exists a countable subfamily $\{\alpha_n : n \in \mathbf{N}\}$, such that $X = \bigcup_{n \in \mathbf{N}} \operatorname{Int}_{\tau^*}(\operatorname{Cl}_{\tau^*}(\operatorname{Int}_{\tau}(\operatorname{Cl}_{\tau}(U_{\alpha_n})))) = \bigcup_{n \in \mathbf{N}} \operatorname{Int}_{\tau^*}(\operatorname{Cl}_{\tau^*}(U_{\alpha_n})) = \bigcup_{n \in \mathbf{N}} \operatorname{Int}_{\tau^*}(\operatorname{Cl}_{\tau^*}(U_{\alpha_n}))$ (by Lemma 1.1(a,c,d)). This implies that (X,τ) is a nearly regular-Lindelöf space and completes the proof.

COROLLARY 3.1. Nearly regular-Lindelöf property is a semiregular property.

Next we prove that the almost regular-Lindelöf and weakly regular-Lindelöf properties are semiregular properties.

Theorem 3.2. Let (X, τ) be a topological space. Then (X, τ) is almost regular-Lindelöf if and only if (X, τ^*) is almost regular-Lindelöf.

Proof. The proof is similar to the proof of Theorem 3.1 above, thus we choose to omit the details. \blacksquare

COROLLARY 3.2. Almost regular-Lindelöf property is a semiregular property.

Theorem 3.3. Let (X, τ) be a topological space. Then (X, τ) is weakly regular-Lindelöf if and only if (X, τ^*) is weakly regular-Lindelöf.

Proof. The proof is similar to the proof of Theorem 3.1 above on using the fact that $\operatorname{Cl}_{\tau}\left(\bigcup_{n\in\mathbb{N}}(\operatorname{Int}_{\tau}(\operatorname{Cl}_{\tau}(U_{\alpha_n}))\right)\subseteq\operatorname{Cl}_{\tau}\left(\bigcup_{n\in\mathbb{N}}U_{\alpha_n}\right)=\operatorname{Cl}_{\tau^*}\left(\bigcup_{n\in\mathbb{N}}U_{\alpha_n}\right)$. Thus the details are omitted

Corollary 3.3. Weakly regular-Lindelöf property is a semiregular property.

REFERENCES

- Balasubramanian, G., On some generalizations of compact spaces, Glasnik Math., 17(37) (1982), 367–380.
- [2] Cameron, D. E., Properties of S-closed spaces, Proc. Amer. Math. Soc., 72 (1978), 581-586.
- [3] Cammaroto, F. and Lo Faro, G., Spazi weakly compact, Riv. Mat. University Parma, 14(7) (1981), 383–395.
- [4] Cammaroto, F. and Santoro, G., Some counterexamples and properties on generalizations of Lindelöf spaces, Internat. J. Math. and Math. Sci., 19(4) (1996), 737–746.
- [5] Frolik, Z., Generlization of compact and Lindelöf spaces, Czechoslovak Math. J., 9(84) (1959), 172–217.
- [6] Mršević, M., Reilly, I. L. and Vamanamurthy, M. K., On semi-regularization topologies, J. Australian Math. Soc. (series A), 38 (1985), 40–54.
- [7] Mršević, M., Reilly, I. L. and Vamanamurthy, M. K., On nearly Lindelöf spaces, Glasnik Mat., 21(41) (1986), 407–414.
- [8] Singal, M. K. and Mathur, A., On nearly compact spaces, Boll. Un. Mat. Ital., 4(2) (1969), 702-710.
- [9] Willard, S and Dissanayake, U.N.B., The almost Lindelöf degree, Canad. Math. Bull., 27(4) (1984), 452–455.

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