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A MAPPING THEOREM ON ℵ-SPACES

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Abstract. In this paper we give a mapping theorems on ℵ-spaces by means of strong compact-covering mappings and σ-mappings. As an application, we get characterizations of quotient (pseudo-open) σ-images of metric spaces.

ℵ-spaces form one class of generalized metric spaces, and play an important role in metrization theory. The concept of σ-mappings was introduced by S.Lin in [5], and by using it, σ-spaces are characterized as σ-images of metric spaces. In [17], g-metrizable spaces are characterized as weak-open σ images of metric spaces. The purpose of this paper is to establish the relationships between metric spaces and ℵ-spaces by means of strong compact-covering mappings and σ-mappings, and get characterizations of quotient (pseudo-open) σ-images of metric spaces.

In this paper all spaces are regular and T_1, all mappings are continuous and surjective. N denotes the set of natural numbers, ω denotes $N \cup \{0\}$. For a collection $\mathcal{P}$ of subsets of a space X and a mapping $f\colon X \to Y$, denote $f(\mathcal{P}) = \{f(P) : P \in \mathcal{P}\}$. For two families $\mathcal{A}$ and $\mathcal{B}$ of subsets of X, denote $\mathcal{A} \wedge \mathcal{B} = \{A \cap B : A \in \mathcal{A}$ and $B \in \mathcal{B}\}$.

Definition 1. Let $\mathcal{P}$ be a cover of a space X.

(1) $\mathcal{P}$ is a k-network for X [9] if for each compact subset K of X and its open neighborhood V, there exists a finite subcollection $\mathcal{P}'$ of $\mathcal{P}$ such that $K \subset \cup\mathcal{P}' \subset V$.

(2) $\mathcal{P}$ is called a cs-network for X if for each $x \in X$, its open neighborhood V and a sequence $\{x_n\}$ converging to x, there exists $P \in \mathcal{P}$ such that $\{x_n : n \geq m\} \cup \{x\} \subset P \subset V$ for some $m \in N$.

(3) $\mathcal{P}$ is called a cs^*-network for X if for each $x \in X$, its open neighborhood V and a sequence $\{x_n\}$ converging to x, there exist $P \in \mathcal{P}$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k} : k \in N\} \cup \{x\} \subset P \subset V$.

A space X is called an ℵ-space if X has a σ-locally finite k-network.

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Definition 2. Let $f\colon X \to Y$ be a mapping.

(1) f is called a σ-mapping [5] if there exists a base $\mathcal{B}$ for X such that $f(\mathcal{B})$ is a σ-locally finite collection of subsets of Y.

(2) f is called a strong sequence-covering mapping [15] if each convergent sequence (including its limit point) of Y is the image of some convergent sequence (including its limit point) of X.

(3) f is called a sequence-covering mapping [10] if each convergent sequence (including its limit point) of Y is the image of some compact subset of X.

(4) f is a sequentially-quotient mapping [11] (resp. subsequence-covering mapping [8]) if, whenever K is a convergent sequence (including its limit) in Y, then there is a convergent sequence L (including its limit) in X (resp. compact subset L of X) such that $f(L)$ is a subsequence of K.

(5) f is a compact-covering mapping if each compact subset of Y is the image of some compact subset of X.

(6) f is a strong compact-covering mapping [6] if it is both strong sequence-covering and compact-covering.

Theorem 3. *The following are equivalent for a space X:*

(1) *X is an $\aleph$-space;*

(2) *X is a strong compact-covering σ-image of a metric space;*

(3) *X is a strong sequence-covering σ-image of a metric space;*

(4) *X is a sequence-covering σ-image of a metric space;*

(5) *X is a subsequence-covering σ-image of a metric space;*

(6) *X is a sequentially-quotient σ-image of a metric space;*

(7) *X is a compact-covering σ-image of a metric space.*

Proof. (1) $\Longrightarrow$ (2). Suppose X is an $\aleph$-space. Then X has a σ-locally finite cs-network by Theorem 4 in [4]. Let $\mathcal{P} = \bigcup\{\mathcal{P}_i : i \in N\}$ be a σ-locally finite cs-network for X, where each $\mathcal{P}_i = \{P_\alpha : \alpha \in A_i\}$ is a locally finite collection of subsets of X which is closed under finite intersections and $X \in \mathcal{P}_i \subset \mathcal{P}_{i+1}$. For each $i \in N$, endow A_i with the discrete topology; then A_i is a metric space. Put

$$M = \Big\{\alpha = (\alpha_i) \in \prod_{i\in N} A_i : \{P_{\alpha_i} : i \in N\} \subset \mathcal{P} \text{ forms a network at some point } x(\alpha) \in X\Big\},$$

and endow M with the subspace topology induced from the usual product topology of the collection $\{A_i : i \in N\}$ of metric spaces. Then M is a metric space. Since X is Hausdorff, $x(\alpha)$ is unique in X for each $\alpha \in M$. We define $f\colon M \to X$ by $f(\alpha) = x(\alpha)$ for each $\alpha \in M$. Since $\mathcal{P}$ is a σ-locally finite cs-network for X, f is surjective. For each $\alpha = (\alpha_i) \in M$, $f(\alpha) = x(\alpha)$. Suppose V is an open neighborhood of $x(\alpha)$ in X; there exists $n \in N$ such that $x(\alpha) \in P_{\alpha_n} \subset V$. Set $W = \{c \in M : \text{the n-th coordinate of } c \text{ is } \alpha_n\}$; then W is an open neighborhood of

α in M, and $f(W) \subset P_{\alpha_n} \subset V$. Hence f is continuous. We will show that f is a strong compact-covering σ-mapping.

(i) f is a σ-mapping.

For each $n \in N$ and $\alpha_n \in A_n$, put

$V(\alpha_1, \dots, \alpha_n) = \{\beta \in M$: for each $i \le n$, the i-th coordinate of β is $\alpha_i\}$.

Let $\mathcal{B} = \{V(\alpha_1, \dots, \alpha_n) : \alpha_i \in A_i (i \le n)$ and $n \in N\}$. Then $\mathcal{B}$ is a base for M.

To prove f is a σ-mapping, we only need to check that $f(V(\alpha_1, \dots, \alpha_n)) = \bigcap_{i \le n} P_{\alpha_i}$ for each $n \in N$ and $\alpha_n \in A_n$ because $f(\mathcal{B})$ is σ-locally finite in X by this result.

For each $n \in N$, $\alpha_n \in A_n$ and $i \le n$, $f(V(\alpha_1, \dots, \alpha_n)) \subset P_{\alpha_i}$, hence $f(V(\alpha_1, \dots, \alpha_n)) \subset \bigcap_{i \le n} P_{\alpha_i}$. On the other hand, for each $x \in \bigcap_{i \le n} P_{\alpha_i}$, there is $\beta = (\beta_j) \in M$ such that $f(\beta) = x$. For each $j \in N$, $P_{\beta_j} \in \mathcal{P}_j \subset \mathcal{P}_{j+n}$, and thus there is $\alpha_{j+n} \in A_{j+n}$ such that $P_{\alpha_{j+n}} = P_{\beta_j}$. Set $\alpha = (\alpha_j)$, then $\alpha \in V(\alpha_1, \dots, \alpha_n)$ and $f(\alpha) = x$. Thus $\bigcap_{i \le n} P_{\alpha_i} \subset f(V(\alpha_1, \dots, \alpha_n))$. Hence $f(V(\alpha_1, \dots, \alpha_n)) = \bigcap_{i \le n} P_{\alpha_i}$. Therefore, f is a σ-mapping.

(ii) f is strong sequence-covering.

For each sequence $\{x_n\}$ converging to x_0, we can assume that all x_n's are distinct, and that $x_n \ne x_0$ for each $n \in N$. Set $K = \{x_m : m \in \omega\}$. Suppose V is an open neighborhood of K in X. A subcollection $\mathcal{A}$ of $\mathcal{P}_i$ is said to have property $F(K, V)$ if:

(a) $\mathcal{A}$ is finite;

(b) for each $P \in \mathcal{A}$, $\varnothing \ne P \cap K \subset P \subset V$

(c) for each $z \in K$, there exists a unique $P_z \in \mathcal{A}$ such that $z \in \mathcal{P}_z$

(d) if $x_0 \in P \in \mathcal{A}$, then $K \setminus P$ is finite.

Since $\mathcal{P}$ is a σ-locally finite cs-network for X, there are $\mathcal{A} \subset \mathcal{P}_i$, with $F(K, V)$ property, and we can assume that $\{\mathcal{A} \subset \mathcal{P}_i : \mathcal{A}$ has the property $F(K, X)\} = \{\mathcal{A}_{ij} : j \in N\}$.

For each $n \in N$, put

$$\mathcal{P}_n' = \bigwedge_{i,j \le n} \mathcal{P}_{ij} \ ;$$

then $\mathcal{P}_n' \subset \mathcal{P}_n$ and $\mathcal{P}_n'$ also has the property $F(K, X)$.

For each $i \in N$, $m \in \omega$ and $x_m \in K$, there is $\alpha_{im} \in A_i$ such that $x_m \in P_{\alpha_{im}} \in \mathcal{P}_i'$. Let $\beta_m = (\alpha_{im}) \in \prod_{i \in N} A_i$. It is easy to prove that $\{P_{\alpha_{im}} : i \in N\}$ is a network of x_m in X. Then there is a $\beta_m \in M$ such that $f(\beta_m) = x_m$ for each $m \in \omega$. For each $i \in N$, there is $n(i) \in N$ such that $\alpha_{in} = \alpha_{io}$ when $n \ge n(i)$. Hence the sequence $\{\alpha_{in}\}$ converges to α_{io} in A_i. Thus the sequence $\{\beta_n\}$ converges to β_0 in M. This show that f is strong sequence-covering.

(iii) f is compact-covering.

Since Y has a σ-locally finite cs-network, for each compact subset L of Y, L has a countable cs-network. So L is metrizable. We can prove that f is compact-covering by the proof of Theorem 2 in [6].

(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5), (6) $\Longrightarrow$ (5), (2) $\Longrightarrow$ (7) $\Longrightarrow$ (4) are obvious.

(5) $\Longrightarrow$ (6). Suppose X is the image in the metric space M under a subsequence-covering σ-mapping f. It suffices to show f is sequentially-quotient. For each convergent sequence $\{x_n\}$ of X with $x_n \to x$, set $K = \{x_n : n \in N\} \cup \{x\}$. Since f is subsequence-covering, there exists a compact subset L of M such that $f(L)$ is a subsequence of $\{x_n\}$. Denote this subsequence of $\{x_n\}$ by $\{x_{n_m}\}$. For each $m \in N$, let $f(a_m) = x_{n_m}$ with $a_m \in M$. Then $\{a_m\}$ is a sequence in compact metrizable subspace L of M. Thus $\{a_m\}$ has a convergent subsequence $\{a_{m_i}\}$ with $a_{m_i} \to a$. Obviously, $f(a) = x$. Put $S = \{a_{m_i} : i \in N\} \cup \{a\}$, then $f(S)$ is a subsequence of K. This implies that f is sequentially-quotient.

(6) $\Longrightarrow$ (1). Suppose X is the image of a metric space M under a sequentially-quotient σ-mapping f. Since f is a σ-mapping, there exists a base $\mathcal{B}$ for M such that $f(\mathcal{B})$ is a σ-locally finite collection of subsets of X. Since sequentially-quotient mappings preserve cs*-networks by Proposition 2.7.3 in [3], $f(\mathcal{B})$ is a σ-locally finite cs^*-network for X. Hence X is an $\aleph$-space by [14, Lemma 1.17, Theorem 1.4]. ■

Lemma 4 *Suppose f is a quotient mapping from a k-space M onto a space X. If $\mathcal{P}$ is a k-network for M and $f(\mathcal{P})$ is point-countable in X, then $f(\mathcal{P})$ is a k-network for X.*

Proof. Denote $\mathcal{F} = f(\mathcal{P})$. Suppose $K \subset V$ with K non-empty compact and V open in X. Put
$$\mathcal{A} = \{F \in \mathcal{F} : F \cap K \neq \varnothing \text{ and } F \subset V\}.$$
Then $K \subset \cup\mathcal{A}'$ for some finite $\mathcal{A}' \subset \mathcal{A}$. Otherwise, for any finite $\mathcal{A}' \subset \mathcal{A}$, $K \setminus \cup\mathcal{A}' \neq \varnothing$. For each $x \in K$, put
$$\mathcal{A}_x = \{F \in \mathcal{F} : x \in F \subset V\};$$
then $\mathcal{A}_x$ is countable, and $\mathcal{A} = \cup\{\mathcal{A}_x : x \in K\}$. Denote $\mathcal{A}_x = \{F_i(x) : i \in N\}$ for each $x \in K$. Take $x_1 \in K$; then there exists a infinite subset $D = \{x_n : n \in N\}$ of K such that for each n, $x_{n+1} \in K \setminus \bigcup_{i,j \le n} F_i(x_j)$. So D has a cluster point by the compactness of K. Let x be a cluster point of D, and set $B = D \setminus \{x\}$; then B is not closed in X. Since f is a quotient mapping, $f^{-1}(B)$ is not closed in M. Since M is a k-space, there exists a compact subset L of M such that $f^{-1}(B) \cap L$ is not closed in L. Let $g = f|_L : L \to f(L)$; then g is a closed mapping, and $g^{-1}(B \cap f(L)) = f^{-1}(B) \cap L$. So $B \cap f(L)$ is not closed in $f(L)$. Hence $B \cap f(L)$ is an infinite subset of X, and $D \cap f(L)$ is such. By $K \cap f(L) \neq \varnothing$, $H = L \cap f^{-1}(K)$ is non-empty compact in M and $H \subset f^{-1}(K) \subset f^{-1}(V)$, $H \subset \cup\mathcal{P}' \subset f^{-1}(V)$ for some finite $\mathcal{P}' \subset \mathcal{P}$. Thus $f(H) \subset f(\cup\mathcal{P}') \subset V$. Denote $\mathcal{P}' = \{P_m : m \le q\}$. We can assume that $P_m \cap H \neq \varnothing$ for each $m \le q$; then $f(P_m) \in \mathcal{A}$. Since
$$D \cap f(\cup\mathcal{P}') \supset D \cap f(H) = D \cap f(L),$$
$D \cap f(\cup\mathcal{P}')$ is infinite. Thus $f(P_m)$ includes infinitely many points of D for some $m \le q$. Take $x_j \in D \cap f(P_m)$; then $f(P_m) = F_i(x_j)$ for some $i \in N$. However, there exists $n > i, j$ such that $x_n \in F_i(x_j)$, a contradiction. Hence $K \subset \cup\mathcal{A}' \subset V$ for some finite $\mathcal{A}' \subset \mathcal{A}$. So $\mathcal{F}$ is a k-network for X. ■

The following two corollaries can be proved using Theorem 3, Lemma 4 and [3, Proposition 2.1.16, Theorem 2.7.23].

Corollary 5. *The following are equivalent for a space X:*

(1) *X is a k- and $\aleph$-space;*

(2) *X is a strong compact-covering and quotient σ-image of a metric space;*

(3) *X is a quotient σ-image of a metric space.*

Corollary 6. *The following are equivalent for a space X:*

(1) *X is a Fréchet space and $\aleph$-space;*

(2) *X is a pseudo-open σ-image of a metric space;*

(3) *X is a closed s-image of a metric space.*

Example 7. Let Z be the topological sum of the unit interval $[0, 1]$, and the collection $\{S(x) : x \in [0, 1]\}$ of 2^ω convergent sequences $S(x)$. Let X be the space obtained from Z by identifying the limit point of $S(x)$ with $x \in [0, 1]$, for each $x \in [0, 1]$. Then, from Example 2.9.27 in [3] (or Example 9.8 in [10]), we have the following facts:

(1) X is *a* compact-covering, quotient compact image of *a* locally compact metric space.

(2) X has no point-countable cs-network.

From the fact above, Theorem 4 in [4], Theorem 3 and Corollary 5, the following hold:

(a) A compact-covering compact image of a metric space need not be a compact-covering σ-image of a metric space.

(b) A quotient π-image of a metric space need not be a quotient σ-image of a metric space.

(c) A quotient s-image of a metric space need not be a quotient σ-image of a metric space. But a quotient σ-image of a metric space is a quotient s-image of a metric space.

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