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ASYMPTOTIC PLANARITY OF DRESHER MEAN VALUES

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Abstract. A family of Dresher mean values is asymptotically planar with respect to its two parameters. An asymptotic formula presenting this property holds if: (a) all variables converge to the same value; and, equivalently, because of means homogeneity, (b) for variables with same additive increment converging to infinity.

Suppose $\mathbf{x} = (x_1, x_2, \dots, x_n)$, $\mathbf{a} = (a, a, \dots, a)$, and $\mathbf{q} = (q_1, q_2, \dots, q_n)$ are sequences of nonnegative reals and $a > 0$. Without loss of generality let the weights q_i be normalized by $q_1 + q_2 + \dots + q_n = 1$. The geometric, the harmonic, and the quadratic mean values respectively are

$$G_q(\mathbf{x}) = \prod_{i=1}^n x_i^{q_i}, \quad A_q(\mathbf{x}) = \sum_{i=1}^n q_i x_i, \quad Q_q(\mathbf{x}) = \sqrt{\sum_{i=1}^n q_i x_i^2}.$$

Note that $\sigma_q^2(\mathbf{x}) = Q_q^2(\mathbf{x}) - A_q^2(\mathbf{x})$ is a weighted variance of $\mathbf{x}$, which satisfies $\sigma_q^2(\mathbf{x}+\mathbf{a}) = \sigma_q^2(\mathbf{x})$. Dresher mean values [2] are a two-parameter family of means that increase with each parameter

$$D_{s,t}(\mathbf{x}) = \begin{cases} \left(\sum_{i=1}^n q_i x_i^s \Big/ \sum_{j=1}^n q_j x_j^t \right)^{1/(s-t)}, & \text{if } s \neq t \\ \exp\left(\sum_{i=1}^n q_i x_i^t \log x_i \Big/ \sum_{j=1}^n q_j x_j^t \right), & \text{if } s = t. \end{cases}$$

Theorem. *Dresher mean values for both cases $s \neq t$ and $s = t$ have the unique asymptotic formulas*

$$\begin{aligned} D_{s,t}(\mathbf{x}) &= A_q(\mathbf{x}) + \frac{s+t-1}{2a}(Q_q^2(\mathbf{x}) - A_q^2(\mathbf{x})) + o(Q_q^2(\mathbf{x}-\mathbf{a})) \\ &= A_q(\mathbf{x}) + (s+t-1)(Q_q(\mathbf{x}) - A_q(\mathbf{x})) + o\big(Q_q^2(\mathbf{x}-\mathbf{a})\big) \\ &= G_q(\mathbf{x}) + (s+t)(A_q(\mathbf{x}) - G_q(\mathbf{x})) + o\big(Q_q^2(\mathbf{x}-\mathbf{a})\big), \quad \mathbf{x} \to \mathbf{a}, \end{aligned}$$

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and if $a \to \infty$, then

$$\begin{aligned} D_{s,t}(\mathbf{x}+\mathbf{a}) &= a + A_q(\mathbf{x}) + \frac{s+t-1}{2a}(Q_q^2(\mathbf{x}) - A_q^2(\mathbf{x})) + o(1/a) \\ &= A_q(\mathbf{x}+\mathbf{a}) + (s+t-1)(Q_q(\mathbf{x}+\mathbf{a}) - A_q(\mathbf{x}+\mathbf{a})) + o(1/a) \\ &= G_q(\mathbf{x}+\mathbf{a}) + (s+t)(A_q(\mathbf{x}+\mathbf{a}) - G_q(\mathbf{x}+\mathbf{a})) + o(1/a). \end{aligned}$$

Asymptotic planarity implies Hoehn and Niven property for Dresher mean values

$$D_{s,t}(\mathbf{x}+\mathbf{a}) - a \to A_q(\mathbf{x}), \quad a \to \infty.$$

Proof. Suppose $s \neq t$ and $\mathbf{h} = \mathbf{x} - \mathbf{a}$. Then

$$\begin{aligned} x_i^s &= a^s \left(1 + \frac{h_i}{a}\right)^s = a^s \left(1 + \frac{s}{a}h_i + \frac{s(s-1)}{2a^2}h_i^2 + o\right), \quad h_i \to 0, \\ \sum_{i=1}^n q_i x_i^s &= a^s \left(1 + \frac{s}{a}A_q(\mathbf{h}) + \frac{s(s-1)}{2a^2}Q_q^2(\mathbf{h}) + o\right), \end{aligned}$$

where $o = o\left(h_i^2\right)$ and $o = o\left(Q_q^2(\mathbf{h})\right)$, respectively. Therefore

$$\begin{aligned} \log D_{s,t}(\mathbf{x}) &= \frac{1}{s-t}\left[\log \sum_{i=1}^n q_i x_i^s - \log \sum_{j=1}^n q_j x_j^t\right] \\ &= \frac{1}{s-t}\left[\log a^s + \log\left(1 + \frac{s}{a}A_q(\mathbf{h}) + \frac{s(s-1)}{2a^2}Q_q^2(\mathbf{h}) + o\right)\right. \\ &\qquad \left. - \log a^t - \log\left(1 + \frac{t}{a}A_q(\mathbf{h}) + \frac{t(t-1)}{2a^2}Q_q^2(\mathbf{h}) + o\right)\right] \\ &= \log a + \frac{1}{s-t}\left[\frac{s}{a}A_q(\mathbf{h}) + \frac{s(s-1)}{2a^2}Q_q^2(\mathbf{h}) - \frac{s^2}{2a^2}A_q^2(\mathbf{h})\right. \\ &\qquad \left. - \frac{t}{a}A_q(\mathbf{h}) - \frac{t(t-1)}{2a^2}Q_q^2(\mathbf{h}) + \frac{t^2}{2a^2}A_q^2(\mathbf{h}) + o\right] \\ &= \log a + \frac{1}{a}A_q(\mathbf{h}) - \frac{1}{2a^2}A_q^2(\mathbf{h}) + \frac{s+t-1}{2a^2}\left(Q_q^2(\mathbf{h}) - A_q^2(\mathbf{h})\right) + o \\ &= \log\left[a\left(1 + \frac{1}{a}A_q(\mathbf{h})\right)\left(1 + \frac{s+t-1}{2a^2}\left(Q_q^2(\mathbf{h}) - A_q^2(\mathbf{h})\right)\right)\right] + o. \end{aligned}$$

Since the obtained expression is well defined and continuous at $s = t$, for both cases $s \neq t$ and $s = t$ we have

$$\begin{aligned} D_{s,t}(\mathbf{x}) &= a \exp\left(\frac{1}{a}A_q(\mathbf{h}) - \frac{1}{2a^2}A_q^2(\mathbf{h}) + \frac{s+t-1}{2a^2}\left(Q_q^2(\mathbf{h}) - A_q^2(\mathbf{h})\right) + o\right) \\ &= a\left(1 + \frac{1}{a}A_q(\mathbf{h}) - \frac{1}{2a^2}A_q^2(\mathbf{h}) + \frac{s+t-1}{2a^2}\left(Q_q^2(\mathbf{h}) - A_q^2(\mathbf{h})\right)\right. \\ &\qquad \left. + \frac{1}{2a^2}A_q^2(\mathbf{h}) + o\right) \\ &= a + A_q(\mathbf{h}) + \frac{s+t-1}{2a}\left(Q_q^2(\mathbf{h}) - A_q^2(\mathbf{h})\right) + o\left(Q_q^2(\mathbf{h}) - A_q^2(\mathbf{h})\right). \end{aligned}$$

This gives the first line of the first formula. The third line follows from asymptotic linearity of power mean values [1], particularly

$$(Q_q(\mathbf{x}) - A_q(\mathbf{x}))/(A_q(\mathbf{x}) - G_q(\mathbf{x})) \to 1, \quad \mathbf{x} \to \mathbf{a}. \tag{1}$$

In the second formula the first line follows from the above proof with $a \to \infty$, $o = o(1/a^2)$, and $o = o(1/a)$ in the last unspecified appearance of o. Hoehn and Niven property [2], which is a consequence of asymptotic linearity property [1], states

$$M_q(\mathbf{x} + \mathbf{a}) - A_q(\mathbf{x} + \mathbf{a}) \to 0, \quad a \to \infty,$$

where M is any power mean value. Therefore

$$Q_q(\mathbf{x} + \mathbf{a}) + A_q(\mathbf{x} + \mathbf{a})/2a \to 1, \quad a \to \infty,$$

what implies the second line. The third line follows from the asymptotic linearity formula at infinity, i.e. (1) for the argument $\mathbf{x}+\mathbf{a}$ and $a \to \infty$. (The second formula also follows from the first one and from homogeneity of involved mean values.) ■

CONJECTURE. *Let $\mathbf{x}$ be a sequence of reals and $a > 0$. The unified asymptotic formula for Dresher mean values holds for convergent variables, as well as for an additive infinitely increasing parameter*

$$\begin{aligned} D_{s,t}(\mathbf{a} + \mathbf{x}) &= a + A_q(\mathbf{x}) + \frac{s+t-1}{2a}\sigma_q^2(\mathbf{x}) + o(\frac{\sigma_q^2(\mathbf{x})}{2a}) \\ &= A_q(\mathbf{a} + \mathbf{x}) + (s+t-1)(Q_q(\mathbf{a} + \mathbf{x}) - A_q(\mathbf{a} + \mathbf{x})) + o(\frac{\sigma_q^2(\mathbf{x})}{2a}) \\ &= G_q(\mathbf{a} + \mathbf{x}) + (s+t)(A_q(\mathbf{a} + \mathbf{x}) - G_q(\mathbf{a} + \mathbf{x})) + o(\frac{\sigma_q^2(\mathbf{x})}{2a}), \end{aligned}$$

where either $\mathbf{x} \to \mathbf{0}$ or $a \to \infty$. Infinitesimals $\sigma_q^2(\mathbf{x})/2a$, $Q_q(\mathbf{a} + \mathbf{x}) - A_q(\mathbf{a} + \mathbf{x})$, and $A_q(\mathbf{a} + \mathbf{x}) - G_q(\mathbf{a} + \mathbf{x})$ are equivalent.

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