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DIRECT AND INVERSE THEOREMS FOR SZÂSZ-LUPAS TYPE OPERATORS IN SIMULTANEOUS APPROXIMATION

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Abstract. In this paper we give the direct and inverse theorems for Szâsz-Lupas operators and study the simultaneous approximation for a new modification of the Szâsz operators with the weight function of Lupas operators.

1. Introduction

Let f be a function defined on the interval $[0,\infty)$ with real values. For $f \in [0,\infty)$ and $n \in N$, the Szâsz operator $S_n(f,x)$ is defined as follows:

$$S_n(f,x) = \sum_{k=0}^{\infty} s_{n,k}(x) f(k/n), \quad \text{where } s_{n,k}(x) = \frac{e^{-nx}(nx)^k}{k!}.$$

The Szâsz-type operator $L_n(f,x)$ is defined by

$$L_n(f,x) = \sum_{k=0}^{\infty} s_{n,k}(x) \phi_{n,k}(f),$$

where

$$\phi_{n,k}(f) = \begin{cases} f(0), & \text{for } k=0 \\ n\int_0^\infty s_{n,k}(t) f(t)\,dt, & \text{for } k=1,2,\dots \end{cases}$$

In [10], Mazhar and Totik introduced the Szâsz-type operator and showed some approximation theorems. Lupas proposed a family of linear positive operators mapping $C[0,\infty)$ into $C[0,\infty)$, the class of all bounded and continuous functions on $[0,\infty)$ namely,

$$(B_n f)(x) = \sum_{k=0}^{\infty} p_{n,k}(x) f(k/n), \quad \text{where } p_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}.$$

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Motivated by the integration of Bernstein polynomials of Derriennic [4], Sahai and Prasad [11] modified the operators B_n for function integrable on $[0,\infty)$ as

$$(M_n f)(x) = (n-1)\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k}(t)f(t)\,dt.$$

Now we consider another modification of operators with the weight function of Lupas operators, which are defined as

$$(V_n f)(x) = (n-1)\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^{\infty} p_{n,k}(t)f(t)\,dt. \tag{1.1}$$

The norm $\|.\|_{C_\alpha}$ on the space $C_\alpha[0,\infty) = \{f \in C[0,\infty) : |f(t)| \leq Kt^\alpha$ for some $\alpha > 0$ and $K > 0\}$ is defined by

$$\|f\|_{C_\alpha} = \sup_{0\leq t<\infty} |f(t)|\, t^{-\alpha}.$$

To improve the saturation order $O(n^{-1})$ for the operator (1.1), we use the technique of linear combination as described below:

$$V_n(f,k,x) = \sum_{j=0}^{k} C(j,k)V_{d_j n}(f,x),$$

where

$$C(j,x) = \prod_{\substack{i=0\\ i\neq j}}^{k} \frac{d_j}{d_j - d_i} \text{ for } k \neq 0 \text{ and } C(0,0) = 1$$

and $d_0, d_1, d_2, \ldots, d_k$ are (k+1) arbitrary, fixed and distinct positive integers. For our convenience we shall write the operator (1.1) as

$$V_n(f,x) = \int_0^{\infty} W(n,x,t)f(t)\,dt,$$

where

$$W(n,x,t) = (n-1)\sum_{k=0}^{\infty} s_{n,k}(x)p_{n,k}(t).$$

The function f is said to belong to the generalized Zygmund class Liz (α,k,a,b) if there exists a constant M such that

$$\omega_{2k}(f,\eta,a,b) \leq M\eta^{\alpha k},\ \eta > 0,$$

where $\omega_{2k}(f,\eta,a,b)$ denotes the modulus of continuity of $2k$-th order of $f(x)$ on the interval $[a,b]$. The class Liz $(\alpha,1,a,b)$ is more commonly denoted by Lip*(α,a,b).

Let $f \in C_\alpha[0,\infty)$ and $0 < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < \infty$. Then for $m \in N$ the Steklov mean $f_{\eta,m}$ of the m-th order corresponding to f, for sufficiently small values of $\eta > 0$ is defined by

$$f_{\eta,m}(x) = \eta^{-m}\left(\int_{-\eta/2}^{\eta/2}\right)^m \left\{f(x) + (-1)^{m-1}\triangle^m_{\sum_{i=1}^{m} x_i} f(x)\right\}\prod_{i=1}^{m} dx_i, \tag{1.2}$$

where $x \in [a_1,b_1]$ and $\triangle^m_\eta f(x)$ is the m-th order forward difference with step length η.

The direct results in ordinary and simultaneous approximation for such type of modified Szâsz-Mirakyan operators were studied by many researchers see e.g. [2], [5], [6] and [12].

2. Auxiliary results

In this section, we shall give some basic results, which will be useful in proving the main results.

Lemma 2.1. [9] *For $m \in N \cup \{0\}$, let the m-th order moment for the Szâsz operator be defined by*

$$\mu_{n,m}(x) = \sum_{k=0}^{\infty} s_{n,k}(x) \left(\frac{k}{n} - x\right)^m .$$

Then we have $\mu_{n,0}(x) = 1$, $\mu_{n,1}(x) = 0$ and

$$n\mu_{n,m+1}(x) = x\left(\mu'_{n,m}(x) + m\mu_{n,m-1}(x)\right), \quad \textit{for } n \in N.$$

Consequently,

(i) $\mu_{n,m}(x)$ is a polynomial in x of degree $[m/2]$;

(ii) for every $x \in [0,\infty)$, $\mu_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$, where $[\beta]$ denotes the integral part of β.

Lemma 2.2. *Let the m-th moment for the Szâsz operator be defined by*

$$\mu_{n,m}(x) = (n-1)\sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} p_{n,k}(t)(t-x)^m\, dt.$$

Then

(i) $\mu_{n,0}(x) = 1$, $\mu_{n,1}(x) = \frac{(1+2x)}{(n-2)}$, $n > 2$;

(ii) $(n-m-2)\mu_{n,m+1}(x) = x\left[\mu'_{n,m}(x) + m(2+x)\mu_{n,m-1}(x)\right] + (m+1) \times (1+2x)\mu_{n,m}(x)$

(iii) $\mu_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$ for all $x \in [0,\infty)$.

Proof. By the definition of $\mu_{n,m}(x)$, we can easily obtain (i). Now the proof of (ii) goes as follows:

$$x\mu'_{n,m}(x) = (n-1)\sum_{k=0}^{\infty} x s'_{n,k}(x) \int_0^{\infty} p_{n,k}(t)(t-x)^m\, dt - mx\mu_{n,m-1}(x).$$

Using relations $t(1+t)p'_{n,k}(t) = (k-nt)p_{n,k}(t)$ and $xs'_{n,k}(x) = (k-nx)s_{n,k}(x)$, we get

$$\begin{aligned} &x\left[\mu'_{n,m}(x) + m\mu_{n,m-1}(x)\right] \\ &\quad = (n-1)\sum_{k=0}^{\infty} (k-nx)\, s_{n,k}(x) \int_0^{\infty} p_{n,k}(t)(t-x)^m\, dt \end{aligned}$$

$$= (n-1)\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^{\infty}\left[(k-nt)+n\,(t-x)\right]p_{n,k}(t)(t-x)^m\,dt$$
$$= (n-1)\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^{\infty} t\,(1+t)\,p'_{n,k}(t)(t-x)^m\,dt + n\mu_{n,m+1}(x)$$
$$= (n-1)\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^{\infty}\left[(1+2x)(t-x)+(t-x)^2+x(1+x)\right]p'_{n,k}(t)(t-x)^m\,dt$$
$$+ n\mu_{n,m+1}(x)$$
$$= -(m+1)(1+2x)\mu_{n,m}(x) - (m+2)\mu_{n,m+1}(x) - mx(1+x)\mu_{n,m-1}(x)$$
$$+ n\mu_{n,m+1}(x)$$

This leads to proof of (ii). The proof of (iii) easily follows from (i) and (ii). ∎

LEMMA 2.3. *If f is differentiable r times ($r = 1, 2, 3, \dots$) on $[0,\infty)$, then we have*

$$\left(V_n^{(r)}f\right)(x) = \frac{n^r(n-r-1)!}{(n-2)!}\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^{\infty} p_{n,k}(t)f^{(r)}(t)\,dt.$$

Proof. By Leibnitz's theorem in (1.1)

$$\left(V_n^{(r)}f\right)(x) = (n-1)\sum_{i=0}^{r}\sum_{k=i}^{\infty}\binom{r}{i}\frac{(-1)^{r-i}n^i e^{-nx}(nx)^{k-i}}{(k-1)!}\int_0^{\infty} p_{n,k}(t)f(t)\,dt$$
$$= (n-1)\sum_{k=0}^{\infty}\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}n^r s_{n,k}(x)\int_0^{\infty} p_{n,k+i}(t)f(t)\,dt$$
$$= (n-1)\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^{\infty}(-1)^r\left\{\sum_{i=0}^{r}\binom{r}{i}(-1)^i n^r p_{n,k+i}(t)(t)\right\}f(t)\,dt.$$

Again using Leibnitz's theorem

$$p_{n-r,k+r}^{(r)}(t) = \frac{(n-1)!}{(n-r-1)!}\sum_{i=0}^{r}(-1)^i\binom{r}{i}p_{n,k+i}(t),$$
$$\left(V_n^{(r)}f\right)(x) = \frac{n^r(n-r-1)!}{(n-2)!}\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^{\infty}(-1)^r p_{n-r,k+r}^{(r)}(t)f(t)\,dt,$$

integrating by parts r times, we get the required result. ∎

LEMMA 2.4. *For the function $f_{\eta,m}(x)$ defined in (1.2), there hold:*

(i) $f_{\eta,m} \in C[a_1,b_1]$;

(ii) $\|f_{\eta,m}^{(r)}\|_{C[a_2,b_2]} \le M_r\eta^{-r}\omega_r(f,\eta,a_1,b_1),\ r = 1,2,\dots,m;$

(iii) $\|f - f_{\eta,m}\|_{C[a_2,b_2]} \le M_{m+1}\omega_m(f,\eta,a_1,b_1);$

(iv) $\|f_{\eta,m}\|_{C[a_2,b_2]} \le M_{m+2}\|f\|_{C[a_2,b_2]} \le M'\|f\|_{C_\alpha}$,

where M_i' are certain constants independent of f and η.

For the proof of the above properties of the function $f_{\eta,m}(x)$ we refer to [12, page 167].

LEMMA 2.5. [8, 9] *There exist polynomials $q_{i,j,r}(x)$ independent of n and k such that*

$$x^r \frac{d^r}{dx^r}\left[e^{-nx}(nx)^k\right] = \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \left|k-nx\right|^j q_{i,j,r}(x)\left[e^{-nx}(nx)^k\right]$$

LEMMA 2.6. *Let $f \in C_\alpha[0,\infty)$. If $f^{(2k+2)}$ exists at a point $x \in (0,\infty)$, then*

$$\lim_{n\to\infty} n^{k+1}\left\{V_n(f,k,x)-f(x)\right\} = \sum_{p=0}^{2k+2} Q(p,k,x) f^{(p)}(x),$$

where $Q(p,k,x)$ is a certain polynomial in x of degree p.

The proof of Lemma 2.6 follows along the lines of [7].

LEMMA 2.7. *Let δ and γ be any two positive numbers and $[a,b] \subset [0,\infty)$. Then, for any $m > 0$ there exists a constant M_m such that*

$$\left\| \int_{|t-x|\ge\delta} V_n(f,x) t^\gamma dt \right\|_{C[a,b]} \le M_m n^{-m}.$$

The proof of this result follows easily by using Schwarz inequality and Lemma 2.7 from [1].

3. Main results

THEOREM 3.1. (Direct Theorem) *Let $f \in C_\alpha[0,\infty)$. Then, for sufficiently large n, there exists a constant M independent of f and n such that*

$$\left\|V_n(f,k,.)-f\right\|_{C[a_2,b_2]} \le \max\left\{C_1\omega_{2k+2}(f;n^{-1/2},a_1,b_1)C_2 n^{-(k+1)}\left\|f\right\|_{C_\alpha}\right\},$$

where $C_1 = C_1(k)$ and $C_2 = C_2(k,f)$.

Proof. By linearity property

$$\begin{aligned}\left\|V_n(f,k,.)-f\right\|_{C[a_2,b_2]} &\le \left\|V_n\left((f-f_{2k+2,\eta}),k,.\right)\right\|_{C[a_2,b_2]}\\ &+\left\|V_n\left(f_{2k+2,\eta},k,.\right)-f_{2k+2,\eta}\right\|_{C[a_2,b_2]}+\left\|f-f_{2k+2,\eta}\right\|_{C[a_2,b_2]}\\ &= A_1+A_2+A_3, \quad \text{say.}\end{aligned}$$

By property (iii) of Steklov mean, we get

$$A_3 \le C_1\omega_{2k+2}(f,\eta,a_1,b_1).$$

Next, by Lemma 2.6, we get

$$A_2 \le C_2 n^{-(k+1)}\left\|f_{2k+2,\eta}\right\|_{C[a_1,b_1]}.$$

Using the interpolation property [5] and properties of Steklov mean,

$$A_2 \leq C_3 n^{-(k+1)}\Big\{\big\|f\big\|_{C_\alpha} + \eta^{-(2k+2)}\omega_{2k+2}(f,\eta)\Big\}.$$

To estimate A_1, we choose a_2, b_2 such that

$$0 < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < \infty.$$

Also let $\psi(t)$ be the characteristic function of the interval $[a_2, b_2]$, then

$$\begin{aligned} A_1 &\leq \big\|V_n\left(\psi(t)\left(f(t) - f_{2k+2,\eta}(t)\right), k, .\right)\big\|_{C[a_3,b_3]} \\ &\quad + \big\|V_n\left(\left(1-\psi(t)\right)\left(f(t) - f_{2k+2,\eta}(t)\right), k, .\right)\big\|_{C[a_3,b_3]} \\ &= A_4 + A_5, \quad \text{say.} \end{aligned}$$

We note that in order to estimate A_4 and A_5, it is sufficient to consider their expressions without the linear combination. It is clear that by Lemma 2.3, we obtain

$$\begin{aligned} &V_n\left(\psi(t)\left(f(t) - f_{2k+2,\eta}(t)\right), x\right) \\ &\qquad = (n-1)\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^\infty p_{n,k}(t)\psi(t)\left(f(t) - f_{2k+2,\eta}(t)\right)\,dt. \end{aligned}$$

Hence,

$$\big\|V_n\left(\psi(t)\left(f(t) - f_{2k+2,\eta}(t)\right), .\right)\big\|_{C[a_1,b_1]} \leq C_4\big\|f - f_{2k+2,\eta}\big\|_{C[a_2,b_2]}.$$

Now for $x \in [a_3, b_3]$ and $t \in [0,\infty)\,/\,[a_2, b_2]$ we can choose an η_1 satisfying $\big|t - x\big| \geq \eta_1$. Therefore by Lemma 2.5 and Schwarz inequality, we have

$$\begin{aligned} I &\equiv \big|V_n\left(\left(1-\psi(t)\right)\left(f(t) - f_{2k+2,\eta}(t)\right), x\right)\big| \\ &\leq (n-1)\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \frac{\big|\phi_{i,j,r}(x)\big|}{x^r}\sum_{k=0}^{\infty} s_{n,k}(x)\big|k - nx\big|^j \times \\ &\qquad \times \int_0^\infty p_{n,k}(t)\left(1-\psi(t)\right)\big|f(t) - f_{2k+2,\eta}(t)\big|\,dt \\ &\leq C_5\big\|f\big\|_{C_\alpha}(n-1)\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \sum_{k=0}^{\infty} s_{n,k}(x)\big|k - nx\big|^j \int_{|t-x|\geq\eta_1} p_{n,k}(t)\,dt \\ &\leq C_5\eta_1^{-2s}\big\|f\big\|_{C_\alpha}(n-1)\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \sum_{k=0}^{\infty} s_{n,k}(x)\big|k - nx\big|^j \times \\ &\qquad \times \left(\int_0^\infty p_{n,k}(t)\,dt\right)^{1/2}\left(\int_0^\infty p_{n,k}(t)(t-x)^{4s}\,dt\right)^{1/2} \\ &\leq C_5\eta_1^{-2s}\big\|f\big\|_{C_\alpha}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \left\{\sum_{k=0}^{\infty} s_{n,k}(x)(k - nx)^{2j}\right\}^{1/2} \times \\ &\qquad \times \left((n-1)\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^\infty p_{n,k}(t)(t-x)^{4s}\,dt\right)^{1/2}. \end{aligned}$$

Hence, by Lemma 2.1 and Lemma 2.2, we have

$$I \le C_6 \|f\|_{C_\alpha} \sum n^{(i+\frac{j}{2}-s)} \le C_6 n^{-q} \|f\|_{C_\alpha}$$

where $q = (s - m/2)$. Now choose $s > 0$ such that $q \ge k+1$. Then $I \le C_6 n^{-(k+1)} \|f\|_{C_\alpha}$.

Therefore by property (iii) of Steklov mean, we get

$$\begin{aligned} A_1 &\le C_7 \|f - f_{2k+2,\eta}\|_{C[a_2,b_2]} + C_6 n^{-(k+1)} \|f\|_{C_\alpha} \\ &\le C_8 \omega_{2k+2}(f, \eta, a_1, b_1) + C_6 n^{-(k+1)} \|f\|_{C_\alpha} \end{aligned}$$

Hence with $\eta = n^{-1/2}$, the theorem follows. ■

THEOREM 3.2. (Inverse Theorem) *If* $0 < \alpha < 2$ *and* $f \in C_\alpha[0,\infty)$ *then in the following statements* $(i) \Rightarrow (ii)$*:*

(*i*) $\|V_n(f,k,x) - f(x)\|_{C[a_1,b_1]} = O\left(n^{-\alpha(k+1)/2}\right)$, *where* $f \in C_\alpha[a,b]$,

(*ii*) $f \in Liz(\alpha, k+1, a_2, b_2)$.

Proof. Let us choose points a', a'', b', b'' in such a way that $a_1 < a' < a'' < a_2 < b_2 < b'' < b' < b_1$. Also suppose $g \in C_0^\infty$ with $supp(g) \subset [a'', b'']$ and $g(x) = 1$ for $x \in [a_a, b_2]$. It is sufficient to show that

$$\|V_n(fg,k,.) - fg\|_{C[a',b']} = O\left(n^{-\alpha(k+1)/2}\right) \Rightarrow (ii). \tag{3.1}$$

Using F in place of fg for all the values of $r > 0$, we get

$$\|\triangle_r^{2k+2} F\|_{C[a'',b'']} \le \|\triangle_r^{2k+2}(F - V_n(F,k,.))\|_{C[a'',b'']} + \|\triangle_r^{2k+2} V_n(F,k,.)\|_{C[a'',b'']} \tag{3.2}$$

By the definition of $\triangle_r^{2k+2}$,

$$\begin{aligned} &\|\triangle_r^{2k+2} V_n(F,k,.)\|_{C[a'',b'']} \\ &= \left\| \int_0^r \cdots \int_0^r V_n\Big(F,k,x + \sum_{i=1}^{2k+2} x_i\Big)\, dx_1 \dots dx_{2k+2} \right\|_{C[a'',b'']} \\ &\le r^{2k+2} \|V_n^{(2k+2)}(F,k,.)\|_{C[a'',b''+(2k+2)r]} \\ &\le r^{2k+2} \Big\{ \|V_n^{(2k+2)}(F - F_{\eta,2k+2},k,.)\|_{C[a'',b''+(2k+2)r]} \\ &\qquad + \|V_n^{(2k+2)}(F_{\eta,2k+2},k,.)\|_{C[a'',b''+(2k+2)r]} \Big\}, \end{aligned} \tag{3.3}$$

where $F_{\eta,2k+2}$ is the Steklov mean of $(2k+2)$-th order corresponding to F. By Lemma 3 from [1], we get

$$\begin{aligned} &\int_0^\infty \left| \frac{\partial^{2k+2}}{\partial x^{2k+2}} W_n(t,x)\, dt \right| \\ &\le \sum_{\substack{2i+j \le 2k+2 \\ i,j \ge 0}} (n-1) \sum_{k=0}^\infty n^i |k - nx|^j \frac{|q_{i,j,2k+2}(x)|}{x^{2k+2}} s_{n,k}(x) \int_0^\infty p_{n,k}(t)\, dt. \end{aligned}$$

Since $\int_0^\infty p_{n,k}(t)\,dt = \frac{1}{n-1}$, by Lemma 2.1,

$$\sum_{k=0}^{\infty} s_{n,k}(x)(k-nx)^{2j} = n^{2j} \sum_{k=0}^{\infty} s_{n,k}(x)\left(\frac{k}{n}-x\right)^{2j} = O(n^j) \tag{3.4}$$

Using Schwarz inequality and Lemma 2.1, we obtain

$$\left\|V_n^{(2k+2)}\left(F-F_{\eta,2k+2},k,.\right)\right\|_{C[a'',b''+(2k+2)r]} \le K_1 n^{k+1}\left\|F-F_{\eta,2k+2}\right\|_{C[a'',b'']} \tag{3.5}$$

By Lemma 2 from [1], we get

$$\int_0^\infty \left[\frac{\partial^k}{\partial x^k} W_n(t,x)\right](t-x)^i dt = 0, \quad \text{for} \quad k > i. \tag{3.6}$$

By Taylor's expansion, we obtain

$$F_{\eta,2k+2}(t) = \sum_{i=0}^{2k+1} \frac{F_{\eta,2k+2}^{(i)}(x)}{i!}(t-x)^i + F_{\eta,2k+2}^{(2k+2)}(\xi)\frac{(t-x)^{2k+2}}{(2k+2)!}, \tag{3.7}$$

where $t < \xi < x$. By (3.6) and (3.7), we get

$$\begin{aligned}&\left\|\frac{\partial^{2k+2}}{\partial x^{2k+2}} V_n\left(F_{\eta,2k+2},k,.\right)\right\|_{C[a'',b''+(2k+2)r]} \\ &\le \sum_{j=0}^{k} \frac{\left|C(j,k)\right|}{(2k+2)!}\left\|F_{\eta,2k+2}^{(2k+2)}\right\|_{C[a'',b'']}\left\|\int_0^\infty \left[\frac{\partial^{2k+2}}{\partial x^{2k+2}} W_{d_j n}(t,x)\right](t-x)^{2k+2}\,dt\right\|_{C[a'',b'']}.\end{aligned}$$

Again applying Schwarz inequality for integration and summation and Lemma 3 from [1], we obtain

$$\begin{aligned} I &\equiv \int_0^\infty \left|\frac{\partial^{2k+2}}{\partial x^{2k+2}} W_n(t,x)\right|(t,x)^{2k+2}\,dt \\ &\le (n-1)\sum_{\substack{2i+j\le 2k+2\\ i,j\ge 0}} \sum_{k=0}^{\infty} n^i s_{n,k}(x)\left|k-nx\right|^j \frac{\left|q_{i,j,2k+2}(x)\right|}{x^{2k+2}}\int_0^\infty p_{n,k}(t)(t-x)^{2k+2}\,dt \\ &\le \sum_{\substack{2i+j\le 2k+2\\ i,j\ge 0}} n^i \frac{\left|q_{i,j,2k+2}(x)\right|}{x^{2k+2}}\left\{\sum_{k=0}^{\infty} s_{n,k}(x)(k-nx)^{2j}\right\}^{1/2}\times \\ &\quad \times\left\{(n-1)\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^\infty p_{n,k}(t)(t-x)^{4k+4}dt\right\}^{1/2}. \end{aligned} \tag{3.8}$$

Using Lemma 2 from [1],

$$(n-1)\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^\infty p_{n,k}(t)(t-x)^{4k+4}\,dt = T_{n,4k+4}(x) = O\left(n^{-(2k+2)}\right). \tag{3.9}$$

Using (3.4) and (3.9) in (3.8), we obtain

$$I \le \sum_{\substack{2i+j\le 2k+2\\ i,j\ge 0}} n^i \frac{\left|q_{i,j,2k+2}(x)\right|}{x^{k+1}} O(n^{j/2}) O\left(n^{-(k+1)}\right) = O(1).$$

Hence

$$\left\|W_n^{(2k+2)}\left(F_{\eta,2k+2},k,.\right)\right\|_{C[a'',b''+(2k+2)r]} \leq K_2\left\|F_{\eta,2k+2}^{(2k+2)}\right\|_{C[a'',b'']}. \tag{3.10}$$

On combining (3.2), (3.3), (3.5) and (3.10) it follows

$$\left\|\triangle_r^{2k+2}F\right\|_{C[a'',b'']} \leq \left\|\triangle_r^{2k+2}\left(F - V_n(F,k,.)\right)\right\|_{C[a'',b'']} + K_3 r^{2k+2}\Big(n^{k+1}\left\|F - F_{\eta,2k+2}\right\|_{C[a'',b'']} + \left\|F_{\eta,(2k+2)}^{(2k+2)}\right\|_{C[a'',b'']}\Big).$$

Since for small values of r the above relation holds, it follows from the properties of $F_{\eta,2k+2}$ and (3.1) that

$$\omega_{2k+2}(F,h,[a'',b'']) \leq K_4\left\{n^{-\alpha(k+1)/2} + h^{2k+2}\left(n^{k+1} + \eta^{-2k+2}\right)_{\omega_{2k+2}}(F,\eta,[a'',b''])\right\}.$$

Choosing η is such a way that $n < \eta^{-2} < 2r$ and following Berens and Lorentz [3], we obtain

$$w_{2k+2}(F,h,[a'',b'']) = O(h^{\alpha(k+1)}). \tag{3.11}$$

Since $F(x) = f(x)$ in $[a_2,b_2]$, from (3.11) we have

$$w_{2k+2}(f,h,[a_2,b_2]) = O(h^{\alpha(k+1)}),\ \text{i.e., } f \in Liz(\alpha,k+1,a_2,b_2).$$

Let us assume (i). Putting $\tau = \alpha(k+1)$, we first consider the case $0 < \tau \leq 1$. For $x \in [a',b']$, we get

$$\begin{aligned} V_n(fg,k,x) - f(x)g(x) &= g(x)V_n\left(\left(f(t)-f(x)\right),k,x\right) + \\ &+ \sum_{j=0}^{k} C(j,k)\int_{a_1}^{b_1} W_{d_j,n}(t,x)f(x)\left(g(t)-g(x)\right)dt + O\left(n^{-k+1)}\right) \\ &= I_1 + I_2 + O\left(n^{-(k+1)}\right), \end{aligned} \tag{3.12}$$

where the O-term holds uniformly for $x \in [a',b']$. Now by assumption

$$\left\|V_n(f,k,.) - f\right\|_{C[a_1,b_1]} = O\left(n^{-\tau/2}\right),$$

we have

$$\left\|I_1\right\|_{C[a',b']} \leq \left\|g\right\|_{C[a',b']}\left\|V_n(f,k,.) - f\right\|_{C[a',b']} \leq K_5 n^{-\tau/2}. \tag{3.13}$$

By the Mean Value Theorem, we get

$$I_2 = \sum_{j=0}^{k} C(j,k)\int_{a_1}^{b_1} W_{d_j,n}(t,x)f(t)\left\{g'(\xi)(t-x)\right\}dt.$$

Once again applying Cauchy-Schwarz inequality and Lemma 2 from [1], we get

$$\begin{aligned} \left\|I_2\right\|_{C[a',b']} &\leq \left\|f\right\|_{C[a_1,b_1]}\left\|g'\right\|_{C[a',b']}\left(\sum_{j=0}^{k}\left|C(j,k)\right|\right)\times \\ &\times \max_{0\leq j\leq k}\left\|\int_0^\infty W_{d_j,n}(t,x)(t-x)^2 dt\right\|_{C[a',b']}^{1/2} = O\left(n^{-\tau/2}\right). \end{aligned} \tag{3.14}$$

Combining (3.12–3.14), we obtain

$$\left\|V_n(fg,k,.) - fg\right\|_{C[a',b']} = O\left(n^{-\tau/2}\right),\ \text{for } 0 < \tau \leq 1.$$

Now to prove the implication for $0 < \tau < 2k+2$, it is sufficient to assume it for $\tau \in (m-1, m)$ and prove if for $\tau \in (m, m+1)$, $(m = 1, 2, 3, \dots, 2k+1)$. Since the result holds for $\tau \in (m-1, m)$, we choose two points x_1, y_1 in such a way that $a_1 < x_1 < a' < b' < y_1 < b_1$. Then in view of assumption $(i) \Rightarrow (ii)$ for the interval $(m-1, m)$ and equivalence of (ii) it follows that $f^{(m-1)}$ exists and belongs to the class $Lip(1-\delta, x_1, y_1)$ for any $\delta > 0$.

Let $g \in C_0^\infty$ be such that $g(x) = 1$ on $[a'', b'']$ and *supp* $g \subset [a'', b'']$. Then with $\chi_2(t)$ denoting the characteristic function of the interval $[x_1, y_1]$, we have

$$\begin{aligned}\left\|V_n(f, g, k, .) - fg\right\|_{C[a',b']} &\le \left\|V_n(g(x)f(t) - f(x)), k, .)\right\|_{C[a',b']} + \\ &+ \left\|V_n(f(t)(g(t) - g(x))\chi(t), k, .)\right\|_{C[a',b']} + O\left(n^{-(k+1)}\right). \quad (3.15)\end{aligned}$$

Now

$$\begin{aligned}\left\|V_n(g(x)(f(t) - f(x)), k, .)\right\|_{C[a',b']} &\le \left\|g\right\|_{C[a'',b'']}\left\|V_n(f, k, .) - f\right\|_{C[a_1,b_1]} \\ &= O\left(n^{-\tau/2}\right). \quad (3.16)\end{aligned}$$

Applying Taylor's expansion of f, we have

$$I_3 \equiv \left\|V_n(f(t)g(t) - g(x))\chi(t), k, .)\right\|_{C[a',b']} =$$

$$\left\|V_n\Big(\Big[\sum_{i=0}^{m-1} \frac{f^{(i)}(x)}{i!}(t-x)^i + \frac{\left\{f^{(m-1)}(\xi) - f^{(m-1)}(x)\right\}}{(m-1)!}\Big](g(t) - g(x))\chi(t), k, .\Big)\right\|_{C[a',b']}$$

where ξ lies between t and x. Since $f^{(m-1)} \in Lip(1-\delta, x_1, y_1)$,

$$\left|f^{(m-1)}(\xi) - f^{(m-1)}(x)\right| \le K_6\left|\xi - x\right|^{1-\delta} \le K_6\left|t - x\right|^{1-\delta},$$

where K_6 is the $Lip(1-\delta, x_1, y_1)$ constant for $f^{(m-1)}$, we have

$$\begin{aligned}I_3 &\le \left\|V_n\Big(\sum_{i=0}^{m-1} \frac{f^{(i)}(x)}{i!}(t-x)^i(g(t) - g(x))\chi(t), k, .\Big)\right\|_{C[a',b']} \\ &+ \frac{K_6}{(m-1)!}\left\|g'\right\|_{C[a'',b'']}\Big(\sum_{j=0}^{k}\left|C(j,k)\right|\Big)\left\|V_{d_j,n}(\left|t-x\right|^{m+1-\delta}\chi(t), .)\right\|_{C[a',b']} \\ &= I_4 + I_5 \quad \text{say.} \quad (3.17)\end{aligned}$$

By Taylor's expansion of g and Lemma 2.6, we have

$$I_4 = O\left(n^{-(k+1)}\right). \quad (3.18)$$

Also, by Hölder's expansion of g and Lemma 2 from [1], we have

$$\begin{aligned}I_5 &\le \frac{K_6}{(m-1)!}\left\|g'\right\|_{C[a'',b'']}\Big(\sum_{j=0}^{k}\left|C(j,k)\right|\Big) \times \\ &\quad \times \max_{0\le j\le k}\left\|\int_{x_1}^{y_1} W_{d_j,n}(t-x)\left|t-x\right|^{m+1-\delta}dt\right\|_{C[a',b']} \\ &\le K_7 \max_{0\le j\le k}\left\|\int_{x_1}^{y_1} W_{d_j,n}(t-x)(t-x)^{2(m+1)}dt\right\|_{C[a',b']}^{\frac{(m+1-\delta)}{2(m+1)}} \\ &= O\left(n^{-(m+1-\delta)/2}\right) = O\left(n^{\tau/2}\right), \quad (3.19)\end{aligned}$$

by choosing δ such that $0 < \delta < m + 1 - \delta$. Combining the estimates (3.15–3.19), we get

$$\left\| V_n(fg, k, .) - fg \right\|_{C[a', b']} = O\left(n^{\tau/2}\right).$$

This completes the proof of the Theorem 3.2. ∎

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