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# HYPERGROUPS OF TYPE U ON THE RIGHT OF SIZE FIVE. PART TWO

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**Abstract.** The hypergroups $H$ of type $U$ on the right can be classified in terms of the family $P_1 = \{1 \circ x \mid x \in H\}$, where $1 \in H$ is the right scalar identity. If the size of $H$ is 5, then $P_1$ can assume only 6 possible values, three of which have been studied in [3]. In this paper, we completely describe other two of the remaining possible cases:

a) $P_1 = \{\{1\}, \{2, 3\}, \{4\}, \{5\}\}$;

b) $P_1 = \{\{1\}, \{2, 3\}, \{4, 5\}\}$.

In these cases, $P_1$ is a partition of $H$ and the equivalence relation associated to it is a regular equivalence on $H$. We find that, apart of isomorphisms, there are exactly 41 hypergroups in case a), and 56 hypergroup in case b).

## 1. Introduction

In this paper we continue the study undertaken in [3] to determine, apart of isomorphisms, the multiplicative tables of the hypergroups of type $U$ on the right of size 5. In that paper this classification is determined according to the possible cases of the family $P_\varepsilon = \{\varepsilon x \mid x \in H\}$, where $\varepsilon$ is the right scalar identity of a hypergroup $H$ of type $U$ on the right. In particular, if $H = \{1, 2, 3, 4, 5\}$ and $\varepsilon = 1$, the possible cases for the family $P_\varepsilon$ are the following:

$C_1: \ 1 \circ 1 = \{1\}; 1 \circ 2 = 1 \circ 3 = 1 \circ 4 = 1 \circ 5 = \{2, 3, 4, 5\}$.

$C_2: \ 1 \circ 1 = \{1\}; 1 \circ 2 = 1 \circ 3 = \{2, 3, 4, 5\}; 1 \circ 4 = 1 \circ 5 = \{4, 5\}$.

$C_3: \ 1 \circ 1 = \{1\}; 1 \circ 2 = 1 \circ 3 = 1 \circ 4 = \{2, 3, 4\}; 1 \circ 5 = \{5\}$.

$C_4: \ 1 \circ 1 = \{1\}; 1 \circ 2 = 1 \circ 3 = \{2, 3\}; 1 \circ 4 = \{4\}; 1 \circ 5 = \{5\}$.

$C_5: \ 1 \circ 1 = \{1\}; 1 \circ 2 = 1 \circ 3 = \{2, 3\}; 1 \circ 4 = 1 \circ 5 = \{4, 5\}$.

$C_6: \ \varepsilon = 1$ is a scalar identity.

In [3], the first three cases have been studied and we obtained, up to isomorphisms, 17 hypergroups. Now we face the fourth and the fifth case. In both cases

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the family $P_\varepsilon$ is a partition of $H$ and the relation $R$ associated to $P_\varepsilon$ is a regular equivalence. By proposition 4.5 of [3], the quotient $H/R$ is a regular and reversible on the right hypergroup, with scalar identity $R(1) = \{1\}$. This result was already exploited in [3] for the analysis of the third case for the family $P_1$. Here we will use it again in the study of the fifth case, in the following way: firstly we search for the multiplicative tables of the regular and reversible on the right hypergroups of size three with scalar identity. Then we reject those tables which can not be quotient hypergroups of hypergroups of type $U$ on the right of size 5. Lastly we use the remaining quotient hypergroups to obtain the tables of hypergroups of type $U$ which we want determine.

The plan of this paper is the following: in the next section we introduce some basic definitions and notations to be used throughout the paper. In the third section we prove that in the fourth case for the partition $P_1$ the hyperproduct $2 \circ 2$ has size $\geq 3$, whence we distinguish two subcases: $2 \circ 2 = \{1, 4, 5\}$ and $2 \circ 2 = \{1, 3, 4, 5\}$. We find 41 hypergroups of type $U$ on the right altogether. In the fourth section, we study the fifth case for the partition $P_1$. We deduce that there exist only three regular and reversible hypergroups that are quotient hypergroups, and these ones give raise to other 56 hypergroups of type $U$ on the right of size 5.

## 2. Basic definitions and results

A *semi-hypergroup* is a non empty set $H$ with a hyperproduct, that is, a possibly multivalued associative product. A *hypergroup* is a semi-hypergroup $H$ such that $xH = Hx = H$ (this condition is called reproducibility).

If a hypergroup $H$ contains an element $\varepsilon$ with the property that, for all $x$ in $H$, one has $x \in x\varepsilon$ (resp., $x \in \varepsilon x$), we say that $\varepsilon$ is a *right identity* (resp., *left identity*) of $H$. If $x\varepsilon = \{x\}$ (resp., $\varepsilon x = \{x\}$), for all $x$ in $H$, then $\varepsilon$ is a *right scalar identity* (resp., *left scalar identity*). The element $\varepsilon$ is said to be an *identity* (resp., *scalar identity*), if it is both right and left identity (resp., right and left scalar identity). If $H$ is a hypergroup with identity $\varepsilon$, then an element $x' \in H$ is called *inverse* of an element $x \in H$, if $\varepsilon \in xx' \cap x'x$.

A hypergroup is said to be *regular* if it has an identity and every element has at least one inverse element. A regular hypergroup is called *reversible on the right*, if for every $x, y, z \in H$ such that $x \in yz$, there exists $z'$ such that $z'$ is inverse element of $z$ and $y \in xz'$. A hypergroup $H$ is said to be of *type $U$ on the right* if it fulfils the following conditions:

$U_1$): $H$ has a right scalar identity $\varepsilon$;

$U_2$): For all $x, y \in H$, $x \in xy \ \Rightarrow \ y = \varepsilon$.

We refer to [2, 4, 5] for other basic concepts and definitions in hypergroup theory. Moreover, we recall from [3] the following results.

PROPOSITION 2.1. *Let $H$ be a hypergroup of type $U$ on the right. For every $x, y, z \in H$ we have:*

1. $x \in \varepsilon y \ \Rightarrow \ zx \subseteq zy$;

*2.* *if* $\varepsilon x = \{x\}$, *then* $z \in xy \;\Rightarrow\; \varepsilon z \subseteq xy$;

*3.* $\varepsilon \in xy \;\Leftrightarrow\; \varepsilon \in yx$;

*4.* *if* $x \in yz$ *and* $y \in xt$, *then* $\varepsilon \in zt \cap tz$.

Proposition 2.2. *Let* $H$ *be a hypergroup of type* $U$ *on the right, such that the family* $P_\varepsilon$ *is a partition of* $H$, *then:*

*1.* *the right scalar identity* $\varepsilon$ *is also left identity;*

*2.* *the relation* $R \subseteq H \times H$ *such that* $xRy \;\Leftrightarrow\; \varepsilon x = \varepsilon y$ *is a regular equivalence;*

*3.* $H/R$ *is a regular and reversible on the right hypergroup, with respect to the hyperproduct* $R(x) \otimes R(y) = \{R(z) \mid z \in xy\}$.

Proposition 2.3. *Let* $H$ *be a hypergroup of type* $U$ *on the right such that the family* $P_\varepsilon$ *is a partition of* $H$ *and let* $R$ *be the relation associated to* $P_\varepsilon$. *Then, for every* $x, y, z \in H$, $a \in R(y)$ *and* $b \in R(z)$, *we have:*

*1.* $R(x) \in R(y) \otimes R(z) \;\Leftrightarrow\; R(x) \cap ab \neq \emptyset \;\Leftrightarrow\; R(x) \subseteq \varepsilon ab$;

*2.* *If* $|R(x)| = 1$, *then* $R(x) \in R(y) \otimes R(z) \;\Leftrightarrow\; x \in ab$.

Proposition 2.4. *Let* $H$ *be a hypergroup of type* $U$ *on the right, of size* 5, *such that the family* $P_\varepsilon$ *is a partition of* $H$. *Then* $H/R$ *does not contain any proper, non-trivial, stable part.*

Finally, we prove an easy lemma that will be useful in next sections.

Lemma 2.1. *Let* $H$ *be a hypergroup of type* $U$ *on the right, such that the family* $P_\varepsilon$ *is a partition of* $H$. *Moreover, let* $|R(x)| \leq 2$, *for every* $x \in H$. *Then, for every* $x, y, a \in H$ *such that* $R(x) = R(y)$, *we have that* $x \in ya \;\Rightarrow\; y \in xa$.

*Proof.* If $x = y$ the result is trivial. If $x \neq y$, from $x \in ya$, one obtains that $a \neq \varepsilon$ and $y \in \varepsilon y = \varepsilon x \subseteq \varepsilon(ya) = (\varepsilon y)a = \{x, y\}a = xa \cup ya$. Therefore $y \in xa$. ∎

## 3. Hypergroups of type $U$ on the right of size five in case $C_4$

In this section, we determine the hypergroups of type $U$ on the right of size 5, starting from the possible cases the hyperproduct $2 \circ 2$ can assume. As we will see later on, there are only two possibilities: $2 \circ 2 = \{1, 4, 5\}$ or $2 \circ 2 = H - \{2\}$. We begin to prove the following results.

Lemma 3.1. *For every* $x, y \in H - \{1\}$, *we have:*

*1.* $3 \in 2 \circ y \Leftrightarrow 2 \in 3 \circ y$;

*2.* *If* $x \in \{4, 5\}$, *then* $2 \in x \circ y \Leftrightarrow 3 \in x \circ y$;

*3.* $1 \circ x \circ y - \{2, 3\} = x \circ y - \{2, 3\}$;

*4.* $2 \circ y - \{2, 3\} = 3 \circ y - \{2, 3\}$;

*5.* *If* $3 \notin 2 \circ 2$, *then* $2 \circ 2 = 2 \circ 3 = 3 \circ 2 = 3 \circ 3$;

*6.* $3 \in 2 \circ 2 \Rightarrow 1 \in 2 \circ 2$.

*Proof.* 1. It follows from Lemma 2.1.

2. $2 \in x \circ y \Rightarrow 1 \circ 2 \subseteq 1 \circ (x \circ y) = (1 \circ x) \circ y = x \circ y \Rightarrow 3 \in x \circ y$.

3. If $z \in 1 \circ x \circ y - \{2,3\}$, then there exists an element $w \in x \circ y$ such that $z \in 1 \circ w$. Since the family $P_1$ is a partition, we have that $1 \circ z = 1 \circ w$, whence $z = w$ because $z \notin \{2,3\}$. Then it results that $z \in x \circ y$, and so $1 \circ x \circ y - \{2,3\} \subseteq x \circ y - \{2,3\}$. The other inclusion is trivial since 1 is left identity.

4. It follows from 3) and the fact that $1 \circ 2 = 1 \circ 3$.

5. It descends from 1) and 3) and the fact that $1 \circ 2 = 1 \circ 3$.

6. At once from the point 1) and Proposition 2.1(4). ■

Proposition 3.1. *1.* $|2 \circ 2| \geq 3$*;*

*2.* $3 \in 2 \circ 2 \Rightarrow 2 \circ 2 = 2 \circ 3 = H - \{2\}$ *and* $3 \circ 2 = 3 \circ 3 = H - \{3\}$*;*

*3.* $3 \notin 2 \circ 2 \Rightarrow 2 \circ 2 = 2 \circ 3 = 3 \circ 2 = 3 \circ 3 = \{1,4,5\}$.

*Proof.* 1. By the way of contradiction, suppose that $|2 \circ 2| \leq 2$. By the preceding lemma, apart of isomorphisms, we have five possible cases: (a) $2 \circ 2 = \{1,3\}$; (b) $2 \circ 2 = \{1\}$; (c) $2 \circ 2 = \{4\}$; (d) $2 \circ 2 = \{1,4\}$; (e) $2 \circ 2 = \{4,5\}$.

In the first case, by Lemma 3.1(1,4), we have that $2 \circ 2 = 2 \circ 3 = \{1,3\}$ and $3 \circ 2 = 3 \circ 3 = \{1,2\}$. In consequence $K = \{1,2,3\}$ is a subhypergroup of $H$ (isomorphic to the D-hypergroup $S_3/S_2$), which is impossible because hypergroups of type $U$ on the right of size 5 do not have proper subhypergroups, see Proposition 2.2 of [3].

In all the other cases, by Lemma 3.1(5) and reproducibility, we can suppose that

$$2 \circ 2 = 2 \circ 3 = 3 \circ 2 = 3 \circ 3 \quad \text{and} \quad 3 \in 2 \circ 4.$$

In particular we have that $2 \circ 2 = 2 \circ 3 \subseteq 2 \circ (2 \circ 4) = (2 \circ 2) \circ 4$. In cases $b$) and $c$), this last inclusion leads to a manifest contradiction.

In the case $d$), for reproducibility, we have that $5 \in 4 \circ 2$. Moreover, from the fact that $\{2\} \cup 2 \circ 4 = 2 \circ (2 \circ 2) = (2 \circ 2) \circ 2 = \{2,3\} \cup 4 \circ 2$ we obtain that $\{3,5\} \subseteq 2 \circ 4$ and $4 \notin 2 \circ 4$. Now, for Lemma 3.1(4), we have that $5 \in 3 \circ 4 \subseteq (2 \circ 4) \circ 4 = 2 \circ (4 \circ 4)$. In consequence there exists an element $x \in 4 \circ 4$ such that $5 \in 2 \circ x$. Since $4 \notin 4 \circ 4$ and $2 \circ \{1,2,3\} = \{1,2,4\}$, we deduce $x = 5$, and so $5 \in 2 \circ 5 \subseteq 2 \circ (4 \circ 2) = (2 \circ 4) \circ 2 \subseteq \{1,3,5\} \circ 2 = \{1,2,3,4\} \cup 5 \circ 2$, which is impossible being $5 \notin 5 \circ 2$.

In the case $e$), by reproducibility, we have that $3 \in (2 \circ 4) \cup (2 \circ 5) = 2 \circ (2 \circ 2) = (2 \circ 2) \circ 2 = (4 \circ 2) \cup (5 \circ 2)$, a contradiction by Lemma 3.1(2).

2. By Lemma 3.1(2) and Proposition 2.1(4), $2 \in 3 \circ 2$ and $1 \in 2 \circ 2$. Moreover, by Lemma 3.1(4) and reproducibility, $5 \notin 2 \circ 2 \Rightarrow 5 \in 4 \circ 2 \subseteq 4 \circ (3 \circ 2) = (4 \circ 3) \circ 2$, that is an absurdity. So $5 \in 2 \circ 2$. Reasoning in a similar way, one obtains that $4 \in 2 \circ 2$, whence the equality $2 \circ 2 = 2 \circ 3 = H - \{2\}$. Analogously one can prove that $3 \circ 2 = 3 \circ 3 = H - \{3\}$.

3. It follows immediately from the point 1) and Lemma 3.1(1, 5). ■

### 3.1. The case $2 \circ 2 = \{1,4,5\}$

In this subsection we find all hypergroups that, besides to $C_4$, fulfil the equality $2 \circ 2 = \{1,4,5\}$. By Proposition 3.1(3), we have that $2 \circ 2 = 2 \circ 3 = 3 \circ 2 = 3 \circ 3 = \{1,4,5\}$. Now, we prove a lemma.

Lemma 3.2. *$3 \in 2 \circ 4 \cap 2 \circ 5$.*

*Proof.* From reproducibility, we can suppose that $3 \in 2 \circ 4$. By the way of contradiction, let $3 \notin 2 \circ 5$. By reproducibility and Lemma 3.1(1), $\{2,3\} \subseteq (4 \circ 5) \cup (5 \circ 5)$. Being $(2 \circ 2) \circ 5 = \{1,4,5\} \circ 5 = \{5\} \cup (4 \circ 5) \cup (5 \circ 5)$, it follows that $3 \in 2 \circ (2 \circ 5)$ and so $4 \in 2 \circ 5$. Moreover $(2 \circ 5) \subseteq 2 \circ (2 \circ 2) = (2 \circ 2) \circ 2 = \{1,4,5\} \circ 2 = \{2,3\} \cup (4 \circ 2) \cup (5 \circ 2) \Rightarrow 4 \in 5 \circ 2$. Since $(2 \circ 2) \circ 4 = \{1,4,5\} \circ 4 = \{4\} \cup 4 \circ 4 \cup 5 \circ 4$ and $2 \circ (2 \circ 4) \supseteq 2 \circ 3 = \{1,4,5\}$, it follows that $5 \in 4 \circ 4$ and so, by Proposition 2.1(4), we have $1 \in (2 \circ 4) \cap (4 \circ 2)$. Finally, from $(2 \circ 4) \circ 2 \supseteq \{1,3\} \circ 2 = H$ it descends $3 \in 2 \circ (4 \circ 2)$ whence $4 \in 4 \circ 2$, that is an absurdity. ■

We note that, in the proof of the previous lemma, it is proved that $3 \in 2 \circ 4$ implies $5 \in 4 \circ 4$. Furthermore, by Lemma 3.1(1) and Proposition 2.1(4), we have respectively $2 \in 3 \circ 4$ and $1 \in 4 \circ 4$ while, by Lemma 3.1(2), from $5 \in 4 \circ 4 \subseteq 4 \circ (2 \circ 2) = (4 \circ 2) \circ 2$, we obtain $\{2,3\} \subseteq 4 \circ 2$. Hence we have that:

$$3 \in 2 \circ 4 \;\Rightarrow\; 2 \in 3 \circ 4;\; 1 \in 4 \circ 4;\; 5 \in 4 \circ 4;\; \{2,3\} \subseteq 4 \circ 2.$$

By exchanging the role of the elements 4 and 5 we also have:

$$3 \in 2 \circ 5 \;\Rightarrow 2 \in 3 \circ 5;\; 1 \in 5 \circ 5;\; 4 \in 5 \circ 5;\; \{2,3\} \subseteq 5 \circ 2.$$

Finally, since $5 \in 4 \circ 4$ and $4 \in 5 \circ 5$, by Proposition 2.1(4), we have also

$$1 \in 4 \circ 5 \cap 5 \circ 4.$$

The preceding results can be summarized in the following incomplete table:

| $\circ$ | 1 | 2,3 | 4 | 5 |
|---|---|---|---|---|
| 1 | $\{1\}$ | $\{2,3\}$ | $\{4\}$ | $\{5\}$ |
| 2 | $\{2\}$ | $\{1,4,5\}$ | $\{3,\dots\}$ | $\{3,\dots\}$ |
| 3 | $\{3\}$ | $\{1,4,5\}$ | $\{2,\dots\}$ | $\{2,\dots\}$ |
| 4 | $\{4\}$ | $\{2,3,\dots\}$ | $\{1,5,\dots\}$ | $\{1,\dots\}$ |
| 5 | $\{5\}$ | $\{2,3,\dots\}$ | $\{1,\dots\}$ | $\{1,4,\dots\}$ |

We go on to complete the table with the following lemma.

Lemma 3.3. *1.* $1 \in 2 \circ 5 \cap 3 \circ 5$*;* $1 \in 5 \circ 2 \cap 5 \circ 3$*;*

*2.* $1 \in 2 \circ 4 \cap 3 \circ 4$*;* $1 \in 4 \circ 2 \cap 4 \circ 3$*;*

*3.* $4 \in 5 \circ 2$ *(whence* $5 \circ 2 = H - \{5\}$*);*

*4.* $5 \in 4 \circ 2$ *(whence* $4 \circ 2 = H - \{4\}$*);*

*5.* $5 \circ 5 = H - \{5\}$*;* $4 \circ 4 = H - \{4\}$*;*

*6.* $4 \circ 5 = H - \{4\}$*;* $5 \circ 4 = H - \{5\}$*;*

*7.* $2 \circ 4 \neq \{1, 3, 4\}$*;* $2 \circ 5 \neq \{1, 3, 5\}$.

*Proof.* 1. By the way of contradiction, suppose $1 \notin 2 \circ 5$. Then $4 \notin 3 \circ 5$ (otherwise, being $3 \in 4 \circ 2$, we should have $1 \in 2 \circ 5$). Similarly $5 \notin 3 \circ 5$ (otherwise $3 \in 5 \circ 2 \Rightarrow 1 \in 2 \circ 5$). Then, by Lemma 3.1(4), $3 \circ 5 = \{2\}$ and $2 \circ 5 = \{3\}$. But $(3 \circ 5) \circ 5 = \{3\}$ while $3 \circ (5 \circ 5) \supseteq \{2, 3\}$, an absurdity. The rest is a consequence of Proposition 2.1(3).

2. Reasoning as in the preceding proof, one obtains the result.

3. $\{2, 3\} = 1 \circ 2 \subseteq (2 \circ 5) \circ 2 = 2 \circ (5 \circ 2) \Rightarrow 4 \in 5 \circ 2$.

4. Similar to the preceding one.

5. If $5 \circ 5 = \{1, 4\}$, the equality $5 \circ (5 \circ 2) = (5 \circ 5) \circ 2$ leads to the contradiction $H = H - \{4\}$. Thus, by Lemma 3.1(2), $5 \circ 5 = H - \{5\}$. Similarly it results $4 \circ 4 = H - \{4\}$.

6. $4 \circ (5 \circ 2) = H \Rightarrow (4 \circ 5) \circ 2 = H$ and so $5 \in (4 \circ 5) \circ 2$, whence, by Lemma 3.1(3), $\{2, 3\} \subseteq 4 \circ 5$. Then $4 \circ 5 \supseteq \{1, 2, 3\}$. Suppose, by the way of contradiction, that $4 \circ 5 = \{1, 2, 3\}$. Then $2 \circ (4 \circ 5) = H - \{3\}$, whence $3 \notin (2 \circ 4) \circ 5 \Rightarrow 2 \circ 4 = \{1, 3\} \Rightarrow 2 \circ (2 \circ 4) = H - \{3\}$. But this is absurd, because $(2 \circ 2) \circ 4 = H$. In the same way, one proves that $5 \circ 4 = H - \{5\}$.

7. If $2 \circ 4 = \{1, 3, 4\}$ then, by Lemma 3.1(4), $5 \notin 3 \circ 4$ and so we obtain $(4 \circ 2) \circ 4 = H - \{5\}$ and $4 \circ (2 \circ 4) = H$, an absurdity. Analogously one can prove that $2 \circ 5 \neq \{1, 3, 5\}$. ∎

Finally we have, up to isomorphisms, the following three hypergroups, when $2 \circ 2 = \{1, 4, 5\}$:

$H_i =$

| $\circ$ | 1 | 2,3 | 4 | 5 |
|---|---|---|---|---|
| 1 | $\{1\}$ | $\{2, 3\}$ | $\{4\}$ | $\{5\}$ |
| 2 | $\{2\}$ | $\{1, 4, 5\}$ | $\{1, 3, \dots\}$ | $\{1, 3, \dots\}$ |
| 3 | $\{3\}$ | $\{1, 4, 5\}$ | $\{1, 2, \dots\}$ | $\{1, 2, \dots\}$ |
| 4 | $\{4\}$ | $H - \{4\}$ | $H - \{4\}$ | $H - \{4\}$ |
| 5 | $\{5\}$ | $H - \{5\}$ | $H - \{5\}$ | $H - \{5\}$ |

$(1 \leq i \leq 3)$ where:

- $H_1$ is obtained for $2 \circ 4 = \{1, 3, 5\}\,; 2 \circ 5 = \{1, 3, 4\}\,; 3 \circ 4 = \{1, 2, 5\}\,; 3 \circ 5 = \{1, 2, 4\}\,;$
- $H_2$ is obtained for $2 \circ 4 = H - \{2\}\,; 2 \circ 5 = \{1, 3, 4\}\,; 3 \circ 4 = H - \{3\}\,; 3 \circ 5 = \{1, 2, 4\}\,;$
- $H_3$ is obtained for $2 \circ 4 = H - \{2\}\,; 2 \circ 5 = H - \{2\}\,; 3 \circ 4 = H - \{3\}\,; 3 \circ 5 = H - \{3\}$.

**3.2. The case $2 \circ 2 = H - \{2\}$**

In this second case, by Proposition 3.1(2), we know that

$$2 \circ 2 = 2 \circ 3 = H - \{2\} \quad \text{and} \quad 3 \circ 2 = 3 \circ 3 = H - \{3\}.$$

Moreover, being for every $x \in H$, $x \circ 2 = x \circ 3$, it is obvious that

$$x \circ (H - \{2\}) = x \circ (H - \{3\}) = x \circ H = H.$$

From this fact, we obtain the inclusion $\{2,3\} \subseteq 4 \circ 2$. In fact, if by absurd $4 \circ 2 \subseteq \{1,5\}$, then we derive the contradiction $5 \notin \{1,5\} \circ 2 \supseteq (4 \circ 2) \circ 2 = 4 \circ (2 \circ 2) = 4 \circ (H - \{2\}) = H$. Therefore $\{2,3\} \cap 4 \circ 2 \neq \emptyset$ and the thesis is a consequence of Lemma 3.1(2).

Moreover, by reproducibility, there exists $x \in H - \{1,2\}$ such that $2 \in x \circ 4$. Obviously $x \in 2 \circ 2$, hence $2 \in 2 \circ (2 \circ 4)$, and so $1 \in 2 \circ 4$. Then, by Proposition 2.1(3), we obtain $1 \in 4 \circ 2$. Consequently all hypergroups of this case satisfy the following condition:

$$\{1,2,3\} \subseteq 4 \circ 2 = 4 \circ 3 \quad \text{and} \quad 1 \in 2 \circ 4 \cap 3 \circ 4.$$

Analogously, exchanging the role of the elements 4 and 5, we can prove that

$$\{1,2,3\} \subseteq 5 \circ 2 = 5 \circ 3 \quad \text{and} \quad 1 \in 2 \circ 5 \cap 3 \circ 5.$$

PROPOSITION 3.2. $4 \circ 2 = 4 \circ 3 = H - \{4\}$ *and* $5 \circ 2 = 5 \circ 3 = H - \{5\}$.

*Proof.* By the preceding remarks, we know that $\{1,2,3\} \subseteq 4 \circ 2 \cap 5 \circ 2$. If, by absurd, $4 \circ 2 = 4 \circ 3 = \{1,2,3\}$, from $H - \{5\} = 4 \circ (4 \circ 2) = (4 \circ 4) \circ 2$ it follows that $\{2,3\} \cap 4 \circ 4 = \emptyset$ and so $4 \circ 4 \subseteq \{1,5\}$.

If $4 \circ 4 = \{1\}$, then $(4 \circ 4) \circ 2 = 4 \circ (4 \circ 2)$ leads to the contradiction $\{2,3\} = H - \{5\}$.

If $4 \circ 4 = \{5\}$, since $4 \circ 5 = 4 \circ (4 \circ 4) = (4 \circ 4) \circ 4 = 5 \circ 4$, from $4 \in 4 \circ (5 \circ 2) = (4 \circ 5) \circ 2 = (5 \circ 4) \circ 2 = 5 \circ (4 \circ 2) = 5 \circ \{1,2,3\} = H - \{4\}$, we obtain a contradiction.

Finally, if $4 \circ 4 = \{1,5\}$, from $(4 \circ 4) \circ 4 = 4 \circ (4 \circ 4)$ we obtain $4 \circ 5 \subseteq \{1,2,3\}$, and so $5 \in (4 \circ 4) \circ 5 = 4 \circ (4 \circ 5) \subseteq 4 \circ \{1,2,3\} = H - \{5\}$, a contradiction.

Exchanging the role of the elements 4 and 5, we can prove that $5 \circ 2 = 5 \circ 3 = H - \{5\}$. ■

In the next propositions we will determine all possible cases assumed from the remaining hyperproducts.

LEMMA 3.4. *1.* $\{2,3\} \subseteq 4 \circ 4 \cap 5 \circ 5$;

*2.* $\{1,5\} \subseteq 2 \circ 4 \cap 3 \circ 4, \quad \{1,4\} \subseteq 2 \circ 5 \cap 3 \circ 5$.

*Proof.* 1. If we suppose that $4 \circ 4 \subseteq \{1,5\}$, then, by Proposition 3.2, we obtain that $H = 4 \circ (H - \{4\}) = 4 \circ (4 \circ 2) = (4 \circ 4) \circ 2 \subseteq \{1,5\} \circ 2 = H - \{5\}$, that is impossible. Then, being $4 \circ 4 \cap \{2,3\} \neq \emptyset$, by Lemma 3.1(2), we derive $\{2,3\} \subseteq 4 \circ 4$. In an analogous way, we can prove that $\{2,3\} \subseteq 5 \circ 5$.

2. By Propositions 2.1(4) and 3.2, we have that $1 \in 2 \circ 4 \cap 2 \circ 5 \cap 3 \circ 4 \cap 3 \circ 5$. Now, if we suppose, by absurd, that $5 \notin 2 \circ 4$ then, by Lemma 3.1(4) and reproducibility, we obtain $5 \notin 3 \circ 4$ and $5 \in 4 \circ 4$. Therefore, by Proposition 3.2, we have $5 \notin (H - \{4\}) \circ 4 = (4 \circ 2) \circ 4 = 4 \circ (2 \circ 4)$, whence $2 \circ 4 = \{1\}$, that is impossible, since the identity $(2 \circ 4) \circ 2 = 2 \circ (4 \circ 2)$ leads to the contradiction $\{2,3\} = H$. Then $\{1,5\} \subseteq 2 \circ 4$. Obviously, by Lemma 3.1(4), we obtain $\{1,5\} \subseteq 3 \circ 4$.

In the same way, we can prove that $\{1,4\} \subseteq 2 \circ 5 \cap 3 \circ 5$. ∎

Proposition 3.3 *For every pair $(x,y)$ of elements in $\{4,5\}$ we have that $\{1,2,3\} \subseteq x \circ y$.*

*Proof.* By Lemma 3.4, we obtain $2 \in 4 \circ 4$ and $4 \in 2 \circ 5$. Thus, by Proposition 2.1(4), we have $1 \in 5 \circ 4 \cap 4 \circ 5$.

Moreover, the equality $(5 \circ 4) \circ 2 = 5 \circ (4 \circ 2)$ implies that $5 \circ 4 \neq \{1\}$, else we derive the contradiction $\{2,3\} = 5 \circ (H - \{4\}) = H$. Now, by absurd, if we suppose that $5 \circ 4 \cap \{2,3\} = \emptyset$, then we have $5 \circ 4 = \{1,4\}$ and the identity $(5 \circ 4) \circ 2 = 5 \circ (4 \circ 2)$ implies that $H - \{4\} = H$, an absurdity. Then, by Lemma 3.1(2), we obtain $\{1,2,3\} \subseteq 5 \circ 4$. By exchanging the elements 5 and 4, we can prove that $\{1,2,3\} \subseteq 4 \circ 5$.

Finally, by Proposition 2.1(4), from the fact that $2 \in 5 \circ 4$ and $5 \in 2 \circ 4$ it follows that $1 \in 4 \circ 4$. So, by Lemma 3.4(1), $\{1,2,3\} \subseteq 4 \circ 4$. Analogously, we can prove that $\{1,2,3\} \subseteq 5 \circ 5$. ∎

By the preceding propositions, we obtain the following partial table:

| $\circ$ | 1 | 2,3 | 4 | 5 |
|---|---|---|---|---|
| 1 | $\{1\}$ | $\{2,3\}$ | $\{4\}$ | $\{5\}$ |
| 2 | $\{2\}$ | $H - \{2\}$ | $\{1,5,\dots\}$ | $\{1,4,\dots\}$ |
| 3 | $\{3\}$ | $H - \{3\}$ | $\{1,5,\dots\}$ | $\{1,4,\dots\}$ |
| 4 | $\{4\}$ | $H - \{4\}$ | $\{1,2,3,\dots\}$ | $\{1,2,3,\dots\}$ |
| 5 | $\{5\}$ | $H - \{5\}$ | $\{1,2,3,\dots\}$ | $\{1,2,3,\dots\}$ |

In the next proposition, we will determine further results about the hyperproducts $x \circ y$ with $x \in \{2,3\}$ and $y \in \{4,5\}$.

Proposition 3.4. *For every $x \in \{2,3\}$ and $y \in \{4,5\}$, we have:*

1. $|x \circ y| \geq 3$;
2. *If $x \circ 4 = \{1,4,5\}$ or $x \circ 5 = \{1,4,5\}$, then $4 \circ 4 = 4 \circ 5 = H - \{4\}$ and $5 \circ 4 = 5 \circ 5 = H - \{5\}$.*

*Proof.* 1. By Lemma 3.4(2), if we suppose that $|x \circ y| < 3$, then the identity $H = x \circ (y \circ x) = (x \circ y) \circ x$ leads to an evident contradiction.

2. Suppose that $x \circ 4 = \{1,4,5\}$ (the case $x \circ 5 = \{1,4,5\}$ can be solved in a similar way). Obviously $H = x \circ (4 \circ 4)$ because $x \in \{2,3\}$ and $\{1,2,3\} \subseteq 4 \circ 4$. So, we obtain $5 \in (x \circ 4) \circ 4 = \{1,4,5\} \circ 4 = \{4\} \cup 4 \circ 4 \cup 5 \circ 4$ whence $5 \in 4 \circ 4$. Therefore $4 \circ 4 = H - \{4\}$. Analogously, from the fact that $H = x \circ (4 \circ 5)$, obtain that $5 \circ 5 = H - \{5\}$.

Finally, from the equality $H = 4 \circ (5 \circ 4)$, there exists $z \in 4 \circ 5$ such that $2 \in z \circ 4$. By hypotheses, necessarily $z = 5$, hence $5 \in 4 \circ 5$ and consequently $4 \circ 5 = H - \{4\}$. Analogously, since $H = 5 \circ (4 \circ 4)$, we deduce that $5 \circ 4 = H - \{5\}$. ∎

Remark 3.1. The equalities $2 \circ 4 = 3 \circ 4 = 2 \circ 5 = 3 \circ 5$ are impossible, or else, by Lemma 3.4(2), we obtain $2 \circ 4 = 3 \circ 4 = 2 \circ 5 = 3 \circ 5 = \{1,4,5\}$ and the identity $(2 \circ 2) \circ 4 = 2 \circ (2 \circ 4)$ leads to the contradiction $H = H - \{3\}$. Therefore, by means of Lemmas 3.1, 3.4 and Proposition 3.4(1), apart of isomorphisms, there are five possible cases for the hyperproducts $x \circ y$, with $x \in \{2,3\}$ and $y \in \{4,5\}$, described in the following partial tables:

$T_1$

| $\circ$ | 4 | 5 |
|---|---|---|
| 2 | $\{1,4,5\}$ | $\{1,3,4\}$ |
| 3 | $\{1,4,5\}$ | $\{1,2,4\}$ |

$T_2$

| $\circ$ | 4 | 5 |
|---|---|---|
| 2 | $\{1,4,5\}$ | $H-\{2\}$ |
| 3 | $\{1,4,5\}$ | $H-\{3\}$ |

$T_3$

| $\circ$ | 4 | 5 |
|---|---|---|
| 2 | $H-\{2\}$ | $H-\{2\}$ |
| 3 | $H-\{3\}$ | $H-\{3\}$ |

$T_4$

| $\circ$ | 4 | 5 |
|---|---|---|
| 2 | $\{1,3,5\}$ | $\{1,3,4\}$ |
| 3 | $\{1,2,5\}$ | $\{1,2,4\}$ |

$T_5$

| $\circ$ | 4 | 5 |
|---|---|---|
| 2 | $\{1,3,5\}$ | $H-\{2\}$ |
| 3 | $\{1,2,5\}$ | $H-\{3\}$ |

We complete this section by listing all possible hypergroups of type $U$, on the right, in this subcase. The tables have been obtained by means of Propositions 3.1(2), 3.2, 3.3 and the preceding partial tables. Associativity has been verified by computer.

1. By Proposition 3.4(2), from the partial tables $T_1$ and $T_2$, we obtain the following two hypergroups:

| $\circ$ | 1 | 2,3 | 4 | 5 |
|---|---|---|---|---|
| 1 | $\{1\}$ | $\{2,3\}$ | $\{4\}$ | $\{5\}$ |
| 2 | $\{2\}$ | $H-\{2\}$ | $\{1,4,5\}$ | $\{1,3,4\}$ |
| 3 | $\{3\}$ | $H-\{3\}$ | $\{1,4,5\}$ | $\{1,2,4\}$ |
| 4 | $\{4\}$ | $H-\{4\}$ | $H-\{4\}$ | $H-\{4\}$ |
| 5 | $\{5\}$ | $H-\{5\}$ | $H-\{5\}$ | $H-\{5\}$ |

| $\circ$ | 1 | 2,3 | 4 | 5 |
|---|---|---|---|---|
| 1 | $\{1\}$ | $\{2,3\}$ | $\{4\}$ | $\{5\}$ |
| 2 | $\{2\}$ | $H-\{2\}$ | $\{1,4,5\}$ | $H-\{2\}$ |
| 3 | $\{3\}$ | $H-\{3\}$ | $\{1,4,5\}$ | $H-\{3\}$ |
| 4 | $\{4\}$ | $H-\{4\}$ | $H-\{4\}$ | $H-\{4\}$ |
| 5 | $\{5\}$ | $H-\{5\}$ | $H-\{5\}$ | $H-\{5\}$ |

2. The partial tables $T_3$ and $T_4$ give rise in all to 20 hypergroups, which can be obtained by completing the following two tables,

| $\circ$ | 1 | 2,3 | 4 | 5 |
|---|---|---|---|---|
| 1 | $\{1\}$ | $\{2,3\}$ | $\{4\}$ | $\{5\}$ |
| 2 | $\{2\}$ | $H-\{2\}$ | $H-\{2\}$ | $H-\{2\}$ |
| 3 | $\{3\}$ | $H-\{3\}$ | $H-\{3\}$ | $H-\{3\}$ |
| 4 | $\{4\}$ | $H-\{4\}$ | $\{1,2,3,\dots\}$ | $\{1,2,3,\dots\}$ |
| 5 | $\{5\}$ | $H-\{5\}$ | $\{1,2,3,\dots\}$ | $\{1,2,3,\dots\}$ |

| $\circ$ | 1 | 2,3 | 4 | 5 |
|---|---|---|---|---|
| 1 | $\{1\}$ | $\{2,3\}$ | $\{4\}$ | $\{5\}$ |
| 2 | $\{2\}$ | $H-\{2\}$ | $\{1,3,5\}$ | $\{1,3,4\}$ |
| 3 | $\{3\}$ | $H-\{3\}$ | $\{1,2,5\}$ | $\{1,2,4\}$ |
| 4 | $\{4\}$ | $H-\{4\}$ | $\{1,2,3,\dots\}$ | $\{1,2,3,\dots\}$ |
| 5 | $\{5\}$ | $H-\{5\}$ | $\{1,2,3,\dots\}$ | $\{1,2,3,\dots\}$ |

by means of the following ten partial tables of the hyperproducts $x \circ y$, with $x, y \in \{4,5\}$:

| | |
|---|---|
| $\{1,2,3\}$ | $\{1,2,3\}$ |
| $\{1,2,3\}$ | $\{1,2,3\}$ |

| | |
|---|---|
| $\{1,2,3\}$ | $\{1,2,3\}$ |
| $\{1,2,3\}$ | $H-\{5\}$ |

| | |
|---|---|
| $\{1,2,3\}$ | $\{1,2,3\}$ |
| $H-\{5\}$ | $\{1,2,3\}$ |

| | |
|---|---|
| $\{1,2,3\}$ | $\{1,2,3\}$ |
| $H-\{5\}$ | $H-\{5\}$ |

| | |
|---|---|
| $\{1,2,3\}$ | $H-\{4\}$ |
| $\{1,2,3\}$ | $H-\{5\}$ |

| | |
|---|---|
| $\{1,2,3\}$ | $H-\{4\}$ |
| $H-\{5\}$ | $\{1,2,3\}$ |

| | |
|---|---|
| $H-\{4\}$ | $\{1,2,3\}$ |
| $\{1,2,3\}$ | $H-\{5\}$ |

| | |
|---|---|
| $\{1,2,3\}$ | $H-\{4\}$ |
| $H-\{5\}$ | $H-\{5\}$ |

| | |
|---|---|
| $H-\{4\}$ | $\{1,2,3\}$ |
| $H-\{5\}$ | $H-\{5\}$ |

| | |
|---|---|
| $H-\{4\}$ | $H-\{4\}$ |
| $H-\{5\}$ | $H-\{5\}$ |

3. With the partial table $T_5$ we obtain 16 hypergroups, which are obtained with by completing the following table,

| $\circ$ | 1 | 2,3 | 4 | 5 |
|---|---|---|---|---|
| 1 | $\{1\}$ | $\{2,3\}$ | $\{4\}$ | $\{5\}$ |
| 2 | $\{2\}$ | $H-\{2\}$ | $\{1,3,5\}$ | $H-\{2\}$ |
| 3 | $\{3\}$ | $H-\{3\}$ | $\{1,2,5\}$ | $H-\{3\}$ |
| 4 | $\{4\}$ | $H-\{4\}$ | $\{1,2,3,\dots\}$ | $\{1,2,3,\dots\}$ |
| 5 | $\{5\}$ | $H-\{5\}$ | $\{1,2,3,\dots\}$ | $\{1,2,3,\dots\}$ |

by means of the 16 partial tables determined by choosing the hyperproducts $4\circ4$ and $4\circ5$ in $\{\{1,2,3\}, H-\{4\}\}$ and the hyperproducts $5\circ4$ and $5\circ5$ in $\{\{1,2,3\}, H-\{5\}\}$.

Finally, we have shown the following result:

THEOREM 3.1. *Apart of isomorphisms, there exist exactly forty one hypergroups in case $C_4$.*

## 4. Hypergroups of type *U* on the right of size five in case $C_5$

We discuss this case, by obtaining first the tables of quotient hypergroups $H/R$ where $R$ is the relation associated to the partition $P_1 = \{1 \circ x \mid x \in H\}$. From Proposition 2.2, we know that $H/R$ is a regular and reversible on the right hypergroup, with the class $R(1) = \{1\}$ as scalar identity. In next propositions, we will determine the hyperproducts $R(x) \otimes R(y)$, while $x, y$ vary in $H - \{1\}$ (we recall that the hyperproduct $\otimes$ is defined in Proposition 2.2(3)). Now, to make easier the notation, we put $R(x) = \overline{x}$ and $\overline{H} = H/R$. Obviously in this case we have $\overline{1} = \{1\}$, $\overline{2} = \overline{3} = \{2,3\}$ and $\overline{4} = \overline{5} = \{4,5\}$.

PROPOSITION 4.1 *For every $\overline{x}, \overline{y} \in \overline{H} - \{\overline{1}\}$ we have:*

1. *If $\overline{x} \neq \overline{y}$, then $\overline{x} \in \overline{x} \otimes \overline{x} \cup \overline{x} \otimes \overline{y}$;*
2. *$\overline{1} \in \overline{x} \otimes \overline{y} \;\Leftrightarrow\; \overline{1} \in \overline{y} \otimes \overline{x}$;*
3. *$\overline{x} \otimes \overline{y} \neq \{\overline{1}\}$;*
4. *$\overline{x} \in \overline{x} \otimes \overline{y} \;\Rightarrow\; \overline{1} \in \overline{y} \otimes \overline{y}$;*
5. *$\overline{x} \in \overline{x} \otimes \overline{x} \;\Rightarrow\; \overline{x} \otimes \overline{x} = \overline{H}$;*
6. *$\overline{x} \notin \overline{x} \otimes \overline{x} \;\Rightarrow\; \overline{x} \otimes \overline{x} = \overline{H} - \{\overline{x}\}$;*
7. *$|\overline{x} \otimes \overline{y}| \geq 2$.*
8. *If $\overline{x} \neq \overline{y}$ and $\overline{x} \otimes \overline{x} = \overline{H}$, then $\overline{1} \in \overline{x} \otimes \overline{y} \cap \overline{y} \otimes \overline{x}$.*

*Proof.* 1. Immediately from the reproducibility in $H$.

2. It follows from Propositions 2.3(2) and 2.1(3).

3. By the way of contradiction, we suppose that $\overline{x} \otimes \overline{y} = \{\overline{1}\}$. By Proposition 2.4, we have that $\overline{x} \neq \overline{y}$, or else the set $\{\overline{1}, \overline{x}\}$ is stable part of $\overline{H}$. By the point 1), we have that $\overline{x} \in \overline{x} \otimes \overline{x}$, whence the contradiction $\overline{1} \in (\overline{x} \otimes \overline{x}) \otimes \overline{y} = \overline{x} \otimes (\overline{x} \otimes \overline{y}) = \{\overline{x}\}$.

4. Let $\overline{x} = \{x, z\}$. From the hypothesis $\overline{x} \in \overline{x} \otimes \overline{y}$ we deduce $x \in z \circ y$ and $z \in x \circ y$, therefore $x \in x \circ (y \circ y)$. So we have $1 \in y \circ y$, and finally $\overline{1} \in \overline{y} \otimes \overline{y}$.

5. If $\overline{x} \in \overline{x} \otimes \overline{x}$, by the point 4), we have that $\overline{1} \in \overline{x} \otimes \overline{x}$. Moreover, by Proposition 2.4, the hypergroup $\overline{H}$ has no proper stable parts, thus $\overline{x} \otimes \overline{x} \neq \{\overline{1}, \overline{x}\}$. Consequently $\overline{x} \otimes \overline{x} = \overline{H}$.

6. If $|\overline{x} \otimes \overline{x}| = 1$, by the hypotheses and the point 3), we can suppose that $\overline{x} \otimes \overline{x} = \{\overline{y}\}$, with $\overline{y} \notin \{\overline{1}, \overline{x}\}$. Obviously $\overline{y} \notin \overline{x} \otimes \overline{y}$ or else we have $\overline{y} \in \overline{x} \otimes \overline{y} = \overline{x} \otimes (\overline{x} \otimes \overline{x}) = (\overline{x} \otimes \overline{x}) \otimes \overline{x} = \overline{y} \otimes \overline{x}$ and from the point 4), we deduce that $\overline{1} \in \overline{x} \otimes \overline{x}$. Now, by reproducibility in $\overline{H}$ and the point 1), we have $\overline{x} \otimes \overline{y} = \{\overline{1}, \overline{x}\}$, whence we obtain $\overline{y} \otimes \overline{y} = (\overline{x} \otimes \overline{x}) \otimes \overline{y} = \overline{x} \otimes (\overline{x} \otimes \overline{y}) = \{\overline{x}, \overline{y}\}$. But this fact contradicts the point 5). Therefore $|\overline{x} \otimes \overline{x}| \neq 1$ and clearly $\overline{x} \otimes \overline{x} = \overline{H} - \{\overline{x}\}$.

7. If $\overline{x} = \overline{y}$, the result follows from 5) and 6). If $\overline{x} \neq \overline{y}$ and, by absurd, suppose that $|\overline{x} \otimes \overline{y}| = 1$, from the point 3), we have that $\overline{x} \otimes \overline{y} = \{\overline{x}\}$ or $\overline{x} \otimes \overline{y} = \{\overline{y}\}$. In the first case, the points 5) and 6) and the identity $\overline{x} \otimes (\overline{y} \otimes \overline{y}) = (\overline{x} \otimes \overline{y}) \otimes \overline{y}$ lead to the contradiction $\overline{H} = \{\overline{x}\}$. Also in the second case, we obtain a clear contradiction, by considering the identity $(\overline{x} \otimes \overline{x}) \otimes \overline{y} = \overline{x} \otimes (\overline{x} \otimes \overline{y})$.

8. Without loss of generality, we can suppose $\overline{2} \otimes \overline{2} = \overline{H}$ and so $\{1,3\} \subseteq 2 \circ 2 = 2 \circ 3$, $\{1,2\} \subseteq 3 \circ 2 = 3 \circ 3$ and $(2 \circ 2) \cap \{4,5\} \neq \emptyset \neq (3 \circ 3) \cap \{4,5\}$.

We note that there exists $a \in \{4,5\}$ such that $a \in (2 \circ 2) \cap (3 \circ 3)$. In fact if $2 \circ 2 = \{1, 3, a\}$, then $a \in 2 \circ 2 = 2 \circ 3 \subseteq 2 \circ (2 \circ 2) = (2 \circ 2) \circ 2 = \{2\} \cup (3 \circ 2) \cup (a \circ 2)$ and so $a \in 3 \circ 2 = 3 \circ 3$. By the points 5) and 6), we obtain that $\overline{4} \otimes \overline{4} \supseteq \{\overline{1}, \overline{2}\}$, and so $a \circ a \supseteq \{1, b\}$ with $b \in \{2,3\}$. Since $a \in b \circ 2 \subseteq (a \circ a) \circ 2 = a \circ (a \circ 2)$, it follows that $\overline{1} \in \overline{4} \otimes \overline{2} \cap \overline{2} \otimes \overline{4}$, completing the proof. ■

Note that, in consequence of the preceding proposition, we know that $\overline{2} \otimes \overline{2} \in \{\{\overline{1}, \overline{4}\}, \overline{H}\}$ and $\overline{4} \otimes \overline{4} \in \{\{\overline{1}, \overline{2}\}, \overline{H}\}$. Now we are going to determine all possible cases which the hyperproducts $\overline{2} \otimes \overline{4}$ and $\overline{4} \otimes \overline{2}$ can assume.

Proposition 4.2. *If $\overline{x} \notin \overline{x} \otimes \overline{x}$, for every $\overline{x} \in \left\{\overline{2}, \overline{4}\right\}$, then $\overline{2} \otimes \overline{4} = \overline{4} \otimes \overline{2} = \{\overline{2}, \overline{4}\}$.*

*Proof.* By Proposition 4.1(6), it follows that $\overline{2} \otimes \overline{2} = \left\{\overline{1}, \overline{4}\right\}$ and $\overline{4} \otimes \overline{4} = \left\{\overline{1}, \overline{2}\right\}$.

Now we observe that, by the identities $(\overline{4} \otimes \overline{4}) \otimes \overline{4} = \overline{4} \otimes (\overline{4} \otimes \overline{4})$ and $(\overline{2} \otimes \overline{2}) \otimes \overline{2} = \overline{2} \otimes (\overline{2} \otimes \overline{2})$, taking in account the Proposition 4.1(1) we obtain $\{\overline{2}, \overline{4}\} \subseteq \overline{2} \otimes \overline{4} \cap \overline{4} \otimes \overline{2}$ and so, by Proposition 4.1(2), we have $\overline{2} \otimes \overline{4} = \overline{4} \otimes \overline{2} \in \left\{\left\{\overline{2}, \overline{4}\right\}, \overline{H}\right\}$.

Suppose $\overline{2} \otimes \overline{4} = \overline{H}$, then $2 \circ 2 = 2 \circ 3 = 3 \circ 2 = 3 \circ 3 = \{1,4,5\}$ and $4 \circ 4 = 4 \circ 5 = 5 \circ 4 = 5 \circ 5 = \{1,2,3\}$, or else if, for example, $2 \circ 2 = 2 \circ 3 = \{1,4\}$, we have $2 \circ (4 \circ 4) \subseteq 2 \circ \{1,2,3\} = \{1,2,4\}$ while $(2 \circ 4) \circ 4 \supseteq 1 \circ 4 = \{4,5\}$. Now, since $2 \circ 4 \cap \{4,5\} \neq \emptyset$, we obtain $3 \in 4 \circ 4 = 5 \circ 4 \subseteq (2 \circ 4) \circ 4 = 2 \circ (4 \circ 4) = H - \{3\}$, a contradiction. ■

Proposition 4.3. *If there exists $\overline{x} \in \left\{\overline{2}, \overline{4}\right\}$ such that $\overline{x} \in \overline{x} \otimes \overline{x}$, then $\overline{2} \otimes \overline{4} = \overline{4} \otimes \overline{2} = \overline{H}$.*

*Proof.* Without loss of generality, we can suppose $\overline{4} \in \overline{4} \otimes \overline{4}$. By Proposition 4.1(5, 6), we have $\overline{4} \otimes \overline{4} = \overline{H}$ and $\overline{2} \otimes \overline{2} \supseteq \{\overline{1}, \overline{4}\}$.

By Proposition 4.1(8), $\overline{1} \in \overline{4} \otimes \overline{2} \cap \overline{2} \otimes \overline{4}$. Since $2 \circ 2 \cap \{4, 5\} \neq \emptyset$ and $5 \in 4 \circ 4 = 4 \circ 5$, we have $5 \in 4 \circ (2 \circ 2) = (4 \circ 2) \circ 2$ and so it follows that $\overline{2} \in \overline{4} \otimes \overline{2}$. We remark that if $\overline{2} \in \overline{2} \otimes \overline{2}$, in a similar way, we obtain $\overline{4} \in \overline{2} \otimes \overline{4}$.

Now the proof splits into two parts.

(I) $\overline{2} \otimes \overline{2} = \{\overline{1}, \overline{4}\}$.

By Proposition 4.1(1, 8), we have $\{\overline{1}, \overline{2}\} \subseteq \overline{2} \otimes \overline{4}$ and by the identity $(\overline{2} \otimes \overline{2}) \otimes \overline{2} = \overline{2} \otimes (\overline{2} \otimes \overline{2})$, it follows that $\overline{4} \in \overline{2} \otimes \overline{4} \Leftrightarrow \overline{4} \in \overline{4} \otimes \overline{2}$ and so $\overline{2} \otimes \overline{4} = \overline{4} \otimes \overline{2}$. Assume, by absurd, that $\overline{4} \notin \overline{2} \otimes \overline{4}$. Then for every $a \in \{2, 3\}$ and $b \in \{4, 5\}$, we have $(a \circ b) \cap \{4, 5\} = \emptyset$ and so, by reproducibility in $H$, $2 \circ 2 = 2 \circ 3 = 3 \circ 2 = 3 \circ 3 = \{1, 4, 5\}$. Moreover, for every $a \in \{2, 3\}$, we have $4 \circ 4 \subseteq (a \circ a) \circ 4 = a \circ (a \circ 4) \subseteq a \circ \{1, 2, 3\} = \{1, 4, 5, a\}$. It follows that $\{2, 3\} \cap 4 \circ 4 = \emptyset$ which quickly leads to a contradiction. So $\overline{2} \otimes \overline{4} = \overline{4} \otimes \overline{2} = \overline{H}$ and we obtain the claim.

(II) $\overline{2} \otimes \overline{2} = \overline{H}$.

We know that $\overline{2} \otimes \overline{4} \supseteq \{\overline{1}, \overline{4}\}$ and $\overline{4} \otimes \overline{2} \supseteq \{\overline{1}, \overline{2}\}$. Assume $\overline{2} \otimes \overline{4} = \{\overline{1}, \overline{4}\}$.

Since $(\overline{2} \otimes \overline{4}) \otimes \overline{2} = \{\overline{1}, \overline{4}\} \otimes \overline{2} = \{\overline{2}\} \cup \overline{4} \otimes \overline{2}$ and $\overline{2} \otimes (\overline{4} \otimes \overline{2}) \supseteq \overline{2} \otimes \overline{2} = \overline{H}$ it follows that $\overline{4} \in \overline{4} \otimes \overline{2}$ and so $\overline{4} \otimes \overline{2} = \overline{H}$.

At this point we have that:

1. $(2 \circ a) \cup (3 \circ a) = H, \quad \forall a \in \{2, 3\}$; (in fact $H = 1 \circ (2 \circ a) = (2 \circ a) \cup (3 \circ a)$)
2. $(4 \circ x) \cup (5 \circ x) = H, \quad \forall x \in \{2, 3, 4, 5\}$; (the proof is similar to 1.)
3. $(a \circ x) = \{1, 4, 5\}, \quad \forall a \in \{2, 3\}$ and $x \in \{4, 5\}$; (if for example, $2 \circ 4 = 2 \circ 5 = \{1, 4\}$ then $2 \circ (2 \circ 5) = 2 \circ \{1, 4\} = \{1, 2, 4\}$ while $(2 \circ 2) \circ 5 \supseteq 1 \circ 5 \ni 5$)
4. $2 \circ 2 = 2 \circ 3 \neq \{1, 3, 4, 5\}$; (otherwise $(2 \circ 2) \circ 4 = H$ while $2 \circ (2 \circ 4) \subseteq \{1, 2, 4, 5\}$)
5. $3 \circ 2 = 3 \circ 3 \neq \{1, 2, 4, 5\}$; (similar to 4.)

Let $2 \circ 2 = \{1, 3, x\}$ and $3 \circ 2 = \{1, 2, y\}$, with $\{x, y\} = \{4, 5\}$. Since $(2 \circ 2) \circ 2 = H - \{x\}$ while $x \in 2 \circ (2 \circ 2)$ we obtain a contradiction. Hence $\overline{2} \otimes \overline{4} = \overline{H}$. At this point, changing the role of $\overline{2}$ and $\overline{4}$ we obtain that $\overline{4} \otimes \overline{2} = \overline{H}$. This completes the proof. ∎

Remark 4.1. By Propositions 4.1(5, 6), 4.2, 4.3 and 4.4, the possible quotient hypergroups $\overline{H}$ of a hypergroup of type $U$ on the right of size 5, whose partition associated to the identity 1 is $P_1 = \{\{1\}, \{2, 3\}, \{4, 5\}\}$, are the following ones:

$\overline{H}_1$

| $\otimes$ | $\overline{1}$ | $\overline{2}$ | $\overline{4}$ |
|---|---|---|---|
| $\overline{1}$ | $\{\overline{1}\}$ | $\{\overline{2}\}$ | $\{\overline{4}\}$ |
| $\overline{2}$ | $\{\overline{2}\}$ | $\{\overline{1}, \overline{4}\}$ | $\{\overline{2}, \overline{4}\}$ |
| $\overline{4}$ | $\{\overline{4}\}$ | $\{\overline{2}, \overline{4}\}$ | $\{\overline{1}, \overline{2}\}$ |

$\overline{H}_2$

| $\otimes$ | $\overline{1}$ | $\overline{2}$ | $\overline{4}$ |
|---|---|---|---|
| $\overline{1}$ | $\{\overline{1}\}$ | $\{\overline{2}\}$ | $\{\overline{4}\}$ |
| $\overline{2}$ | $\{\overline{2}\}$ | $\{\overline{1}, \overline{4}\}$ | $\{\overline{1}, \overline{2}, \overline{4}\}$ |
| $\overline{4}$ | $\{\overline{4}\}$ | $\{\overline{1}, \overline{2}, \overline{4}\}$ | $\{\overline{1}, \overline{2}, \overline{4}\}$ |

$\overline{H}_2^*$

| $\otimes$ | $\overline{1}$ | $\overline{2}$ | $\overline{4}$ |
|---|---|---|---|
| $\overline{1}$ | $\{\overline{1}\}$ | $\{\overline{2}\}$ | $\{\overline{4}\}$ |
| $\overline{2}$ | $\{\overline{2}\}$ | $\{\overline{1},\overline{2},\overline{4}\}$ | $\{\overline{1},\overline{2},\overline{4}\}$ |
| $\overline{4}$ | $\{\overline{4}\}$ | $\{\overline{1},\overline{2},\overline{4}\}$ | $\{\overline{1},\overline{2}\}$ |

$\overline{H}_3$

| $\otimes$ | $\overline{1}$ | $\overline{2}$ | $\overline{4}$ |
|---|---|---|---|
| $\overline{1}$ | $\{\overline{1}\}$ | $\{\overline{2}\}$ | $\{\overline{4}\}$ |
| $\overline{2}$ | $\{\overline{2}\}$ | $\{\overline{1},\overline{2},\overline{4}\}$ | $\{\overline{1},\overline{2},\overline{4}\}$ |
| $\overline{4}$ | $\{\overline{4}\}$ | $\{\overline{1},\overline{2},\overline{4}\}$ | $\{\overline{1},\overline{2},\overline{4}\}$ |

In particular, we observe that the quotient hypergroups $\overline{H}_2$ and $\overline{H}_2^*$ are isomorphic. Moreover, if we denote by $F(\overline{H}_2)$ and $F(\overline{H}_2^*)$ the families of hypergroups of type $U$ on the right, of partition $P_1 = \{\{1\},\{2,3\},\{4,5\}\}$ and respectively of quotient hypergroup $\overline{H}_2$ and $\overline{H}_2^*$, then every hypergroup in $F(\overline{H}_2)$ is isomorphic to a hypergroup in $F(\overline{H}_2^*)$ and vice versa. In fact, it is easy to verify that if $(H,\circ)$ is a hypergroup in $F(\overline{H}_2)$ and $j$ is the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 5 & 2 & 3 \end{pmatrix}$, then the hyperproduct $\diamond$ such that $x \diamond y = \{j(z) \in H \mid z \in j(x) \circ j(y)\}$, for every $x, y \in H$, is such that $(H,\diamond)$ is a hypergroup in the family $F(\overline{H}_2^*)$ isomorphic to $(H,\circ)$ (an isomorphism is the same map $j$).

In next propositions, we are going to use the tables $\overline{H}_1$, $\overline{H}_2$ and $\overline{H}_3$ to obtain all distinct hypergroups of this case.

Theorem 4.1. *Apart of isomorphisms, there exists a unique hypergroup of type $U$ on the right, whose partition associated to the identity is $P_1 = \{\{1\},\{2,3\},\{4,5\}\}$ and whose quotient hypergroup is $\overline{H}_1$. This hypergroup is given by the following hyperproduct table:*

| $\circ$ | 1 | 2,3 | 4,5 |
|---|---|---|---|
| 1 | 1 | $\{2,3\}$ | $\{4,5\}$ |
| 2 | 2 | $\{1,4\}$ | $\{3,5\}$ |
| 3 | 3 | $\{1,5\}$ | $\{2,4\}$ |
| 4 | 4 | $\{2,5\}$ | $\{1,3\}$ |
| 5 | 5 | $\{3,4\}$ | $\{1,2\}$ |

*Proof.* Obviously, in the hypergroups that have $P_1$ as partition and $\overline{H}_1$ as quotient hypergroup, taking into account the Proposition 2.3(1), it results:

- $\overline{2} \otimes \overline{4} = \{\overline{2},\overline{4}\} \Rightarrow (2 \circ b) \cup (3 \circ b) = \{2,3,4,5\}, \forall b \in \{4,5\}$;
- $\overline{4} \otimes \overline{2} = \{\overline{2},\overline{4}\} \Rightarrow (4 \circ a) \cup (5 \circ a) = \{2,3,4,5\}, \forall a \in \{2,3\}$;
- $\overline{2} \otimes \overline{2} = \{\overline{1},\overline{4}\} \Rightarrow (2 \circ a) \cup (3 \circ a) = \{1,4,5\}, \forall a \in \{2,3\}$;
- $\overline{4} \otimes \overline{4} = \{\overline{1},\overline{2}\} \Rightarrow (4 \circ b) \cup (5 \circ b) = \{1,2,3\}, \forall b \in \{4,5\}$.

In consequence, we can prove the following assertions (where the proof is omitted, it means that it can be obtained by analogous arguments):

- $2 \circ 2 = 2 \circ 3 = \{1,4\}$ (in fact, if $2 \circ 2 = 2 \circ 3 = \{1,4,5\}$, then we should have $(2 \circ 2) \circ 4 = H$, while $1 \notin 2 \circ 4 \Rightarrow 2 \notin 2 \circ (2 \circ 4)$, that is an absurdity);

- $3 \circ 2 = 3 \circ 3 = \{1, 5\}$ (as the preceding argument, taking into account that $(2 \circ 2) \cup (3 \circ 2) = \{1, 4, 5\}$);
- $2 \circ 4 = 2 \circ 5 = \{3, 5\}$ (in fact, $(2 \circ 2) \circ 2 = \{2, 3, 5\} \Rightarrow 2 \circ (2 \circ 2) = \{2\} \cup (2 \circ 4) = \{2, 3, 5\} \Rightarrow 2 \circ 4 = \{3, 5\}$);
- $3 \circ 4 = 3 \circ 5 = \{2, 4\}$;
- $5 \circ 2 = 5 \circ 3 = \{3, 4\}$ (in fact, from $(2 \circ 4) \circ 2 = \{3, 5\} \circ 2 = \{1, 5\} \cup (5 \circ 2)$

and $2 \circ (4 \circ 2) \subseteq 2 \circ \{2, 3, 5\}$ it follows that $2 \notin 5 \circ 2$ and so $5 \circ 2 = 5 \circ 3 = \{3, 4\}$;

- $4 \circ 2 = 4 \circ 3 = \{2, 5\}$;
- $4 \circ 4 = 4 \circ 5 = \{1, 3\}$ (in fact $(2 \circ 2) \circ 4 = \{1, 4\} \circ 4 = \{4, 5\} \cup (4 \circ 4)$

while $2 \circ (2 \circ 4) = 2 \circ \{3, 5\} = \{1, 3, 4, 5\}$ and so $4 \circ 4 = \{1, 3\}$);

- $5 \circ 4 = 5 \circ 5 = \{1, 2\}$.

So the table is complete and the hypergroup in the claim is obtained. ■

We remark that the hypergroup found in the preceding theorem is also of type C on the right (i.e. a hypergroup $H$ with a right scalar identity $\varepsilon$ such that for all $x, y, z \in H, xy \cap xz \neq \emptyset \Rightarrow \varepsilon y = \varepsilon z$, [9]), and that there is only another one hypergroup having the same property, as shown in [8], which is the one belonging to the case $C_1$, see [3].

Theorem 4.2. *Apart of isomorphisms, there exist eleven hypergroups of type U on the right, whose partition associated to the identity is* $P_1 = \{\{1\}, \{2, 3\}, \{4, 5\}\}$ *and whose quotient hypergroup is* $\overline{H}_2$.

*Proof.* Also in this case, by using the same preceding tecniques, one can prove that:

- $\{2, 3\} \subseteq b_1 \circ b_2, \forall b_1, b_2 \in \{4, 5\}$ (it follows from $\overline{4} \otimes (\overline{2} \otimes \overline{2}) = (\overline{4} \otimes \overline{2}) \otimes \overline{2}$, noting that: $b \circ 4 = b \circ 5, \forall b \in \{4, 5\}$, $b \circ (2 \circ 2) = \{b\} \cup b \circ b$ and $(b \circ 2) \circ 2 \supseteq \{2, 3\}$);
- $4 \circ 2 \cup 5 \circ 2 = 4 \circ 3 \cup 5 \circ 3 = H$;
- $2 \circ 4 \cup 3 \circ 4 = 2 \circ 5 \cup 3 \circ 5 = H$;
- $4 \circ 4 \cup 5 \circ 4 = 4 \circ 5 \cup 5 \circ 5 = H$;
- $2 \circ 2 \cup 3 \circ 2 = 2 \circ 3 \cup 3 \circ 3 = \{1, 4, 5\}$.

Then, being $\overline{4} \otimes \overline{4} = \overline{H}_2$, we obtain

$$4 \circ 4 = 4 \circ 5 = \{1, 2, 3, 5\} \quad \text{and} \quad 5 \circ 4 = 5 \circ 5 = \{1, 2, 3, 4\}.$$

Moreover, $\overline{2} \otimes \overline{2} = \left\{\overline{1}, \overline{4}\right\} \Rightarrow a_1 \circ a_2 \subseteq \{1, 4, 5\}, \forall a_1, a_2 \in \{2, 3\}$. If we suppose that $2 \circ 2 = \{1, 4\}$, then from $(2 \circ 2) \circ 2 = \{2, 3\} \cup (4 \circ 2)$ and $2 \circ (2 \circ 2) = \{2\} \cup (2 \circ 4)$, it follows that $4 \notin 2 \circ 4$, whence $2 \circ 4 = \{1, 3, 5\}$, that is:

$$2 \circ 2 = \{1, 4\} \Rightarrow 2 \circ 4 = \{1, 3, 5\}.$$

Now, suppose $2 \circ 4 = \{1, 3, 5\}$ and $2 \circ 2 = \{1, 4, 5\}$. Thus obtain $2 \circ (2 \circ 2) = H - \{4\}$ and $(2 \circ 2) \circ 2 = H$, that is an absurdity. So it must be $|2 \circ 2| = 2$. Since it is not restrictive to suppose $2 \circ 2 = \{1, 4\}$, it results:

$$2 \circ 2 = \{1, 4\} \Leftrightarrow 2 \circ 4 = \{1, 3, 5\}.$$

On the other hand, if $|2 \circ 2| = 3$, then we have:

$$2 \circ 2 = \{1,4,5\} \Leftrightarrow 2 \circ 4 = \{1,3,4,5\}\,.$$

Analogously, considering the hyperproduct $3 \circ 3$, one proves that:

$$3 \circ 3 = \{1,5\} \Leftrightarrow 3 \circ 5 = \{1,2,4\}$$

and

$$3 \circ 3 = \{1,4,5\} \Leftrightarrow 3 \circ 5 = \{1,2,4,5\}\,.$$

Note that

$$2 \circ 2 = \{1,4\} \Rightarrow 3 \in 4 \circ 2$$

(indeed $4 \circ (2 \circ 2) = H \Rightarrow (4 \circ 2) \circ 2 = H \Rightarrow 3 \in 4 \circ 2$) and analogously

$$3 \circ 3 = \{1,5\} \Rightarrow 2 \in 5 \circ 2.$$

Finally, up to isomorphisms, one obtains the following hypergroups:

$H_i =$

| $\circ$ | 1 | 2,3 | 4,5 |
|---|---|---|---|
| 1 | 1 | $\{2,3\}$ | $\{4,5\}$ |
| 2 | 2 | $\{1,4\}$ | $\{1,3,5\}$ |
| 3 | 3 | $\{1,5\}$ | $\{1,2,4\}$ |
| 4 | 4 | C | $H-\{4\}$ |
| 5 | 5 | D | $H-\{5\}$ |

$(1 \leq i \leq 3)$

where:

- $H_1$ is obtained for $C = \{1,3,5\}\,, D = \{1,2,4\}\,;$
- $H_2$ for $C = \{1,3,5\}\,, D = H - \{5\}\,;$
- $H_3$ for $C = H - \{4\}\,, D = H - \{5\}$.

$H_i =$

| $\circ$ | 1 | 2,3 | 4,5 |
|---|---|---|---|
| 1 | 1 | $\{2,3\}$ | $\{4,5\}$ |
| 2 | 2 | $\{1,4\}$ | $\{1,3,5\}$ |
| 3 | 3 | $\{1,4,5\}$ | $H-\{3\}$ |
| 4 | 4 | C | $H-\{4\}$ |
| 5 | 5 | D | $H-\{5\}$ |

$(4 \leq i \leq 8)$

where:

- $H_4$ is obtained for $C = \{1,3,5\}\,, D = \{1,2,4\}\,;$
- $H_5$ for $C = \{1,3,5\}\,, D = H - \{5\}\,;$
- $H_6$ for $C = H - \{4\}\,, D = \{1,2,4\}\,;$

- $H_7$ for $C = H - \{4\}, D = H - \{5\}$;
- $H_8$ for $C = H - \{4\}, D = \{1, 3, 4\}$.

$H_i =$

| $\circ$ | 1 | 2,3 | 4,5 |
|---|---|---|---|
| 1 | 1 | $\{2,3\}$ | $\{4,5\}$ |
| 2 | 2 | $\{1,4,5\}$ | $H-\{2\}$ |
| 3 | 3 | $\{1,4,5\}$ | $H-\{3\}$ |
| 4 | 4 | C | $H-\{4\}$ |
| 5 | 5 | D | $H-\{5\}$ |

$(9 \leq i \leq 11)$

where:

- $H_9$ is obtained for $C = \{1, 3, 5\}, D = \{1, 2, 4\}$;
- $H_{10}$ for $C = \{1, 3, 5\}, D = H - \{5\}$;
- $H_{11}$ for $C = H - \{4\}, D = H - \{5\}$. ■

We complete this section by computing all the hypergroups of type $U$ on the right such that the partition $P_1 = \{\{1\}, \{2, 3\}, \{4, 5\}\}$ and the associated quotient is $\overline{H}_3$. In this case, we must have the following properties:

1. $1 \in x \circ y$, $x \circ y \cap \{2, 3\} \neq \emptyset$ and $x \circ y \cap \{4, 5\} \neq \emptyset$, for every $x, y \in H - \{1\}$;
2. $(x \circ a) \cup (x \circ b) = H - \{x\}$, for every $x \in H$, $a \in \{2, 3\}$ and $b \in \{4, 5\}$;
3. $(a \circ z) \cup (b \circ z) = H$, for every $\{a, b\} \in \{\{2, 3\}, \{4, 5\}\}$ and $z \in H - \{1\}$;
4. $(x \circ y) \circ z = x \circ (y \circ z) = H$, for every $x \in H$ and $y, z \in H - \{1\}$.

Now define on the set $H = \{1, 2, 3, 4, 5\}$ the hyperproduct $\star$ such that:

a) 1 is a right scalar identity;

b) $1 \star 2 = 1 \star 3 = \{2, 3\}$ and $1 \star 4 = 1 \star 5 = \{4, 5\}$;

c) $x \star 2 = x \star 3$ and $x \star 4 = x \star 5$, for every $x \in H - \{1\}$;

d) $x \notin x \star y$, for every $x, y \in H - \{1\}$;

e) the above conditions 1), 2), 3) are verified.

It is not difficult to see that the following properties hold:

- the hyperproduct $\star$ is reproducible;
- $x \star (y \star z) = H$, for every $x, y, z \in H - \{1\}$;
- $(x \star y) \star z = x \star (y \star z)$ if $1 \in \{x, y, z\}$;
- $(x \star y) \star z = x \star (y \star z)$ if $x \star y = H - \{x\}$.

The following lemma is essential to establish when the hyperproduct $\star$ is associative; if this is true, then $(H, \star)$ is a hypergroup of type $U$ on the right, whose partition associated to identity is $P_1 = \{\{1\}, \{2, 3\}, \{4, 5\}\}$ and with quotient hypergroup $\overline{H}_3$.

Lemma 4.1. *Let* $\{a_1, a_2\} = \{2, 3\}$, $\{b_1, b_2\} = \{4, 5\}$, $x \star y = \{1, a_1, b_1\}$ *and* $z \in H - \{1\}$ , *then:*

$$(x \star y) \star z = H \Leftrightarrow [b_1 \in a_1 \star a_1 = a_1 \star a_2] \text{ and } [a_1 \in b_1 \star b_1 = b_1 \star b_2].$$

*Proof.* If $z \in \{2, 3\} = \{a_1, a_2\}$, then $(x \star y) \star z = \{1, a_1, b_1\} \star z = \{2, 3\} \cup (a_1 \star z) \cup (b_1 \star z) = H$ gives $b_1 \in a_1 \star a_1 = a_1 \star a_2$. Analogously, if $z \in \{4, 5\} = \{b_1, b_2\}$, we obtain $a_1 \in b_1 \star b_1 = b_1 \star b_2$.

Vice versa, if $b_1 \in a_1 \star a_1 = a_1 \star a_2$ and $a_1 \in b_1 \star b_1 = b_1 \star b_2$ then, for every $z \in \{z_1, z_2\} \in \{\{a_1, a_2\}, \{b_1, b_2\}\}$ we obtain $(x \star y) \star z = \{1, a_1, b_1\} \star z = \{z_1, z_2\} \cup (a_1 \star z) \cup (b_1 \star z) = H$. ■

Now we are able to prove the following:

Theorem 4.3. *Apart of isomorphisms, there exist forty four hypergroups of type* $U$ *on the right, whose partition associated to the identity is* $P_1 = \{\{1\}, \{2, 3\}, \{4, 5\}\}$ *and whose quotient hypergroup is* $\overline{H}_3$.

*Proof.* We know that for all $x, y \in H - \{1\}$ , $3 \leq |x \star y| \leq 4$. If $x \star y = H - \{x\}$ we call the hyperproduct $x \star y$ *full (F.)*; in particular, if $x = y$, we call it *diagonally full (D.F.)*.

It is easy to see that at least two hyperproducts are *D.F.*. In fact otherwise we can suppose $2 \star 2 = \{1, 3, 4\}$ and $3 \star 3 = \{1, 2, 5\}$. By Lemma 4.1, we obtain the contradiction $5 \in 2 \star 2$. Let

$H =$

| $\star$ | 1 | 2,3 | 4,5 |
|---|---|---|---|
| 11 | $\{1\}$ | $\{2,3\}$ | $\{4,5\}$ |
| 2 | $\{2\}$ | $A$ | $X$ |
| 3 | $\{3\}$ | $B$ | $Y$ |
| 4 | $\{4\}$ | $C$ | $Z$ |
| 5 | $\{5\}$ | $D$ | $W$ |

We have the following possible cases:

1. $A, B, Z, W$ are *D.F.*:

| $\star$ | 1 | 2,3 | 4,5 |
|---|---|---|---|
| 1 | $\{1\}$ | $\{2,3\}$ | $\{4,5\}$ |
| 2 | $\{2\}$ | $H-\{2\}$ | $X$ |
| 3 | $\{3\}$ | $H-\{3\}$ | $Y$ |
| 4 | $\{4\}$ | $C$ | $H-\{4\}$ |
| 5 | $\{5\}$ | $D$ | $H-\{5\}$ |

and the hyperproducts $X, Y, C, D$:

(a) are all $F.$; we obtain one hypergroup:

| $\star$ | 1 | 2,3 | 4,5 |
|---|---|---|---|
| 1 | $\{1\}$ | $\{2,3\}$ | $\{4,5\}$ |
| 2 | $\{2\}$ | $H-\{2\}$ | $H-\{2\}$ |
| 3 | $\{3\}$ | $H-\{3\}$ | $H-\{3\}$ |
| 4 | $\{4\}$ | $H-\{4\}$ | $H-\{4\}$ |
| 5 | $\{5\}$ | $H-\{5\}$ | $H-\{5\}$ |

(b) exactly three are $F.$; apart of isomorphisms, we obtain the hypergroup

| $\star$ | 1 | 2,3 | 4,5 |
|---|---|---|---|
| 1 | $\{1\}$ | $\{2,3\}$ | $\{4,5\}$ |
| 2 | $\{2\}$ | $H-\{2\}$ | $\{1,3,4\}$ |
| 3 | $\{3\}$ | $H-\{3\}$ | $H-\{3\}$ |
| 4 | $\{4\}$ | $H-\{4\}$ | $H-\{4\}$ |
| 5 | $\{5\}$ | $H-\{5\}$ | $H-\{5\}$ |

(c) exactly two are $F.$; we obtain four hypergroups, where:

| |
|---|
| $X=\{1,3,4\}\,;Y=\{1,3,5\}\,;C=H-\{4\}\,;D=H-\{5\}$ |
| $X=\{1,3,4\}\,;Y=H-\{3\}\,;C=H-\{4\}\,;D=\{1,3,4\}$ |
| $X=\{1,3,4\}\,;Y=H-\{3\}\,;C=H-\{4\}\,;D=\{1,2,4\}$ |
| $X=\{1,3,5\}\,;Y=H-\{3\}\,;C=H-\{4\}\,;D=\{1,2,4\}$ |

(d) exactly one is $F.$; we obtain two hypergroups:

| |
|---|
| $X=H-\{2\}\,;Y=\{1,2,4\}\,;C=\{1,2,5\}\,;D=\{1,3,4\}$ |
| $X=H-\{2\}\,;Y=\{1,2,4\}\,;C=\{1,3,5\}\,;D=\{1,2,4\}$ |

(e) no hyperproduct is $F.$; we obtain two hypergroups:

| |
|---|
| $X=\{1,3,4\}\,;Y=\{1,2,5\}\,;C=\{1,2,5\}\,;D=\{1,3,4\}$ |
| $X=\{1,3,4\}\,;Y=\{1,2,5\}\,;C=\{1,3,5\}\,;D=\{1,2,4\}$ |

Therefore the case 1. gives rise to 10 hypergroups.

2. $B,Z,W$ are $D.F.$ and $A=2\star 2=2\star 3=\{1,3,4\}$:

| $\star$ | 1 | 2,3 | 4,5 |
|---|---|---|---|
| 1 | $\{1\}$ | $\{2,3\}$ | $\{4,5\}$ |
| 2 | $\{2\}$ | $\{1,3,4\}$ | $X$ |
| 3 | $\{3\}$ | $H-\{3\}$ | $Y$ |
| 4 | $\{4\}$ | $C$ | $H-\{4\}$ |
| 5 | $\{5\}$ | $D$ | $H-\{5\}$ |

In this case, by reproducibility $5 \in X = 2 \star 4 = 2 \star 5$ and, by Lemma 4.1, there are no hyperproducts equal to $\{1,2,5\}$ or else $5 \in 2 \star 2 = A$. Concerning the hyperproducts $X, Y, C, D$ we have the following possibilities:

(a) All are $F$.; we obtain 1 hypergroup:

| $\star$ | 1 | 2,3 | 4,5 |
|---|---|---|---|
| 1 | $\{1\}$ | $\{2,3\}$ | $\{4,5\}$ |
| 2 | $\{2\}$ | $\{1,3,4\}$ | $H-\{2\}$ |
| 3 | $\{3\}$ | $H-\{3\}$ | $H-\{3\}$ |
| 4 | $\{4\}$ | $H-\{4\}$ | $H-\{4\}$ |
| 5 | $\{5\}$ | $H-\{5\}$ | $H-\{5\}$ |

(b) Exactly three are $F$.; we obtain five hypergroups:

| |
|---|
| $X=\{1,3,5\}\,; Y=H-\{3\}\,; C=H-\{4\}\,; D=H-\{5\}$ |
| $X=H-\{2\}\,; Y=\{1,2,4\}\,; C=H-\{4\}\,; D=H-\{5\}$ |
| $X=H-\{2\}\,; Y=H-\{3\}\,; C=\{1,3,5\}\,; D=H-\{5\}$ |
| $X=H-\{2\}\,; Y=H-\{3\}\,; C=H-\{4\}\,; D=\{1,2,4\}$ |
| $X=H-\{2\}\,; Y=H-\{3\}\,; C=H-\{4\}\,; D=\{1,3,4\}$ |

(c) Exactly two are $F$.; we obtain eight hypergroups:

| |
|---|
| $X=\{1,3,5\}\,; Y=\{1,2,4\}\,; C=H-\{4\}\,; D=H-\{5\}$ |
| $X=\{1,3,5\}\,; Y=H-\{3\}\,; C=\{1,3,5\}\,; D=H-\{5\}$ |
| $X=\{1,3,5\}\,; Y=H-\{3\}\,; C=H-\{4\}\,; D=\{1,2,4\}$ |
| $X=\{1,3,5\}\,; Y=H-\{3\}\,; C=H-\{4\}\,; D=\{1,3,4\}$ |
| $X=H-\{2\}\,; Y=\{1,2,4\}\,; C=H-\{4\}\,; D=\{1,3,4\}$ |
| $X=H-\{2\}\,; Y=\{1,2,4\}\,; C=H-\{4\}\,; D=\{1,2,4\}$ |
| $X=H-\{2\}\,; Y=\{1,2,4\}\,; C=\{1,3,5\}\,; D=H-\{5\}$ |
| $X=H-\{2\}\,; Y=H-\{3\}\,; C=\{1,3,5\}\,; D=\{1,2,4\}$ |

(d) Exactly one is $F.$; we obtain five hypergroups:

| |
|---|
| $X=\{1,3,5\}\,;Y=\{1,2,4\}\,;C=\{1,3,5\}\,;D=H-\{5\}$ |
| $X=\{1,3,5\}\,;Y=\{1,2,4\}\,;C=H-\{4\}\,;D=\{1,2,4\}$ |
| $X=\{1,3,5\}\,;Y=\{1,2,4\}\,;C=H-\{4\}\,;D=\{1,3,4\}$ |
| $X=\{1,3,5\}\,;Y=H-\{3\}\,;C=\{1,3,5\}\,;D=\{1,2,4\}$ |
| $X=H-\{2\}\,;Y=\{1,2,4\}\,;C=\{1,3,5\}\,;D=\{1,2,4\}$ |

(e) No hyperproduct is $F.$; we obtain one hypergroup:

| |
|---|
| $X=\{1,3,5\}\,;Y=\{1,2,4\}\,;C=\{1,3,5\}\,;D=\{1,2,4\}$ |

Then the case 2. gives rise in all to twenty hypergroups.

3. Only two hyperproducts are $D.F.$. In this last case, by Lemma 4.1, we can suppose that $B$ and $Z$ are $D.F.$ (if, for example, $Z$ and $W$ were $D.F.$ then $A=2\star 2=\{1,3,4\}$ and $B=3\star 3=\{1,2,5\}$ and so $5\in 2\star 2$, a contradiction) and distinguish two possibilities:

$$A=\{1,3,4\}\,,W=\{1,2,4\} \quad \text{or} \quad A=\{1,3,4\}\,,W=\{1,3,4\}\,.$$

If $A=\{1,3,4\}$ and $W=\{1,2,4\}$ then we have:

| $\star$ | 1 | 2,3 | 4,5 |
|---|---|---|---|
| 1 | $\{1\}$ | $\{2,3\}$ | $\{4,5\}$ |
| 2 | $\{2\}$ | $\{1,3,4\}$ | $X$ |
| 3 | $\{3\}$ | $H-\{3\}$ | $Y$ |
| 4 | $\{4\}$ | $C$ | $H-\{4\}$ |
| 5 | $\{5\}$ | $D$ | $\{1,2,4\}$ |

By Lemma 4.1, there are no hyperproducts equal to $\{1,3,5\}$ or $\{1,2,5\}$ and so, it is easy to see that at least two hyperproducts are $F.$. There are the following possibilities for the other hyperproducts $X,Y,C,D$:

(a) All are $F.$; we obtain one hypergroup:

| $\star$ | 1 | 2,3 | 4,5 |
|---|---|---|---|
| 1 | $\{1\}$ | $\{2,3\}$ | $\{4,5\}$ |
| 2 | $\{2\}$ | $\{1,3,4\}$ | $H-\{2\}$ |
| 3 | $\{3\}$ | $H-\{3\}$ | $H-\{3\}$ |
| 4 | $\{4\}$ | $H-\{4\}$ | $H-\{4\}$ |
| 5 | $\{5\}$ | $H-\{5\}$ | $\{1,2,4\}$ |

(b) Exactly three are $F$.; we obtain two hypergroups:

| |
|---|
| $X = H - \{2\}\,; Y = \{1,2,4\}\,; C = H - \{4\}\,; D = H - \{5\}$ |
| $X = H - \{2\}\,; Y = H - \{3\}\,; C = H - \{4\}\,; D = \{1,3,4\}$ |

(c) Exactly two are $F$.; we obtain one hypergroup:

| |
|---|
| $X = H - \{2\}\,; Y = \{1,2,4\}\,; C = H - \{4\}\,; D = \{1,3,4\}$ |

In case $A = \{1,3,4\}$ and $W = \{1,3,4\}$ we have:

| $\star$ | 1 | 2,3 | 4,5 |
|---|---|---|---|
| 1 | $\{1\}$ | $\{2,3\}$ | $\{4,5\}$ |
| 2 | $\{2\}$ | $\{1,3,4\}$ | $X$ |
| 3 | $\{3\}$ | $H - \{3\}$ | $Y$ |
| 4 | $\{4\}$ | $C$ | $H - \{4\}$ |
| 5 | $\{5\}$ | $D$ | $\{1,3,4\}$ |

In this case, by reproducibility, $5 \in X$ and $2 \in D$. As regards the remaining hyperproducts $X, Y, C, D$, we have the following possibilities:

(a) All are $F$.; we obtain one hypergroup:

| $\star$ | 1 | 2,3 | 4,5 |
|---|---|---|---|
| 1 | $\{1\}$ | $\{2,3\}$ | $\{4,5\}$ |
| 2 | $\{2\}$ | $\{1,3,4\}$ | $X$ |
| 3 | $\{3\}$ | $H - \{3\}$ | $Y$ |
| 4 | $\{4\}$ | $C$ | $H - \{4\}$ |
| 5 | $\{5\}$ | $D$ | $\{1,3,4\}$ |

(b) Exactly three are $F$.; we obtain two hypergroups:

| |
|---|
| $X = \{1,3,5\}\,; Y = H - \{3\}\,; C = H - \{4\}\,; D = H - \{5\}$ |
| $X = H - \{2\}\,; Y = \{1,2,4\}\,; C = H - \{4\}\,; D = H - \{5\}$ |

(c) Exactly two are $F$.; we obtain four hypergroups:

| |
|---|
| $X = \{1,3,5\}\,; Y = \{1,2,4\}\,; C = H - \{4\}\,; D = H - \{5\}$ |
| $X = \{1,3,5\}\,; Y = H - \{3\}\,; C = \{1,3,5\}\,; D = H - \{5\}$ |
| $X = H - \{2\}\,; Y = \{1,2,4\}\,; C = \{1,3,5\}\,; D = H - \{5\}$ |
| $X = \{1,3,5\}\,; Y = H - \{3\}\,; C = H - \{4\}\,; D = \{1,2,4\}$ |

(d) Exactly one is $F$.; we obtain two hypergroups:

| |
|---|
| $X = \{1,3,5\}\,; Y = \{1,2,4\}\,; C = H - \{4\}\,; D = \{1,2,4\}$ |
| $X = \{1,3,5\}\,; Y = \{1,2,4\}\,; C = \{1,3,5\}\,; D = H - \{5\}$ |

(e) No hyperproduct is $F$.; we obtain one hypergroup:

$$\boxed{X = \{1,3,5\}\,; Y = \{1,2,4\}\,; C = \{1,3,5\}\,; D = \{1,2,4\}}$$

So the case 3. gives rise in all to fourteen hypergroups. ∎

Finally, we obtain the following result:

Theorem 4.4. *Apart of isomorphisms, there exist exactly fifty six hypergroups in case $C_5$.*

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