МАТЕМАТИЧКИ ВЕСНИК
61 (2009), 153–158

UDK 512.552
оригинални научни рад
research paper

SOME DECOMPOSITIONS OF SEMIGROUPS

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Abstract. In this paper we will introduce the notion of a-connected elements of a semigroup, a-connected semigroups, and weakly externally commutative semigroup, and we prove that a weakly externally commutative semigroup is a semilattice of a-connected semigroups. Undefined notions can be found in [4].

Let $(S,\cdot)$ be a semigroup and $a\in S$. We define a binary operation (sandwich operation) $\circ$ on the set S by $x\circ y = xay$, where $x,y\in S$. Then S becomes a semigroup with respect to this operation. We denote it by (S,a), and we refer to (S,a) (for any $a\in S$) as a variant of $(S,\cdot)$. Variants of semigroups of binary relations have been studied by Blyth and Hickey [1], Hickey [2,3].

A semigroup S is called Archimedean if, for every couple $a,b\in S$, there exists $n\in Z^+$ such that $a^n\in SbS$.

Let S be a commutative semigroup, $a\in S$, then (S,a) is also a commutative semigroup. By above mentioned (S,a) is a semilattice of Archimedean semigroups, i.e. $S=\bigcup_{\alpha\in Y} S_\alpha$, Y is a semilattice, S_α are Archimedean semigroups for every $\alpha\in Y$. Now, if $x,y\in S_\alpha$, then there exists $n\in Z^+$ such that

$$\underbrace{x\circ x\circ\cdots\circ x}_{n}\in y\circ S \quad\Longleftrightarrow\quad x^n a^{n-1}\in yaS\,.$$

This gives the motivation for the following

Definition 1. Let S be a semigroup and $a\in S$. The elements $x,y\in S$ are a-connected if there exist $n,m\in Z^+$ such that $(xa)^n\in yaS$ and $(ya)^m\in xaS$. The semigroup S is a-connected if x,y are a-connected for all $x,y\in S$.

We remark that if $(xa)^n\in yaS$ and $(ya)^m\in xaS$, then $(xa)^p\in yaS$, $(ya)^p\in xaS$ where $p=\max\{n,m\}$, $m,n,p\in Z^+$.

AMS Subject Classification: 20M10.

Keywords and phrases: Semigroups; semilattice; Archimedean semigroups; externally commutative semigroup.

In [5] S. Lajos introduced the concept of external commutativity to semigroups. A semigroup S is called an externally commutative semigroup if it satisfies the permutation identity $xyz = zyx$. It has been shown in [6] that simple semigroups and cancellative semigroups are all externally commutative semigroups.

In [7] M. Yamada gave the construction of arbitrary externally commutative semigroups.

In [6] we have introduced the concept of weakly external commutativity.

Definition 2. If in a semigroup S there exist an element a so that for all $x, y \in S$

$$xay = yax, \tag{1}$$

holds, then a semigroup S is called a weakly externally commutative semigroup.

Example 1. Let the semigroup S be a given by the table

	1	2	3	4
1	2	1	1	1
2	1	2	2	2
3	1	2	2	2
4	1	2	3	4

Then S is not an externally commutative semigroup since $3 \cdot 4 \cdot 4 = 2 \neq 4 \cdot 4 \cdot 3 = 3$. However S is a weakly externally commutative semigroup because, $x \cdot 1 \cdot y = y \cdot 1 \cdot x$, $x \cdot 2 \cdot y = y \cdot 2 \cdot x$, $x \cdot 3 \cdot y = y \cdot 3 \cdot x$ for all $x, y \in S$.

Clearly, every externally commutative semigroup S is a weakly externally commutative.

Example 2. Let K be a commutative monoid, T semigroup with zero. Let $\varphi\colon T - \{0\} \longrightarrow K$ be an arbitrary homomorphism. Let $S = K \cup T - \{0\}$ and multiplication on S defined by:

$$A \circ B = \begin{cases} AB, & \text{for} \quad AB \neq 0 \quad \text{in } T \\ \varphi(A)\varphi(B), & \text{for} \quad AB = 0 \quad \text{in } T, \end{cases}$$

$A \circ c = \varphi(A)c$, $c \circ A = c\varphi(A)$, $c \circ d = cd$, for each $c, d \in K$.

It is not hard to prove that $(S, \circ)$ is a semigroup. Moreover, if $A, B \in T - \{0\}$, $s \in K$ are arbitrary elements then,

$$\begin{aligned} A \circ s \circ B &= (\varphi(A)s) \circ B = (\varphi(A)s)\varphi(B) = \varphi(A)s\varphi(B) \\ &= \varphi(B)s\varphi(A) = (B \circ s)\varphi(A) = B \circ s \circ A. \end{aligned}$$

Consequently, S is a weakly externally commutative semigroup. It is clear that S will not be commutative or externally commutative if T is not such.

In [7] we have proved the the following result.

Lemma 1. *Let S be a weakly externally commutative semigroup, then the set*

$$B = \{a \in S \mid (\forall x, y \in S)\, xay = yax\}$$

is an ideal in S.

Lemma 2. *Let S be a weakly externally commutative semigroup, $x, y \in S$ and $a \in B$. Then for $k \in Z^+$ we have*

$$(xay)^{2k} = (xa)^{2k-1}y^{2k-1}xay, \quad (xay)^{2k+1} = (xa)^{2k+1}y^{2k+1}. \tag{2}$$

Proof. We prove this lemma by induction. For $k = 1$, since by Lemma 1 $xay \in B$, it follows that

$$(xay)^2 = xayxay, \quad (xay)^3 = xay(xay)xay = xaxa(xay)yy = (xa)^3y^3.$$

Suppose that (2) holds, then

$$\begin{aligned}(xay)^{2k+2} &= (xay)^{2k+1}xay = (xa)^{2k+1}y^{2k+1}xay,\\ (xay)^{2k+3} &= (xay)^{2k+2}xay = (xa)^{2k+1}y^{2k+1}(xay)xay\\ &= (xa)^{2k+1}xa(xay)y^{2k+1}y = (xa)^{2k+3}y^{2k+3}. \quad \blacksquare\end{aligned}$$

Remark 1. From Lemma 2 it follows that

$$(xay)^m \in (xa)^{m-1}S \tag{3}$$

for each $x, y \in S$, $a \in B$ and $m \in Z^+$.

Theorem 1. *Let S be a weakly externally commutative semigroup, $a \in B$ arbitrary fixed element. Then S is a semilattice of a-connected semigroups.*

Proof. We define a relation ρ on S by

$$x\rho y \iff (\exists n \in Z^+)\ (xa)^n \in yaS,\ (ya)^n \in xaS. \tag{4}$$

From $(xa)^2 = xaxa \in xaS$ it follows that ρ is a reflexive relation. Clearly ρ is a symmetric relation. Let $x, y \in S$ be elements such that $x\rho y$ and $y\rho z$. Then

$$(\exists n \in Z^+)\ (xa)^n \in yaS,\ (ya)^n \in xaS,$$

and

$$(\exists m \in Z^+)\ (ya)^m \in zaS,\ (za)^m \in yaS,$$

There exist $t, s \in S$ such that $(xa)^n = yat,\ (za)^m = yas$. Now by (3) we have

$$\begin{aligned}(xa)^{(n+1)(m+1)} &= (xa)^{n(m+1)}(xa)^{m+1} = (yat)^{m+1}(xa)^{m+1}\\ &\in (ya)^mS(xa)^{m+1} \subseteq zaS,\\ (za)^{(n+1)(m+1)} &= (za)^{n(m+1)}(za)^{m+1} = (yas)^{n+1}(za)^{n+1}\\ &\in (ya)^nS(za)^{n+1} \subseteq xaS,\end{aligned}$$

whence $x\rho z$ so ρ is a transitive relation.

Hence, ρ is an equivalence relation.

Clearly, from $x\rho y$ we have that $(xa)^{2n+1} \in yaS$. Let $z \in S$ be an arbitrary element. Since $a \in B$ and B is an ideal, by Lemma 2 we obtain

$$\begin{aligned}(xza)^{2n+2} &= x(zax)^{2n+1}za = x(xaz)^{2n+1}za = x(xa)^{2n+1}z^{2n+1}za \\ &= x(xa)^{2n+1}zz^{2n+1}a \in xyaSzz^{2n+1}a \\ &= xzaSyz^{2n+1}a = yzaSxz^{2n+1}a \subseteq yzaS\,.\end{aligned}$$

Analogously, $(yza)^{2n+2} \in xzaS$, so $xz\rho yz$. Hence ρ is a right congruence on S.

Similarly,

$$(zxa)^{2n+2} = z(xaz)^{2n+1}xa = z(xa)^{2n+1}z^{2n+1}xa \in zyaSz^{2n+1}xa \subseteq zyaS\,,$$

and analogously $(zya)^{2n+2} \in zxaS$. Hence, $zx\rho zy$ and the equivalence relation ρ is a left congruence on S.

By what has been said above it follows that ρ is a congruence on S.

Let $x \in S$. Then, since $ax^2, xa \in B$, we obtain

$$(x^2a)^3 = xx(ax^2)ax^2a = xa(ax^2)xx^2a \in xaS\,,$$

and

$$(xa)^3 = xa(xa)xa = xxa(xa)a \in x^2aS\,.$$

Thus $x\rho x^2$. Hence ρ is a band congruence on S.

Let $x, y \in S$, then

$$(xya)^2 = x(yax)ya = y(yax)xa = yy(ax)xa = yxa(ax)y \in yxaS\,.$$

Analogously, $(yxa)^2 \in xyaS$. Consequently $xy\rho yx$. So ρ is a semilattice congruence on S, whence S is a semilattice of a-connected semigroups. ■

Corollary 1. *Any externally commutative semigroup S is a semilattice of a-connected semigroups for every $a \in S$.*

Proof. Any element $a \in S$ satisfies $xay = yax$ for all $x, y \in S$. ■

A semigroup S is called a medial semigroup if it is satisfies the permutation identity $xyzt = xzyt$.

Example 3. Let a semigroup S be given by the table

	1	2	3
1	2	2	2
2	2	2	2
3	3	3	3 .

The semigroup S given by the above table is a medial semigroup and S is not externally commutative semigroup since $2 \cdot 1 \cdot 3 = 2 \neq 3 = 3 \cdot 1 \cdot 2$.

Theorem 2. *A medial semigroup S is a band of a-connected semigroups for each $a \in S$.*

Proof. On the medial semigroup S, for arbitrary fixed $a \in S$ we define the relation ρ, given by (4). By Theorem 1 we see that ρ is an equivalence relation.

Let $x, y \in S$ be elements such that $x\rho y$. Then

$$(\exists n \in Z^+)\ (xa)^n \in yaS,\ (ya)^n \in xaS.$$

If $z \in S$ is an arbitrary element, then by mediality

$$(xza)^{n+1} = (xa)^n z^{n+1} xa \in yaSzz^n xa = yzaSz^n xa \subseteq yzaS\,.$$

Dually, $(yza)^{n+1} \in xzaS$. Similarly, $(zxa)^{n+1} \in zyaS, (zya)^{n+1} \in zxaS$. Hence ρ is a congruence relation on S.

Let $x \in S$ be an arbitrary element, then

$$(x^2a)^2 = x^2ax^2a = xax^3a \in xaS,$$
$$(xa)^2 = xaxa = x^2aa \in x^2aS,$$

whence ρ is a band congruence on S. By the definition of ρ, each ρ-class is a-connected. Thus S is a band of a-connected semigroups. ■

Definition 3. Let S be a semigroup and $a \in S$, elements $x, y \in S$ are simply a-connected if

$$xa \in yaS, \quad ya \in xaS.$$

Let S be a semigroup and $a \in S$. The semigroup S is said to be simply a-connected if every two elements are simply a-connected.

For example, a group G is simply a-connected, for every $a \in G$.

Example 4. The semigroup S given by the table

	1	2	3
1	2	2	1
2	2	2	2
3	2	2	3

,

is simply 1-connected and it is not a group. Moreover, S is trivially simply 2-connected since 2 is a zero on S and S is not simply 3-connected because $1 \cdot 3 \cdot 3 \neq 3 \cdot 3 \cdot 1$.

The semigroup S in Example 2 is not simply a-connected since, for example, $1 \cdot 1 \notin 3 \cdot 1 \cdot S$, $1 \cdot 2 \notin 3 \cdot 2 \cdot S$, $2 \cdot 3 \notin 3 \cdot 3 \cdot S$.

Remark 2. Let S be an arbitrary semigroup and $a \in S$, then the relation η defined on S by

$$x\eta y \Longleftrightarrow xa \in yaS^1,\ ya \in xaS^1$$

is, clearly, a left congruence relation on S. If S is commutative semigroup, then η is a semilattice congruence and so a commutative semigroup is a semilattice of simply a-connected semigroups, for every $a \in S$.

Example 5. Let the semigroup S be given by the table

	2	3	4	5	6
2	3	2	2	2	3
3	2	3	3	3	2
4	2	3	3	3	2
5	2	3	3	5	6
6	3	2	2	6	5 .

Since S is a commutative semigroup, it is a-connected for every $a \in S$. If $a = 2$, then $\eta = S \times S$. If $a = 5$, then η-classes are $S_\alpha = \{2,3,4\}$, $S_\beta = \{5,6\}$ and $Y = \{\alpha, \beta\}$ is a semilattice.

Acknowledgement. We would like to thank to Professor Boris Novikov for suggestions.

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(received 24.03.2008, in revised form 20.11.2008)

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