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ALMOST MENGER AND RELATED SPACES

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Abstract. In this paper we consider the notion of almost Menger property which is similar to the familiar property of Menger and prove that we can use regular open sets instead of open sets in the definition of almost Menger property. We give conditions for a space X^n to be almost Menger. In the similar way, we consider almost γ-sets and the almost star-Menger property.

1. Introduction and definitions

In this paper we consider almost Menger and related spaces similar to the well-known property of Menger. For this we use the closures of open sets. The idea is not completely new. Tkachuk in [11] and Scheepers in [9] and implicitly in [10] considered a property similar to the classical notion of Rothberger [8] using the closures of open sets. In [5], Kočinac introduced the notion of almost Menger property. In [6], Di Maio and Kočinac considered the almost Menger property in hyperspaces. We will show that the almost Menger property is different from the Menger property. In Section 2 we also show that we can replace open sets with regular open sets in the definition of almost Menger spaces. In Section 3 we have the characterization of almost Menger property in all finite powers. In Sections 4 and 5 we consider almost γ-sets and almost star-Menger spaces in the similar manner as almost Menger spaces in Section 2.

We will assume that all topological spaces in this paper are Hausdorff.

We first recall the classical notions of Menger [7] and Gerlits-Nagy [3] spaces.

Definition 1.1. A topological space $\langle X, T\rangle$ is Menger if for each sequence $(\mathcal{U}_n : n \in \mathbf{N})$ of open covers of X there exists a sequence $(\mathcal{V}_n : n \in \mathbf{N})$ such that for every $n \in \mathbf{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbf{N}} \mathcal{V}_n$ is a cover of X.

Let us recall that an open cover $\mathcal{U}$ of X is an ω-cover if for each finite subset F of X there exists $U \in \mathcal{U}$ such that $F \subset U$ and X is not a member of $\mathcal{U}$.

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An open cover $\mathcal{U}$ of X is a γ-cover if it is infinite and for every $x \in X$ the set $\{U \in \mathcal{U} : x \notin U\}$ is finite.

We shall use the symbols $\mathcal{O}$, Ω and Γ to denote the collections of all open-, ω- and γ-covers, respectively, of X.

Gerlits and Nagy [3] defined the notion of γ-sets in the following way.

Definition 1.2. A topological space $\langle X, T\rangle$ is a γ-set if for each sequence $(\mathcal{U}_n : n \in \mathbf{N})$ of ω-covers there exists a sequence $(V_n : n \in \mathbf{N})$ such that for every $n \in \mathbf{N}$, $V_n \in \mathcal{U}_n$ and $\{V_n : n \in \mathbf{N}\}$ is a γ-cover of X.

In [4], Kočinac used the operator St to introduce the following notion (for star selection principles see also [1]). Let us recall that for a subset A of a space X and a collection $\mathcal{P}$ of subsets of X $St(A, \mathcal{P})$ denotes the set $\bigcup\{P \in \mathcal{P} : A \cap P \neq \emptyset\}$.

We will use the following definition.

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of an infinite set X.

Definition 1.3. The symbol $S^*_{fin}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbf{N})$ of elements of $\mathcal{A}$ there exists a sequence $(\mathcal{V}_n : n \in \mathbf{N})$ such that for every $n \in \mathbf{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbf{N}}\{St(V, \mathcal{U}_n) : V \in \mathcal{V}_n\} \in \mathcal{B}$.

In this notation we have the following definition [4].

Definition 1.4. A space X is said to have the star-Menger property if it satisfies selection hypothesis $S^*_{fin}(\mathcal{O}, \mathcal{O})$.

2. Properties of almost Menger spaces

The following notion was introduced in [5].

Definition 2.1. A topological space $\langle X, T\rangle$ is almost Menger if for each sequence $(\mathcal{U}_n : n \in \mathbf{N})$ of open covers of X there exists a sequence $(\mathcal{V}_n : n \in \mathbf{N})$ such that for every $n \in \mathbf{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup\{\mathcal{V}_n^{'} : n \in \mathbf{N}\}$ is a cover of X, where $\mathcal{V}_n^{'} = \{\overline{V} : V \in \mathcal{V}_n\}$.

From the definition it is obvious that every Menger space is an almost Menger space. The following example shows that the inverse does not hold.

Example 1. Let X be the Euclidean plane with the deleted radius topology (see [11, Example 77]). Since X is not Lindelöf, X does not have a Menger property. In order to prove that X is almost Menger, we will use the fact that the closure of every open set in the deleted radius topology is the same as in the usual Euclidean topology and that the Euclidean plane with the Euclidean topology is σ-compact and therefore has the Menger property (hence, the almost Menger property, too).

We will see that almost Menger spaces are equivalent with Menger spaces in the class of regular spaces.

Theorem 2.1. *Let X be a regular space. If X is an almost Menger space, then X is a Menger space.*

Proof. Let $(\mathcal{U}_n : n \in \mathbf{N})$ be a sequence of open covers of X. Since X is a regular space, there exists for each n an open cover $\mathcal{V}_n$ of X such that $\mathcal{V}_n^{'} = \{\overline{V} : V \in \mathcal{V}_n\}$ is a refinement of $\mathcal{U}_n$. By assumption, there exists a sequence $(\mathcal{W}_n : n \in \mathbf{N})$ such that for each n, $\mathcal{W}_n$ is a finite subset of $\mathcal{V}_n$ and $\bigcup\{\mathcal{W}_n^{'} : n \in \mathbf{N}\}$ is a cover of X, where $\mathcal{W}_n^{'} = \{\overline{W} : W \in \mathcal{W}_n\}$. For every $n \in \mathbf{N}$ and every $W \in \mathcal{W}_n$ we can choose $U_W \in \mathcal{U}_n$ such that $\overline{W} \subset U_W$. Let $\mathcal{U}_n^{'} = \{U_W : W \in \mathcal{W}_n\}$.

We shall prove that $\bigcup\{\mathcal{U}_n^{'} : n \in \mathbf{N}\}$ is an open cover of X. Let $x \in X$. There exists $n \in \mathbf{N}$ and $\overline{W} \in \mathcal{W}_n^{'}$ such that $x \in \overline{W}$. By construction, there exists $U_W \in \mathcal{U}_n^{'}$ such that $\overline{W} \subset U_W$. Therefore, $x \in U_W$. ■

Kočinac proved in [4] that in the class of paracompact spaces the notions of Menger and star-Menger spaces are equivalent. So, we have the following corollary.

Corollary 2.1. *For a paracompact space X the following are equivalent:*

(a) X is a Menger space;

(b) X is a star-Menger space;

(c) X is an almost Menger space.

The next theorem shows that we can replace open sets with regular open sets in the description of almost Menger space.

A subset B of a topological space X is called regular open (regular closed) if $B = int(\overline{B})(B = \overline{int(B)})$.

Theorem 2.2. *A topological space X is almost Menger if and only if for each sequence $(\mathcal{U}_n : n \in \mathbf{N})$ of covers of X by regular open sets, there exists a sequence $(\mathcal{V}_n : n \in \mathbf{N})$ such that for every $n \in \mathbf{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup\{\mathcal{V}_n^{'} : n \in \mathbf{N}\}$ is a cover of X, where $\mathcal{V}_n^{'} = \{\overline{V} : V \in \mathcal{V}_n\}$.*

Proof. $(\Rightarrow)$: Let $(\mathcal{U}_n : n \in \mathbf{N})$ be a sequence of covers of X by regular open sets. Since every regular open set is open, $(\mathcal{U}_n : n \in \mathbf{N})$ is a sequence of open covers.

By assumption, there exists a sequence $(\mathcal{V}_n : n \in \mathbf{N})$ such that for every $n \in \mathbf{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup\{\mathcal{V}_n^{'} : n \in \mathbf{N}\}$ is a cover of X, where $\mathcal{V}_n^{'} = \{\overline{V} : V \in \mathcal{V}_n\}$.

$(\Leftarrow)$: Let $(\mathcal{U}_n : n \in \mathbf{N})$ be a sequence of open covers of X. Let $(\mathcal{U}_n^{'} : n \in \mathbf{N})$ be a sequence defined by $\mathcal{U}_n^{'} = \{int(\overline{U}) : U \in \mathcal{U}_n\}$. Then each $\mathcal{U}_n^{'}$ is a cover of X by regular open sets. Indeed, each $int(\overline{U})$ is a regular open set (see [2]), and $U \subset int(\overline{U})$ since U is an open set.

Then there exists a sequence $(\mathcal{V}_n : n \in \mathbf{N})$ such that for every $n \in \mathbf{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n^{'}$ and $\bigcup\{\mathcal{V}_n^{'} : n \in \mathbf{N}\}$ is a cover of X, where $\mathcal{V}_n^{'} = \{\overline{V} : V \in \mathcal{V}_n\}$.

By construction, for each $n \in \mathbf{N}$ and $V \in \mathcal{V}_n$ there exists $U_V \in \mathcal{U}_n$ such that $\overline{V} = \overline{int(\overline{U_V})}$.

Since U_V is an open set, $\overline{U_V}$ is a regular closed set (see [2]). Therefore $\overline{U_V} = \overline{int(\overline{U_V})}$. Hence, $\bigcup_{n\in\mathbf{N}}\{U_V : V \in \mathcal{V}_n\} = X$, so X is an almost Menger space. ∎

Definition 2.2. Let X and Y be topological spaces. A function $f : X \to Y$ is almost continuous if for each regular open set $B \subset Y$, $f^{-1}(B)$ is an open set in X.

Theorem 2.3. *Let X be an almost Menger space, and Y be a topological space. If $f : X \to Y$ is an almost continuous surjection, then Y is an almost Menger space.*

Proof. Let $(\mathcal{U}_n : n \in \mathbf{N})$ be a sequence of covers of Y by regular open sets. Let $\mathcal{U}_n' = \{f^{-1}(U) : U \in \mathcal{U}_n\}$ for each $n \in \mathbf{N}$. Then $(\mathcal{U}_n' : n \in \mathbf{N})$ is a sequence of open covers of X, since f is an almost continuous surjection. Since X is an almost Menger space, there exists a sequence $(\mathcal{V}_n : n \in \mathbf{N})$ such that for every $n \in \mathbf{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n'$ and $\bigcup\{\mathcal{V}_n' : n \in \mathbf{N}\}$ is a cover of X, where $\mathcal{V}_n' = \{\overline{V} : V \in \mathcal{V}_n\}$. For each $n \in \mathbf{N}$ and $V \in \mathcal{V}_n$ we can choose $U_V \in \mathcal{U}_n$ such that $V = f^{-1}(U_V)$. Let $\mathcal{W}_n = \{\overline{U_V} : V \in \mathcal{V}_n\}$. We will prove that $\bigcup\{\mathcal{W}_n : n \in \mathbf{N}\}$ is a cover for X.

If $y = f(x) \in Y$, then there exists $n \in \mathbf{N}$ and $V \in \mathcal{V}_n$ such that $x \in \overline{V}$. Since $V = f^{-1}(U_V)$, $x \in \overline{f^{-1}(U_V)} \subset f^{-1}(\overline{U_V})$. Hence, $y = f(x) \in \overline{U_V} \in \mathcal{W}_n$. ∎

3. Characterization of almost Menger spaces in all finite powers

Theorem 3.1. *X^n is an almost Menger space for each $n \in \mathbf{N}$ if and only if the topological space X satisfies selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbf{N})$ of ω-covers of X there exists a sequence $(\mathcal{V}_n : n \in \mathbf{N})$ such that $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for every $F \subset X$ there exists $n \in \mathbf{N}$ and $V \in \mathcal{V}_n$ such that $F \subset \overline{V}$.*

Proof. $(\Rightarrow)$: Let $k \in \mathbf{N}$ be fixed and let $(\mathcal{U}_n : n \in \mathbf{N})$ be a sequence of open covers of X^k, where each $\mathcal{U}_n = \{U_{nj} : j \in J_n\}$, where J_n is an infinite countable index set.

Let $F \subset X$ be a finite set. Then F^k is a finite subset of X^k, so F^k is a compact set. Since $\mathcal{U}_n$ is an open cover of X^k there exists finite subset J_n^F of J_n such that $F^k \subset \bigcup_{j\in J_n^F} U_{nj}$. By the Wallace theorem (see 3.2.10. in [2]), there exists an open set V_F in X such that $F \subset V_F$ and $V_F^k \subset \bigcup_{j\in J_n^F} U_{nj}$.

Let $\mathcal{V}_n = \{V_F : F \subset X \text{ finite}\}$, then for each finite subset F of X there exists $V_F \in \mathcal{V}_n$ such that $F \subset V_F$. By assumption, there exists a sequence $(\mathcal{W}_n : n \in \mathbf{N})$ such that for each $n \in \mathbf{N}$, $\mathcal{W}_n$ is a finite subset of $\mathcal{V}_n$ and $\{\mathcal{W}_n' : n \in \mathbf{N}\}$ is an ω-cover of X, where $\mathcal{W}_n' = \{\overline{W} : W \in \mathcal{W}_n\}$.

Let for each $n \in \mathbf{N}$ $\mathcal{W}_n$ has K_n elements.

Let $\mathcal{H}_n = \{\overline{U_{nj}} : j \in J_n^{F_i}, i \in K_n\}$, then the sequence $(\mathcal{H}_n : n \in \mathbf{N})$ is such that:

(i): For each $n \in \mathbf{N}$ if we denote $I_n = \{j \in J_n : j \in J_n^{F_i}, i \in K_n\}$, then I_n is a finite subset of J_n and $\mathcal{H}_n = \{\overline{U_{nj}} : j \in I_n\}$.

(ii): Let $x = (x_1, \dots, x_k) \in X^k$. Then $F = \{x_1, \dots, x_k\}$ is a finite subset of X, so there is $n \in \mathbf{N}$ and $W \in \mathcal{W}_n$ such that $F \subset \overline{W}$.

Let $W = V_{F_i}$ for some $i \in K_n$. Then:

$F^k \subset \overline{V_{F_i}}^k \subset \overline{V_{F_i}^k} \subset \overline{\bigcup_{j \in J_n^{F_i}} U_{nj}} \subset \bigcup_{j \in J_n^{F_i}} \overline{U_{nj}}$.

Then there exists $j \in J_n^{F_i}$ such that $F^k \subset \overline{U_{nj}}$, hence $x \in \overline{U_{nj}}$ for $\overline{U_{nj}} \in \mathcal{H}_n$. Therefore X^k is an almost Menger space.

($\Leftarrow$): Let $(\mathcal{U}_n : n \in \mathbf{N})$ be a sequence of ω-covers of X where each $\mathcal{U}_n = \{U_{nk} : k \in K_n\}$, where K_n is an infinite countable index set.

Let $\mathbf{N} = N_1 \cup N_2 \cup \dots \cup N_n \cup \dots$ be a partition of $\mathbf{N}$ into countably many pairwise disjoint infinite subsets. For every $i \in \mathbf{N}$ and every $j \in N_i$ let $\mathcal{V}_j = \{U^i : U \in \mathcal{U}_j\}$. Obviously, the sequence $\{\mathcal{V}_j : j \in N_i\}$ is a sequence of open covers of X^i. Since X^i is an almost Menger space, for every $i \in \mathbf{N}$ one can choose a sequence $(\mathcal{W}_j : j \in N_i)$ so that for each j there exists finite subset $I_j \subset K_j$ such that:

(1) $\mathcal{W}_j = \{\overline{U_{jk}^i} : k \in I_j\}$;

(2) $\{\mathcal{W}_j : j \in N_i\}$ is a cover of X^i.

We shall prove that $\{\overline{U_{jk}} : k \in I_j, j \in \mathbf{N}\}$ is an ω-cover for X. Indeed, let $F = \{x_1, x_2, \dots, x_p\}$ be a finite subset of X. Then $(x_1, x_2, \dots, x_p) \in X^p$, so there is some $l \in N_p$ such that $(x_1, x_2, \dots, x_p) \in \mathcal{W}_l$. So, we can find $k \in I_l$ such that $(x_1, x_2, \dots, x_p) \in \overline{U_{lk}^p} = \overline{U_{lk}}^p$. It is clear that $F \subset \overline{U_{lk}}$. ■

4. Almost γ-sets

We say that a cover $\mathcal{U}$ of X is an almost γ-cover if it is infinite and for every $x \in X$, $\{U \in \mathcal{U} : x \notin \overline{U}\}$ is finite.

Definition 4.1. A topological space X is an almost γ-set if for each sequence $(\mathcal{U}_n : n \in \mathbf{N})$ of ω-covers of X there exists a sequence $(V_n : n \in \mathbf{N})$ such that for every $n \in \mathbf{N}$, $V_n \in \mathcal{U}_n$ and $\{V_n : n \in \mathbf{N}\}$ is an almost γ-cover for X.

Theorem 4.1. *A topological space X is almost γ-set if and only if for each sequence $(\mathcal{U}_n : n \in \mathbf{N})$ of ω-covers of X by regular open sets, there exists a sequence $(V_n : n \in \mathbf{N})$ such that for every $n \in \mathbf{N}$, $V_n \in \mathcal{U}_n$ and $\{V_n : n \in \mathbf{N}\}$ is an almost γ-cover of X.*

Proof. ($\Rightarrow$): This is evident.

($\Leftarrow$): Let $(\mathcal{U}_n : n \in \mathbf{N})$ be a sequence of ω-covers of X . Let $(\mathcal{U}_n' : n \in \mathbf{N})$ be a sequence defined by $\mathcal{U}_n' = \{int(\overline{U}) : U \in \mathcal{U}_n\}$. Then each $\mathcal{U}_n'$ is an ω-cover of X by regular open sets. Indeed, each $int(\overline{U})$ is a regular open set and $U \subseteq int(\overline{U})$ since U is an open set.

Then there exists a sequence $(V_n : n \in \mathbf{N})$ such that for every $n \in \mathbf{N}$, $V_n \in \mathcal{U}_n'$ and $\{V_n : n \in \mathbf{N}\}$ is an almost γ-cover for X.

By the same argument as in the proof of Theorem 2.1, $\overline{U} = \overline{int(\overline{U})}$, so $\overline{V_n} = \overline{U_n}$ for some $U_n \in \mathcal{U}_n$ and X is an almost γ-set. ■

THEOREM 4.2. *Let X be an almost γ-set and let Y be a topological space. If $f : X \to Y$ is an almost continuous surjection, then Y is an almost γ-set.*

Proof. Let $(\mathcal{U}_n : n \in \mathbf{N})$ be a sequence of ω-covers of Y by regular open sets. Let $\mathcal{U}_n^{'} = \{f^{-1}(U) : U \in \mathcal{U}_n\}$, then each $\mathcal{U}_n^{'}$ is an ω-cover of X. Indeed, if F is a finite subset of X then $f(F)$ is a finite subset of Y. So, by assumption, there exists $U \in \mathcal{U}_n$ such that $f(F) \subset U$. Then $F \subset f^{-1}(U)$. Since f is almost continuous, $f^{-1}(U)$ is an open set for every $U \in \mathcal{U}_n$, so $\mathcal{U}_n^{'}$ is really an ω-cover of X.

Since X is an almost γ-set there is a sequence $(V_n^{'} : n \in \mathbf{N})$ such that: for every $n \in \mathbf{N}$ there exists $U_n \in \mathcal{U}_n$ such that $V_n^{'} = f^{-1}(U_n)$ and $\{V_n^{'} : n \in \mathbf{N}\}$ is an almost γ-cover for X.

For each $n \in \mathbf{N}$, let $V_n = U_n$ such that $f^{-1}(U_n) = V_n^{'}$. If $y = f(x) \in Y$ then there exists $n_0 \in \mathbf{N}$ such that:

$\forall n \in \mathbf{N}, n > n_0 \Rightarrow x \in \overline{V_n'}$.

Since $\overline{V_n'} = \overline{f^{-1}(V_n)} \subseteq f^{-1}(\overline{V_n})$, we have that $\forall n > n_0$ $y \in \overline{V_n}$. Hence, Y is an almost γ-set. ■

5. Almost star-Menger spaces

DEFINITION 5.1. A topological space X is an almost star-Menger space if for each sequence $(\mathcal{U}_n : n \in \mathbf{N})$ of open covers of X there exists a sequence $(\mathcal{V}_n : n \in \mathbf{N})$ such that for each $n \in \mathbf{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\{\overline{St(\bigcup \mathcal{V}_n, \mathcal{U}_n)} : n \in \mathbf{N}\}$ is a cover of X.

THEOREM 5.1. *A topological space X is almost star-Menger if and only if for each sequence $(\mathcal{U}_n : n \in \mathbf{N})$ of covers of X by regular open sets there exists a sequence $(\mathcal{V}_n : n \in \mathbf{N})$ such that for each $n \in \mathbf{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\{\overline{St(\bigcup \mathcal{V}_n, \mathcal{U}_n)} : n \in \mathbf{N}\}$ is a cover of X.*

Proof. $(\Rightarrow)$: Since every regular open set is open, it is obvious.

$(\Leftarrow)$: Let $(\mathcal{U}_n : n \in \mathbf{N})$ be a sequence of open covers of X. Let $\mathcal{U}_n^{'} = \{int(\overline{U}) : U \in \mathcal{U}_n\}$. Then each $\mathcal{U}_n^{'}$ is a cover of X by regular open sets. Indeed, each $int(\overline{U})$ is a regular open set (see [2]) and $U \subseteq int(\overline{U})$ since U is an open set.

Then there exists a sequence $(\mathcal{V}_n : n \in \mathbf{N})$ such that for every $n \in \mathbf{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n^{'}$ and $\{\overline{St(\bigcup \mathcal{V}_n, \mathcal{U}_n')} : n \in \mathbf{N}\}$ is a cover of X.

Claim 1. $St(U, \mathcal{U}_n) = St(int(\overline{U}), \mathcal{U}_n)$ for each $U \in \mathcal{U}_n$.

Proof of Claim 1. Since $U \subset int(\overline{U})$, it is obvious that $St(U, \mathcal{U}_n) \subset St(int(\overline{U}), \mathcal{U}_n)$.

Let $x \in St(int(\overline{U}), \mathcal{U}_n)$. Then there exists $V \in \mathcal{U}_n$ such that $x \in V$ and $V \cap int(\overline{U}) \neq \emptyset$. Then we have $V \cap U \neq \emptyset$ which implies $x \in St(U, \mathcal{U}_n)$.

For every $V \in \mathcal{V}_n$ we can choose $U_V \in \mathcal{U}_n$ such that $V = int(\overline{U_V})$. Let $\mathcal{W}_n = \{U_V : V \in \mathcal{V}_n\}$. We shall prove that $\{\overline{St(\bigcup \mathcal{W}_n, \mathcal{U}_n)} : n \in \mathbf{N}\}$ is a cover of X.

Let $x \in X$. Then there exists $n \in \mathbf{N}$ such that $x \in \overline{St(\bigcup \mathcal{V}_n, \mathcal{U}_n')}$. For every neighborhood V of x, we have $V \cap St(\bigcup \mathcal{V}_n, \mathcal{U}_n') \neq \emptyset$. Then there exists $U \in \mathcal{U}_n$ such that $(V \cap int(\overline{U}) \neq \emptyset) \wedge (\bigcup \mathcal{V}_n \cap int(\overline{U}) \neq \emptyset) \Rightarrow (V \cap U \neq \emptyset) \wedge (\bigcup \mathcal{V}_n \cap U \neq \emptyset)$. By Claim 1 we have that $\bigcup \mathcal{W}_n \cap U \neq \emptyset$, so $x \in \overline{St(\bigcup \mathcal{W}_n, \mathcal{U}_n)}$. ∎

THEOREM 5.2. *Let X be an almost star-Menger topological space and let Y be a topological space. If $f : X \to Y$ is an almost continuous surjection, then Y is an almost star-Menger space.*

Proof. Let $(\mathcal{U}_n : n \in \mathbf{N})$ be a sequence of covers of Y by regular open sets. Let $\mathcal{U}_n' = \{f^{-1}(U) : U \in \mathcal{U}_n\}$. Then each $\mathcal{U}_n'$ is an open cover of X since f is almost continuous and each $U \in \mathcal{U}_n$ is a regular open set. Since X is almost star-Menger, there is a sequence $(\mathcal{V}_n' : n \in \mathbf{N})$ such that for every $n \in \mathbf{N}$, $\mathcal{V}_n'$ is a finite subset of $\mathcal{U}_n'$ and $\{\overline{St(\bigcup \mathcal{V}_n', \mathcal{U}_n')} : n \in \mathbf{N}\}$ is a cover for X.

Let $\mathcal{V}_n = \{U : f^{-1}(U) \in \mathcal{V}_n'\}$ and $x \in X$. Then $f^{-1}(\bigcup \mathcal{V}_n) = \bigcup \mathcal{V}_n'$ and there is $n \in \mathbf{N}$ such that $x \in \overline{St(f^{-1}(\bigcup \mathcal{V}_n), \mathcal{U}_n')}$. If $y = f(x) \in Y$, then $y \in f(\overline{St(f^{-1}(\bigcup \mathcal{V}_n), \mathcal{U}_n')}) \subseteq \overline{f(St(f^{-1}(\bigcup \mathcal{V}_n), \mathcal{U}_n'))} \subseteq \overline{St(\bigcup \mathcal{V}_n, \mathcal{U}_n)}$. We will prove the last inclusion:

Suppose that $f^{-1}(\bigcup \mathcal{V}_n) \cap f^{-1}(U) \neq \emptyset$. Then also $f(f^{-1}(\bigcup \mathcal{V}_n)) \cap f(f^{-1}(U)) \neq \emptyset$, so $\bigcup \mathcal{V}_n \cap U \neq \emptyset$.

So, the sequence $(\mathcal{V}_n : n \in \mathbf{N})$ witnesses that X is an almost star-Menger space. ∎

6. Closing remarks

In this paper we did not consider almost Rothberger spaces, but all properties from Section 2 concerning the almost Menger property can be investigated for the almost Rothberger property applying quite similar techniques for their proofs.

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