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SOME RESULTS IN FIXED POINT THEORY CONCERNING GENERALIZED METRIC SPACES

Ali Fora, Azzeddine Bellour and Adnan Al-Bsoul

Abstract. In this paper we shall study the fixed point theory in generalized metric spaces (gms). One of our results will be a generalization of Kannan's fixed point theorem in the ordinary metric spaces, and Das's fixed point theorem in gms.

1. Introduction

In 2000 A. Branciari [1] introduced the concept of a generalized metric space on the lines of ordinary metric space, where the triangle inequality of a metric space has been replaced by an inequality involving three terms instead of two (tetrahedral inequality). It is easy to see that, every metric space is a generalized metric space (gms), but the converse need not be true. In this paper, we shall study the fixed point theory in generalized metric spaces. Our results will be generalized Kannan's fixed point theorem (Theorem 1) and Das's Theorem [2].

Throughout this paper, $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{R}_+$ denotes the set of all nonnegative real numbers, $\mathbb{N}$ denotes the set of all natural numbers, and X denotes a nonempty set.

Definition 1. Let X be a nonempty set and let $d : X \times X \longrightarrow \mathbb{R}_+$ be a mapping such that for all $x, y \in X$, and for all $w, z \in X$ with $z \neq x$, $z \neq w$, $w \neq y$, we have the following properties:

$$d(x,y) = 0 \quad \text{iff} \quad x = y \tag{1.1}$$

$$d(x,y) = d(y,x) \quad \text{(symmetry)} \tag{1.2}$$

$$d(x,y) \leq d(x,z) + d(z,w) + d(w,y) \quad \text{(tetrahedral inequality).} \tag{1.3}$$

Then we say that (X, d) is a generalized metric space or gms.

Definition 2. Let (X, d) be a gms. A sequence (x_n) in X is said to be a Cauchy sequence if for any $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$,

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$n \geq n_\varepsilon$, one has $d(x_n, x_{n+m}) < \varepsilon$. (X, d) is called complete if every Cauchy sequence in X is convergent.

Let $T : X \longrightarrow X$ be a mapping where (X, d) is gms. For each $x \in X$ let

$$O(x) = \{x, Tx, T^2x, T^3x, \dots\}$$

which will be called the orbit of T at x. $O(x)$ is called T-orbitally complete if and only if every Cauchy sequence in $O(x)$ converges to a point in X.

The next result will be useful in the proof of our main results.

Lemma 1. *Let (X, d) be a gms, let $x_i \in X$, $x_{i-1} \neq x_i$, $1 \leq i \leq n$, $n \geq 3$, $x_0 = x$, $x_n = y$. Then, either*

$$d(x, y) \leq \sum_{i=1}^{n} d(x_{i-1}, x_i),$$

or

$$d(x, y) \leq \sum_{i=1}^{n-2} d(x_{i-1}, x_i) + d(x_{n-2}, y).$$

Proof. Using the tetrahedral inequality, by induction on k, for all $k \in \mathbb{N} \cup \{0\}$ and for all $t_i \in X$, $0 \leq i \leq 2k+3$ with $t_i \neq t_{i+1}$ we have

$$d(t_0, t_{2k+3}) \leq \sum_{i=1}^{2k+3} d(t_{i-1}, t_i). \tag{1}$$

Now, let $x_i \in X$ with $x_{i-1} \neq x_i$ for all $i, = 1, \dots, n$, $n \geq 3, x_0 = x$, $x_n = y$. If $n-3$ is even, then there exists $k \in \mathbb{N} \cup \{0\}$ such that $n = 2k+3$. Hence by (1), we have

$$d(x, y) \leq \sum_{i=1}^{n} d(x_{i-1}, x_i).$$

If $n-3$ is odd, then there exists $k \in \mathbb{N} \cup \{0\}$, such that $n-1 = 2k+3$. Hence by (1), we have

$$d(x, y) \leq \sum_{i=1}^{n-2} d(x_{i-1}, x_i) + d(x_{n-2}, y). \qquad \blacksquare$$

2. Our contribution concerning fixed point theory

In this section we shall give a generalization of the following Kannan's fixed point theorem.

Theorem 1. (Kannan's fixed point theorem) *Let $T : X \longrightarrow X$ where (X, d) is a complete metric space and T satisfy the condition*

$$d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)],$$

where $0 \leq \beta < \frac{1}{2}$ and $x, y \in X$. Then T has a unique fixed point in X.

For the next result, let Φ denote the class of all nondecreasing upper semicontinuous functions $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for all $t > 0$.

LEMMA 2. *Let $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a nondecreasing function such that the sequence $(\varphi^n(t))$ converges to 0 for all $t > 0$. Then*

i) $\varphi(t) < t$ *for all* $t > 0$;

ii) $\varphi(0) = 0$.

Proof. i) Suppose the contrary. Then there exists $t > 0$ such that $\varphi(t) \geq t$. This implies that $\varphi^2(t) \geq \varphi(t) \geq t$ and hence $\varphi^n(t) \geq t$ for all $n \in \mathbb{N}$. Consequently $(\varphi^n(t))$ cannot converge to 0, a contradiction.

ii) Suppose that $\varphi(0) > 0$, then by (i), $\varphi(0) \leq \varphi^2(0) = \varphi(\varphi(0)) < \varphi(0)$, a contradiction. So $\varphi(0) = 0$. ■

THEOREM 2. *Let (X, d) be a gms, let $T : X \longrightarrow X$ be a mapping such that*

$$d(Tx, Ty) \leq \varphi(\max\{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx)\}) \tag{2}$$

where $\varphi \in \Phi$, and if there exists $x \in X$ such that $O(x)$ is orbitally complete, then T has a unique fixed point in X.

Proof. Define the sequence (x_n) inductively as follows: $x_0 = x$, $x_n = Tx_{n-1}$ for all $n \geq 1$. So, by (2) we have,

$$d(T^n x, T^{n+1} x) \leq \varphi(\max\{d(T^{n-1} x, T^n x), \\ d(T^{n-1} x, T^n x), d(T^n x, T^{n+1} x), d(T^n x, T^n x)\})$$

which implies that

$$d(T^n x, T^{n+1} x) \leq \varphi(d(T^{n-1} x, T^n x)).$$

Then, for all $n \in \mathbb{N}$,

$$d(T^n x, T^{n+1} x) \leq \varphi^n(d(x, Tx)) \tag{3}$$

If there exists $n < m$ such that $x_n = x_m$, let $y = T^n x$ then $T^k y = y$ where $k = m - n$. Since $k \geq 1$, we have

$$d(y, Ty) = d(T^k y, T^{k+1} y) \leq \varphi^k(d(y, Ty)).$$

Since $\varphi(t) < t$ for all $t > 0$, so $d(y, Ty) = 0$ and hence y is a fixed point of T. Assume that $x_n \neq x_m$ for all $n \neq m$, so we have

$$d(Tx, T^3 x) \leq \varphi(\max\{d(x, T^2 x), d(x, Tx), d(T^2 x, T^3 x), d(T^2 x, Tx)\}),$$

which implies that

$$d(Tx, T^3 x) \leq \varphi(M)$$

where $M = \max\{d(x, T^2 x), d(x, Tx)\}$.

In general, if n is a positive integer, then

$$d(T^n x, T^{n+2} x) \leq \varphi^n(M). \tag{4}$$

Then, for all $m \geq n + 3$ by Lemma 1, either

$$d(T^n x, T^m x) \leq d(T^n x, T^{n+1} x) + d(T^{n+1} x, T^{n+2} x) + \cdots + d(T^{m-1} x, T^m x),$$

or

$$d(T^n x, T^m x) \leq d(T^n x, T^{n+1} x) + d(T^{n+1} x, T^{n+2} x) + \cdots + d(T^{m-2} x, T^m x).$$

Then, either

$$d(T^n x, T^m x) \leq \sum_{k=n}^{m-1} \varphi^k(d(x, Tx)), \tag{5}$$

or

$$d(T^n x, T^m x) \leq \sum_{k=n}^{m-3} \varphi^k(d(x, Tx)) + \varphi^{m-2}(M). \tag{6}$$

Thus, by (3), (4), (5), and (6) we have

$$d(T^n x, T^m x) \leq \sum_{k=n}^{m-1} \varphi^k(M)$$

for all $m, n \in \mathbb{N}$, $m \geq n$. Since

$$\sum_{k=n}^{\infty} \varphi^k(M) \overset{n\to\infty}{\longrightarrow} 0,$$

(x_n) is a Cauchy sequence, but $O(x)$ is $T-$orbitally complete, hence (x_n) converges to $p \in X$.

The point p is a fixed point of T. To see this we have two cases under consideration.

Case 1. (x_n) does not converge to Tp. Then, there exists a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \neq Tp$, for all $k \in \mathbb{N}$. Hence

$$d(p, Tp) \leq d(p, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, Tp),$$

then if $k \longrightarrow \infty$, we get

$$d(p, Tp) \leq \overline{\lim_{k\to\infty}}\, d(x_{n_k}, Tp). \tag{7}$$

On the other hand, we have from (2)

$$\begin{aligned} d(x_n, Tp) &= d(Tx_{n-1}, Tp) \\ &\leq \varphi(\max\{d(x_{n-1}, p), d(x_{n-1}, x_n), d(p, Tp), d(p, x_n)\}). \end{aligned}$$

Let $n \longrightarrow \infty$, we get

$$\overline{\lim_{n\to\infty}}\, d(x_n, Tp) \leq \varphi(d(p, Tp)). \tag{8}$$

Hence, by (7) and (8), we have $d(p, Tp) = 0$ and $p = Tp$.

Case 2. (x_n) converges to Tp. Suppose that $p \neq Tp$, then there exists a subsequence (x_{n_k}) such that $x_{n_k} \in X - \{p, Tp\}$ for all $k \in \mathbb{N}$, hence

$$d(p, Tp) \leq d(p, x_{n_k}) + d(x_{n_k}, x_{n_{k+1}}) + d(x_{n_{k+1}}, Tp). \tag{9}$$

As $k \longrightarrow \infty$ in (9), we get $Tp = p$, a contradiction.

Then in all cases p is a fixed point of T.

For the uniqueness, assume that $w \neq p$ is also a fixed point of T. From (2),

$$d(p, w) = d(Tp, Tw) \leq \varphi(\max\{d(p, w), d(p, Tp), d(w, Tw), d(w, Tp)\})$$

which implies that

$$d(p,w) \leq \varphi(d(p,w)),$$

hence $p = w$, a contradiction. Therefore, T has a unique fixed point p. ■

COROLLARY 1. *Let (X,d) be a gms, let $T : X \longrightarrow X$ be a mapping such that*

$$d(Tx,Ty) \leq q \max\{d(x,y), d(x,Tx), d(y,Ty), d(y,Tx)\}$$

where $0 \leq q < 1$, and if there exists $x \in X$ such that $O(x)$ is orbitally complete, then T has a unique fixed point in X.

Proof. Put $\varphi(t) = qt$ in Theorem 2. ■

The condition: "there exists $x \in X$ such that $O(x)$ is orbitally complete" is necessary; to see this, consider the next example.

EXAMPLE 1. Let $X = (0,1]$, $d(x,y) = |x-y|$, and let $T : X \longrightarrow X$ be a mapping such that $Tx = \frac{x}{2}$ for all $x \in X$. So, $O(a) = \{a, \frac{a}{2}, \frac{a}{2^2}, \ldots, \frac{a}{2^n}, \ldots\}$ for all $a \in X$. Let $x_n = \frac{a}{2^n}$, $n \in \mathbb{N}$, so (x_n) is a Cauchy sequence in $O(a)$, but (x_n) does not converge, then $O(a)$ is not complete. Moreover, T satisfies the condition (2) where $\varphi(t) = \frac{1}{2}t$, and does not have a fixed point in X.

For the next result, let Ψ denote the class of all functions $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ which are nondecreasing and $\sum_{n=1}^{\infty} \sum \psi^n(t) < \infty$ for all $t \geq 0$.

THEOREM 3. *Let (X,d) be a gms, let $T : X \longrightarrow X$ be a continuous mapping such that*

$$d(Tx,T^2x) \leq \psi(d(x,Tx)); \qquad d(Tx,T^3x) \leq \psi(d(x,T^2x)) \tag{10}$$

where $\psi \in \Psi$, and if there exists $x \in X$ such that $O(x)$ is orbitally complete, then T has a fixed point in X.

Proof. Define the sequence (x_n) inductively as follows: $x_0 = x$, $x_n = Tx_{n-1}$ for all $n \geq 1$.

For all $n \in \mathbb{N}$, we have

$$d(T^nx, T^{n+1}x) \leq \psi^n(d(x,Tx)). \tag{11}$$

If $x_n = x_m$ for some $m > n$, then T^nx is a fixed point of T.

Now, assume that $x_n \neq x_m$ for all $n \neq m$. For all $n \in \mathbb{N}$, we have

$$d(T^nx, T^{n+2}x) \leq \psi^n(d(x,T^2x)). \tag{12}$$

Then, for all $m \geq n+3$, either

$$d(T^nx,T^mx) \leq d(T^nx,T^{n+1}x) + d(T^{n+1}x,T^{n+2}x) + \cdots + d(T^{m-1}x,T^mx),$$

or

$$d(T^nx,T^mx) \leq d(T^nx,T^{n+1}x) + d(T^{n+1}x,T^{n+2}x) + \cdots + d(T^{m-2}x,T^mx).$$

Then, either

$$d(T^nx,T^mx) \leq \sum_{k=n}^{m-1} \psi^k(d(x,Tx)), \tag{13}$$

or

$$d(T^n x, T^m x) \leq \sum_{k=n}^{m-3} \psi^k(d(x,Tx)) + \psi^{m-2}(d(x,T^2x)). \tag{14}$$

Thus, by (11), (12), (13), and (14), we have

$$d(T^n x, T^m x) \leq \sum_{k=n}^{m-1} \psi^k(R)$$

for all $n, m \in \mathbb{N}$, where $R = \max\{d(x,Tx), d(x,T^2x)\}$. Since $\sum_{n=1}^{\infty} \psi^n(R) < \infty$, then (x_n) is a Cauchy sequence, but $O(x)$ is T-orbitally complete, hence (x_n) converges to $p \in X$, and by the continuity of T, we have (x_n) converges also to Tp, then p is a fixed point of T. ∎

COROLLARY 2. *Let (X,d) be a gms, let $T : X \longrightarrow X$ be a continuous mapping such that*

$$\min\{d(Tx,Ty), \max\{d(x,Tx), d(y,Ty)\}\} \leq \psi(d(x,y)), \quad \text{and}$$
$$d(x,T^2x) \leq d(x,Tx), \tag{15}$$

where $\psi \in \Psi$, and if there exists $x \in X$ such that $O(x)$ is orbitally complete, then T has a fixed point in X.

Proof. By setting $y = Tx$ in (15), we get

$$\min\{d(Tx,T^2x), \max\{d(x,Tx), d(Tx,T^2x)\}\} \leq \psi(d(x,Tx)),$$

which implies that $d(Tx,T^2x)\} \leq \psi(d(x,Tx))$ for all $x \in X$. Similarly, if we put $y = T^2x$ in (15), we get

$$\min\{d(Tx,T^3x), \max\{d(x,Tx), d(T^2x,T^3x)\}\} \leq \psi(d(x,T^2x)),$$

hence

$$\min\{d(Tx,T^3x), d(x,Tx)\} \leq \psi(d(x,T^2x))$$

for all $x \in X$, which implies that

$$d(Tx,T^3x) \leq \psi(d(x,T^2x)).$$

Then by Theorem 3, T has a fixed point. ∎

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Department of Mathematics, Yarmouk University, Irbid - Jordan

E-mails: afora_jo@yahoo.com, bellourazze123@yahoo.com, albsoulaaa@hotmail.com