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QUASI CONTINUOUS SELECTIONS OF UPPER BAIRE CONTINUOUS MAPPINGS

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Abstract. The paper deals with the existence problem of selections for a closed valued and c-upper Baire continuous multifunction F. The main goal is to find a minimal *usco* multifunction intersecting F and its selection that is quasi continuous everywhere except at points of a nowhere dense set. The methods are based on properties of minimal multifunctions and a cluster multifunction generated by a cluster process with respect to the system of all sets of second category with the Baire property.

In this paper we will study the existence of a quasi continuous selection for a closed valued and upper Baire continuous multifunction F. A multifunction F is upper Baire continuous, if $U \cap F^+(V)$ contains a set of second category with the Baire property, whenever U, V are open and $U \cap F^+(V) \neq \emptyset$ (see Definition 2). If F is upper Baire continuous, then for any open set V the upper inverse image $F^+(V) = \{x : F(x) \subset V\}$ is of the form $(G \setminus S) \cup T$, where G is of second category and open, S, T are of first category and T is a subset of the closure of G. So, this type of continuity seems to be very close to the Baire property of mappings. The upper Baire continuity has the following three nice features:

(1) Any upper Baire continuous multifunction acting from X into a regular space with a countable basis is lower semi continuous on a residual set [7, Th. 2.1].

(2) A compact valued multifunction F acting from a Baire space into a metric one has the Baire property (i.e., $F^+(T)$ has the Baire property for any closed set T) if and only if F is upper Baire continuous everywhere except for at points of a set of first category [7, Th. 3.2].

(3) An upper Baire continuous compact valued multifunction acting from X into a T_1-regular space has a quasi continuous selection [1].

Here, (1) deals with one of the most general generic theorems, (2) is a characterization of some global (measure) property by a local (continuity) property and the last but not least, (3) proved by Cao and Moors [1], deals likely with the most

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general sufficient conditions for the existence of a quasi continuous selection (much stronger than the first result of this kind proved in [7]). Note that the compact valuedness in (3) cannot be omitted, as shown by a multifunction $F : \mathbb{R} \to \mathbb{R}$ defined by letting $F(x) = \{1/x\}$ for $x \neq 0$ and $F(0) = \mathbb{R}$. It can be shown that F is upper semi continuous without any quasi continuous selection, but it has a selection which is continuous everywhere except for at points of a nowhere dense set. Hence, a general question arises is: For a closed valued and c-upper Baire continuous multifunction, is there a reasonable selection/submultifunction? This is the main goal of the present paper and the answer is given in Theorem 2 and Corollary 1. Besides, we also solve the dual problem on whether a lower Baire continuous multifunction has a quasi continuous selection, see Theorem 3.

In the sequel X, Y are topological spaces, $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{R}$ denotes the reals with usual topology. By $\overline{A}$, A° we denote the closure, the interior of A, respectively. A space Y is σ-compact, if $Y = \bigcup_{n=1}^{\infty} C_n$, where C_n's are compact. By a multifunction F we understand a subset of cartesian product $X \times Y$ and it is identified with a mapping $F : X \to Y$ with the values $\{y \in Y : (x, y) \in F\} =: F(x)$ (it can be empty valued at some points). So, we make no difference between a mapping $F : X \to Y$ and its graph $\{(x, y) : y \in F(x)\}$. By $\mathrm{Dom}\,(F)$, we denote the domain of F, i.e., the set of all arguments x such that $F(x)$ is non-empty. If $\mathrm{Dom}\,(F) = A$ ($\mathrm{Dom}\,(F)$ is a dense set), F is said to be defined on A (densely defined). Further, F is locally bounded at a point x, if there is an open set H containing x and a compact set C such that $F(H) := \bigcup\{F(x) : x \in H\} \subset C$ and it is locally bounded on a set A (bounded on A), if it is so at any point of A ($F(A)$ is a subset of some compact set). For a set $C \subset Y$, $F \cap C$ denotes the multifunction defined by letting $(F \cap C)(x) = F(x) \cap C$ for all $x \in X$.

A function f is understood as a special case of a multifunction with values $\{f(x)\}$. A function f is a selection of a multifunction F, if $f(x) \in F(x)$ for all $x \in \mathrm{Dom}\,(f) = \mathrm{Dom}\,(F)$. For any set $W \subset Y$ the upper and lower inverse images of W under F are defined by $F^+(W) = \{x \in X : F(x) \subset W\}$, $F^-(W) = \{x \in X : F(x) \cap W \neq \emptyset\}$.

A multifunction F is *usc* (upper semi continuous) at $x \in \mathrm{Dom}\,(F)$, if for any open set V with $F(x) \subset V$ there is an open set U containing x such that $F(u) \subset V$ for any $u \in U$. Then F is *usc*, if it is so at any point $x \in \mathrm{Dom}(F)$. A multifunction F is *c-usc* (c-upper semi continuous) at $x \in \mathrm{Dom}\,(F)$, if for any open set V with compact complement such that $F(x) \subset V$ there is an open set U containing x such that $F(u) \subset V$ for any $u \in U$. Then F is *c-usc*, if it is so at any point $x \in \mathrm{Dom}(F)$, see [4], [6], [10]. Finally, F is *usco* at x, if $F(x)$ is non-empty compact and F is *usc* at x.

Any non-empty system $\mathcal{E} \subset 2^X \setminus \{\emptyset\}$ is called a cluster system. For some special cluster systems we will use special notation. For example, $\mathcal{O}$, $\mathcal{B}r$ are cluster systems containing all non-empty open sets, all sets of second category with the Baire property, respectively and $\mathcal{E}^\circ = 2^X \setminus \{\emptyset\}$.

In the next two definitions, we introduce notions of an $\mathcal{E}$-cluster point and an upper $\mathcal{E}$-continuity, as basic tools to investigate properties of multifunctions. These concepts were firstly studied in [7], later in [9] and for the functions in [3].

Definition 1. A point $y \in Y$ is an $\mathcal{E}$-cluster point of F at a point x, if for any open sets V containing y and U containing x, there is a set $E \in \mathcal{E}$ such that $E \subset U$ and $F(e) \cap V \neq \emptyset$ for any $e \in E$. The set of all $\mathcal{E}$-cluster points of F at x is denoted by $\mathcal{E}_F(x)$. The multifunction $\mathcal{E}_F$ with values $\mathcal{E}_F(x)$ is called an $\mathcal{E}$-cluster multifunction of F. We will say that a multifunction F has an $\mathcal{E}$-closed graph, if $\mathcal{E}_F \subset F$.

Example 1. The notion of $\mathcal{E}$-closed graphs is more general than that of closed graphs, because if F has a closed graph, then $\mathcal{E}_F \subset \overline{F} = F$ ($\overline{F}$ is the closure of F in $X \times Y$). On the other hand, a multifunction G from $\mathbb{R}$ to $\mathbb{R}$ defined by letting $G(x) = [0,1]$ for x rational and $G(x) = \{0\}$ otherwise has a $\mathcal{B}r$-closed graph ($\mathcal{B}r_G(x) = \{0\}$ for all x), but its graph is not closed. Similarly, the Dirichlet function has an $\mathcal{O}$-closed graph, since its $\mathcal{O}$-cluster multifunction is empty valued.

Lemma 1. *For any net $\{x_t\}$ converging to x and $y_t \in \mathcal{E}_F(x_t)$, $\mathcal{E}_F(x)$ contains all accumulation points of the net $\{y_t\}$.*

Proof. Let y be an accumulation point of $\{y_t\}$. Then for any open sets V containing y and U containing x there are frequently given indices t' such that $x_{t'} \in U$ and $y_{t'} \in V \cap \mathcal{E}_F(x_{t'})$. Hence there is an $E \in \mathcal{E}$ such that $E \subset U$ and $F(e) \cap V \neq \emptyset$ for any $e \in E$. This means $y \in \mathcal{E}_F(x)$. ∎

Definition 2. A multifunction F is u-$\mathcal{E}$-continuous at a point $x \in \mathrm{Dom}\,(F)$, if for any open sets V, U such that $F(x) \subset V$ and $x \in U$ there is a set $E \in \mathcal{E}$ such that $E \subset U \cap \mathrm{Dom}\,(F)$ and $F(E) \subset V$. A multifunction F is u-$\mathcal{E}$-continuous, if it is so at any point of $\mathrm{Dom}\,(F)$. A u-$\mathcal{E}$-continuous function is simply called $\mathcal{E}$-continuous. A multifunction F is c-u-$\mathcal{E}$-continuous, if $U \cap F^+(V)$ contains some $E \in \mathcal{E}$, whenever U, V are open, $Y \setminus V$ is compact and $U \cap F^+(V) \neq \emptyset$. For $\mathcal{E} = \mathcal{B}r$ and $\mathcal{O}$, we have upper Baire continuity, upper quasi continuity, (c-upper Baire continuity, c-upper quasi continuity), respectively.

Remark 1.

(1) By Lemma 1, the multifunction $\mathcal{E}_F$ has a closed graph, hence it has closed values. This means that $\mathcal{E}_F^-(K)$ is closed for any compact set K or equivalently, $\mathcal{E}_F^+(G)$ is open for any open set G with a compact complement, i.e., $\mathcal{E}_F$ is c-*usc*.
(2) If K is compact and $\mathcal{E}_F^-(K)$ is dense in an open set H, then $\overline{H} \subset \mathcal{E}_F^-(K)$.
(3) If Y is σ-compact, then $\mathrm{Dom}\,(\mathcal{E}_F)$ is an F_σ-set.
(4) If f is $\mathcal{E}$-continuous at x, then $f(x) \in \mathcal{E}_f(x)$.
(5) If $\mathcal{B}r_F$ is a densely defined multifunction or F is upper Baire (c-upper Baire) continuous on a dense set, then X is a Baire space.
(6) For any multifunction F, $\mathcal{E}_F \subset \mathcal{E}_F^\circ = \overline{F}$.

Remark 2. The global Baire continuity of a function has a very interesting feature. If X is Baire and Y is regular, then a function f is Baire continuous on an open set G if and only if f is quasi continuous on G, see [9, Th.3]. In mutifunction setting these notions are different. If $F : \mathbb{R} \to \mathbb{R}$ is defined by letting $F(x) = [0, 1]$ for x rational and $F(x) = \{0\}$ otherwise, then F is upper Baire continuous but not upper quasi continuous.

Definition 3. ([2], [5]) A multifunction F is minimal at a point x, if $F(x)$ is non-empty and for any open sets U, V such that U contains x and $V \cap F(x) \neq \emptyset$ there is a non-empty open set $G \subset U \cap \mathrm{Dom}\,(F)$ such that $F(G) \subset V$. The global definition is given by the local one at any point of $\mathrm{Dom}\,(F)$. It is evident that any selection of a minimal multifunction is quasi continuous.

We will use the next theorem which holds under very general conditions and generalizes the result from [7, Th. 5.3].

Theorem 1. ([1]) *Let Y be T_1-regular and F be non-empty compact valued and upper Baire continuous. Then F has a quasi continuous selection.*

Lemma 2. *Let Y be Hausdorff, C be a compact set in Y and let F be closed valued and c-upper Baire continuous. If $\emptyset \neq X^0 \setminus I \subset F^-(C)$, where X^0 is non-empty open and I is of first category, then the multifunction $F \cap C$ is upper Baire continuous on $X^0 \setminus I$.*

Proof. Let $x_0 \in X^0 \setminus I$ and $F(x_0) \cap C \subset V$, $x_0 \in U \subset X^0$ and V, U be arbitrary open. The set $(Y \setminus V) \cap C$ is compact and its complement $V \cup (Y \setminus C)$ is open containing $F(x_0)$. Since F is c-upper Baire continuous, there is an $E \in \mathcal{B}r$ such that $E \subset U \cap \mathrm{Dom}\,(F)$ and $F(e) \subset V \cup (Y \setminus C)$ for any $e \in E$. Then for any $e \in (E \cap X^0) \setminus I \in \mathcal{B}r$ we have $\emptyset \neq F(e) \cap C \subset V \cap C \subset V$. This means that $F \cap C$ is u-upper Baire continuous at x_0. ∎

Lemma 3. *Suppose that the interior of* $\mathrm{Dom}\,(\mathcal{B}r_f)$ *is non-empty, where f is an arbitrary function. If Y is a regular topological space, then $\mathcal{B}r_f$ is minimal on the interior of* $\mathrm{Dom}\,(\mathcal{B}r_f)$.

Proof. Suppose that $\mathcal{B}r_f$ is not minimal at some point $x \in (\mathrm{Dom}\,(\mathcal{B}r_f))^\circ$. Then, there are open sets V, $U \subset (\mathrm{Dom}\,(\mathcal{B}r_f))^\circ$, $x \in U$ and a set $A \subset U$ which is dense in U such that $\mathcal{B}r_f(x) \cap V \neq \emptyset$ and $\mathcal{B}r_f(a) \cap (Y \setminus \overline{V}) \neq \emptyset$ for any $a \in A$. Let $y \in \mathcal{B}r_f(x) \cap V$. Then there is a set $E = (G \setminus S) \cup T \in \mathcal{B}r$, where G is open, S, T are of first category, and $E \subset U \cap \mathrm{Dom}\,(f)$ such that $f(E) \subset V$. The set $G \cap U$ is non-empty, so there is a point $a \in A \cap G \cap U$ such that $\mathcal{B}r_f(a) \cap (Y \setminus \overline{V}) \neq \emptyset$. Pick up $z \in \mathcal{B}r_f(a) \cap (Y \setminus \overline{V})$. Then there is a set $E_0 \in \mathcal{B}r$, $E_0 \subset G \cap U \cap \mathrm{Dom}\,(f)$ such that $f(E_0) \subset Y \setminus \overline{V}$ and E_0 is of the form $E_0 = (G_0 \setminus S_0) \cup T_0$, where G_0 is open and S_0, T_0 are of first category. Since $G \cap U \cap G_0$ is of second category, there is a point $e \in G \cap U \cap G_0 \setminus (S \cup S_0) \subset E$. It follows that $f(e) \in V$. On the other hand, $e \in E_0$ implies $f(e) \in Y \setminus \overline{V}$, which is a contradiction. ∎

Lemma 4. *If F is c-upper Baire continuous, then $F^+(V)$ has the Baire property for any open set V with a compact complement.*

Proof. If not, there is an open set U such that both sets $X_0 := F^+(V)$ and $X \setminus X_0$ are of second category at any point of U. Let $x \in X_0 \cap U$ with $F(x) \neq \emptyset$. By c-upper Baire continuity, there is an $E \in Br$ such that $E \subset U \cap \operatorname{Dom}(F)$ and $F(E) \subset V$. Since E is of second category with the Baire property, $E = (G \setminus I) \cup J$ for some open G and I, J of first category such that $G \cap U \neq \emptyset$ (otherwise $E = \big((G \setminus I) \cup J)\big) \cap U = \big((G \setminus I) \cap U\big) \cup \big(J \cap U\big) = J \cap U$ is of first category). The set $X \setminus X_0$ is of second category at any point of U, so $\big((G \cap U \cap (X \setminus X_0)\big) \setminus I$ is of second category. It follows that there is a point $e \in \big((G \cap U \cap (X \setminus X_0)\big) \setminus I \subset E \subset \operatorname{Dom}(F)$. So $F(e) \not\subset V$, contradicting with $F(E) \subset V$. ■

THEOREM 2. *Let Y be a T_1-regular σ-compact space, $G \subset X$ be non-empty open and let F be closed valued and c-upper Baire continuous on G. Then there are an open set $H \subset G$ and a multifunction F_0 defined on G such that $G \setminus H$ is nowhere dense, $F_0(x) \subset \mathcal{B}r_F(x)$ for any $x \in H$ and the following hold*

(1) F_0 is both minimal usco and locally bounded on H,

(2) $F(x) \cap F_0(x) \neq \emptyset$ for any $x \in H$,

(3) there is a selection g of F which is both quasi continuous and locally bounded on H,

(4) if F has a $\mathcal{B}r$-closed graph, then $F_0 \subset F$.

Proof. Let $Y = \bigcup_{k\in\mathbb{N}} C_k$, and each C_k be compact. Assumption of c-upper Baire continuity guarantees that any non-empty open subset of G is of second category (see Remark 1 (5)), i.e., G is a Baire space. Since $G \subset \bigcup_{k\in\mathbb{N}} F^-(C_k)$ and $F^-(C_k) = X \setminus F^+(Y \setminus C_k)$ has the Baire property (by Lemma 4), there is a sequence $\{H_{k_n}\}_{n\in\mathbb{N}}$ (possibly finite) of non-empty open pairwise disjoint subsets of G such that $I := G \setminus \bigcup_{n\in\mathbb{N}} H_{k_n}$ is of first category and $H_{k_n} \setminus I \subset F^-(C_{k_n})$. Put $H := \bigcup_{n\in\mathbb{N}} H_{k_n}$. Then the set $G \setminus H$ is of first category. Since G is a Baire space, $G \setminus H$ is also nowhere dense. By Lemma 2, $F \cap C_{k_n}$ is compact valued and u-$\mathcal{B}r$-continuous on $H_{k_n} \setminus I$. By Theorem 1, there is a selection f_n of $F \cap C_{k_n}$, which is defined and quasi continuous on $H_{k_n} \setminus I$ (in the relative topology). So f_n is $\mathcal{B}r$-continuous at any point of $H_{k_n} \setminus I$. By Remark 1 (4), $f_n(x) \in \mathcal{B}r_{f_n}(x)$ for any $x \in H_{k_n} \setminus I$.

Define $f : H \setminus I \to Y$ by letting

$$f(x) = f_n(x) \quad \text{for} \quad x \in H_{k_n} \setminus I. \tag{$*$}$$

Since $f_n \subset f \subset F$,

$$f_n \subset \mathcal{B}r_{f_n} \subset \mathcal{B}r_f \subset \mathcal{B}r_F. \tag{$**$}$$

Put $F_0 := \mathcal{B}r_f$ on the domain of $\mathcal{B}r_f$ and $F_0 := F$ otherwise.

(1) Since f_n is bounded by C_{k_n} and $H_{k_n} \setminus I$ is dense in H_{k_n}, $\mathcal{B}r_{f_n}$ is non-empty, compact valued (by Remark 1 (2)) and bounded by C_{k_n} on $\overline{H_{k_n}}$. Since $\mathcal{B}r_{f_n}$ is bounded with a closed graph, $\mathcal{B}r_{f_n}$ is *usco* and bounded on H_{k_n}. It is clear that $\mathcal{B}r_{f_n}(x) = \mathcal{B}r_f(x)$ for any $x \in H_{k_n}$, see ($*$). By Lemma 3, $\mathcal{B}r_{f_n}$ is minimal on H_{k_n}. Hence, F_0 is both *usco* minimal and locally bounded on H.

(2) We will show that $\mathcal{B}r_f(x) \cap F(x) \neq \emptyset$ for any $x \in H$. If not, there is some $a_0 \in H$ such that $\mathcal{B}r_f(a_0) \cap F(a_0) = \emptyset$. By regularity of Y and compactness of $\mathcal{B}r_f(a_0)$, there are two disjoint open sets $V_2 \supset \mathcal{B}r_f(a_0)$ and $V_1 \supset F(a_0)$. Since $\mathcal{B}r_f$ is locally bounded, there are an open set $U \subset H$ containing a_0 and a compact set C such that $\mathcal{B}r_f(U) \subset C$. $\mathcal{B}r_f$ is *usco* at a_0, hence there is an open set $W_1 \subset U$ containing a_0 such that $\mathcal{B}r_f(W_1) \subset \overline{V_2}$. So, $\mathcal{B}r_f(W_1) \subset \overline{V_2} \cap C$. Since F is c-upper Baire continuous at a_0 and $F(a_0) \cap \overline{V_2} \cap C = \emptyset$, there is $E := (G_0 \setminus S) \cup T \in \mathcal{B}r$ such that G_0 is open, S, T are of first category, $E \subset W_1 \cap \mathrm{Dom}\,(F)$ and $F(E) \subset Y \setminus (\overline{V_2} \cap C)$. Since $G_0 \setminus S \subset H = \bigcup_{n \in \mathbb{N}} H_{k_n}$, there is an $m \in \mathbb{N}$ such that $G_0 \cap H_{k_m} \neq \emptyset$. By $(*)$ and $(**)$, for $e \in G_0 \cap H_{k_m} \setminus (I \cup S)$ we have $f(e) \in F(e) \cap \mathcal{B}r_f(e)$, contradicting with $F(E) \subset Y \setminus (\overline{V_2} \cap C)$ and $\mathcal{B}r_f(E) \subset \overline{V_2} \cap C$.

(3) Define a selection g of F by letting $g(x) \in \mathcal{B}r_f(x) \cap F(x)$ if $x \in H$ and $g(x) \in F(x)$ otherwise. It is clear that g is a selection of F which is quasi continuous on H, by Lemma 3. Since $\mathcal{B}r_f$ is locally bounded on H, so is g.

(4) By definition of F_0, $(**)$ and Remark 1 (6) we have

$$F_0 \subset \mathcal{B}r_f \cup F \subset \mathcal{B}r_F \cup F \subset \overline{F} = F. \quad \blacksquare$$

It is worth to formulate Theorem 2 for $\mathrm{Dom}\,(F) = X$. Moreover, by [4], there is a *c-usc* multifunction F which is not *usc* at any point (on the other hand, if F if *c-lsc*, then F is *lsc* everywhere except for at points of a nowhere dense set, see [4]). By the next corollary, F has a submultifunction, which is both minimal *usco* and locally bounded everywhere except for at points of a nowhere dense set ((4) follows also from [4]).

Corollary 1. *Let Y be a T_1-regular σ-compact space and let F and f be defined on X. Then*

(1) if F is closed valued and c-upper Baire continuous, then F has a selection g which is both quasi continuous and locally bounded on an open dense set. Moreover, if Y is metric, then g is continuous everywhere except for at points of a set of first category, by [8].

(2) if f is c-Baire continuous, then f is quasi continuous on an open dense set,

(3) if X is Baire and F is closed valued and c-usc, then F has a submultifunction, which is both minimal usco and locally bounded on an open dense set,

(4) if f is c-continuous, then f is continuous on an open dense set.

Proof. It is sufficient to prove (3). Suppose that F is *c-usc*. The multifunction F_0 in Theorem 2 is minimal *usco* and locally bounded on a dense open set H, hence for any $x \in H$ there is an open set U_0 containing x such that $F_0(U_0) \subset C$, where C is compact. We will show that $F_0(x) \subset F(x)$. If not, there are a point $y \in F_0(x) \setminus F(x)$ and two disjoint open sets $V \supset F(x)$ and W containing y (we use regularity of Y and closed values of F and F_0). The set $C \cap \overline{W}$ is non-empty, compact and disjoint from $F(x)$. Since F is *c-usc*, there is an open set U containing x such that $U \subset U_0$ and $F(U) \subset Y \setminus (C \cap \overline{W})$. Since F_0 is minimal, there is a non-empty open set $H_0 \subset U$ such that $F_0(H_0) \subset W$. Hence $F_0(H_0) \subset C \cap \overline{W}$. So F and F_0 have disjoint values on H_0, contradicting with Theorem 2 (2). ∎

In Theorem 2, c-upper Baire continuity guarantees that X is Baire. Theorem 2 also holds for c-upper quasi continuous, provided X is Baire. Without this assumption it is not valid.

EXAMPLE 2. Define a function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ ($\mathbb{Q}^+$ is the set of all positive rational numbers with the usual topology) by letting $f(x) = \{n\}$, where $x = n/m$ is a rational number in the standard form. Then f is c-quasi continuous, but it is not quasi continuous at any point (comparing this with Theorem 2 (3)).

At the end of the paper we will give an application of our results to the existence of selections of lower Baire continuous multifunctions. A multifunction F is lower Baire continuous at a point $x \in \mathrm{Dom}\,(F)$, if for any open sets V, U, such that $F(x) \cap V \neq \emptyset$ and $x \in U$ there is a set $E \in \mathcal{B}r$ such that $E \subset U \cap \mathrm{Dom}\,(F)$ and $F(e) \cap V \neq \emptyset$ for any $e \in E$. A multifunction F is lower Baire continuous, if it is so at any point of $\mathrm{Dom}\,(F)$. Equivalently, F is lower Baire continuous at x if $\emptyset \neq F(x) \subset \mathcal{B}r_F(x)$. In contrast to Theorem 2, a quasi continuous selection on an open dense set need not exist, as shown in the next example.

EXAMPLE 3. Define a multifunction $F : \mathbb{R} \to \mathbb{R}$ by letting $F(x) = \{n\}$, where $x = n/m$ is a rational number in the standard form and $F(x) = \mathbb{R}$ otherwise. Then F is lower Baire quasi continuous, but any of its selections is not quasi continuous.

The main idea is to find a quasi continuous selection f of $\mathcal{B}r_F$ with a metric range.

THEOREM 3. *Let Y be a σ-compact metric space and F be a closed valued densely defined lower Baire continuous multifunction. Then there is an open dense set H and a function $f : H \to Y$ such that f is quasi continuous, continuous on a residual set A and $f(a) \in F(a)$ for any $a \in A$.*

Proof. Again, $\mathrm{Dom}\,(F)$ is a residual set. Since F is lower Baire continuous and $F \subset \mathcal{B}r_F$, $\mathcal{B}r_F$ is non-empty valued on a residual set. By Remark 1 (3), $\mathcal{B}r_F$ is defined on an F_σ-set. Since X is Baire, $\mathcal{B}r_F$ is defined at least on a dense open set G. By Remark 1 (1), $\mathcal{B}r_F$ is *c-usc* and so is c-upper Baire continuous on G. It follows from Theorem 2 that $\mathcal{B}r_F$ has a selection f, which is quasi continuous on an open dense set $H \subset G$ and $G \setminus H$ is a nowhere dense set. Put $B_n = \{x \in H \cap \mathrm{Dom}\,(F) : d(f(x), F(x)) < 1/n\}$. We will show that $H \setminus B_n$ is a set of first category. Let $x \in H$, V, U be open sets containing $f(x)$ and x respectively such that the diameter of V is less than $\frac{1}{2n}$. By quasi continuity of f, there is a non-empty open set $H_0 \subset U$ such that $f(H_0) \subset V$. Let $h \in H_0$. Since $f(h) \in \mathcal{B}r_F(h)$, there is a set $E \in \mathcal{B}r$ such that $E \subset H_0$ and $F(e) \cap V \neq \emptyset$ for any $e \in E$. Hence $E \subset B_n$, and $H \setminus B_n$ is a set of first category. It follows that $B := \cap_{n=1}^{\infty} B_n = \{x \in H \cap \mathrm{Dom}\,(F) : d(f(x), F(x)) = 0\}$ is residual in H. It is clear that $f(b) \in F(b)$ for any $b \in B$. Since Y is metric, f is continuous on a residual set C, by [8]. Finally, the proof is completed by putting $A = B \cap C$. ∎

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