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FUZZY GRILLS AND INDUCED FUZZY TOPOLOGY

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Abstract. In this paper we introduce a new kind of fuzzy topology as an accompanying structure generated by any given fuzzy grill on the ambient set X in a fuzzy topological space (X, τ). The basic properties of this induced fuzzy topology are discussed here in some detail. We have also shown various relations between the fuzzy grill-oriented topology and the original fuzzy topology under some suitability condition applied on the fuzzy grill under consideration.

1. Introduction and preliminaries

It is well known that the fundamental idea of fuzzy sets was first introduced by Zadeh [14]. Chang [3] was the initiator of the notion of fuzzy topology in 1968. In general topology the notion of grills was first proposed by Choquet [5] in 1947, which has been observed as an excellent tool for studying different topological concepts. In fuzzy setting, the concept of fuzzy grills on fuzzy topological spaces was initiated by Azad [1], basically for the study of proximities in fuzzy spaces. Subsequently, Srivastava and Gupta [12] and Chattopadhyay et al. [4] investigated fuzzy basic proximity by use of fuzzy grills. Recently some researchers are trying to extend these studies to the broader framework of fuzzy topology. In [2], the authors have studied fuzzy compactness, fuzzy almost compactness etc. via fuzzy grills. In this paper, we also use fuzzy grills to generate a new fuzzy topology larger than the original one and to study certain basic properties of this new induced topology. We introduce in Section 2, a closure operator which corresponds to the notion of Kuratowski closure operator in general topology and this operator induces the aforesaid new topology via fuzzy grill. In Section 3, we define a suitability condition which when imposed on a fuzzy grill $\mathcal{G}$, makes the generated fuzzy topology more well-behaved and applicable. We maintain that in the framework of the investigations being undertaken in this paper, many other aspects of fuzzy topology namely fuzzy compactness, fuzzy paracompactness can be studied further, which we propose to do in future.

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Throughout this paper, by an fts X, we mean a fuzzy topological space (X, τ), as initiated by Chang [3]. A *fuzzy set* A in a set X is a function on X into the closed unit interval $[0, 1]$ of the real line. The set $\{x \in X : A(x) > 0\}$ is called the *support of* A and is denoted by $suppA$. The fuzzy sets in X taking on respectively the constant values 0 and 1 are denoted by 0_X and 1_X [14] respectively.

For two fuzzy sets A, B in X, i.e., $A, B \in I^X$ $(I = [0, 1])$, we write $A \leq B$ if $A(x) \leq B(x)$ for each $x \in X$, whereas the notation AqB means that A is *quasi-coincident* [10] with B, i.e., AqB implies $A(x) + B(x) > 1$ for some $x \in X$. The negations of these statements are denoted by $A \not\leq B$ and $A\overline{q}B$ respectively. A *fuzzy singleton or a fuzzy point* [10] with support x and value α $(0 < \alpha \leq 1)$ is denoted by x_α.

A fuzzy set A is non-empty if $A \neq 0_X$. For $A, B \in I^X$, A is called a *q-nbd of* B [10] if BqU for some fuzzy open set U in X, with $U \leq A$; if in addition, A itself is fuzzy open then it is called an open-q-nbd of B. The collection of all open q-nbds of any fuzzy point x_α is denoted by $\mathcal{Q}(x_\alpha)$. For a fuzzy set A in an fts X, the *fuzzy complement*, *fuzzy interior* and *fuzzy closure* of A in X are written as $1 - A$ [or sometimes as $1_X - A$] or $\overline{A}$, $intA$ and clA respectively.

A subfamily $\mathcal{B}$ of the fuzzy topology τ of an fts (X, τ) is a *base* for τ [13] iff for each fuzzy singleton x_α in (X, τ) and for each open q-nbd U of x_α, $x_\alpha qB$, for some $B \in \mathcal{B}$ with $B \leq U$.

We recall from [10] that x_α is called an *adherence point* of a fuzzy set A if every q-nbd of x_α is quasi-coincident with A and $cl(A)$ is the union of all adherence points of A.

A fuzzy point x_α is called an *accumulation point* of a fuzzy set A [10] if x_α is an adherence point of A and every q-nbd of x_α and A are quasi-coincident at some point different from x, whenever $x_\alpha \leq A$. The union of all the accumulation points of A is called the *fuzzy derived set* of A, denoted by A^d. It is also known from [10] that for any fuzzy set A in an fts X, $cl(A) = A \bigvee A^d$.

2. Fuzzy grills and generated fuzzy topology

Definition 2.1. [1] A non-void collection $\mathcal{G}$ of fuzzy sets in an fts (X, τ) is called a fuzzy grill on X if

(i) $0_X \notin \mathcal{G}$

(ii) $A \in \mathcal{G}$, $B \in I^X$ and $A \leq B \Rightarrow B \in \mathcal{G}$ and

(iii) $A, B \in I^X$ and $A \bigvee B \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$.

Definition 2.2. For any two fuzzy sets A and B in an fts X, we define $(A+B)$ and $(A - B)$ to be the fuzzy sets given by

$$(A + B)(x) = \begin{cases} A(x) + B(x), & \text{if } A(x) + B(x) \leq 1 \\ 1, & \text{if } A(x) + B(x) > 1. \end{cases}$$

and

$$(A - B)(x) = \begin{cases} A(x) - B(x), & \text{if } A(x) > B(x) \\ 0, & \text{if } A(x) \leq B(x), \end{cases}$$

where $x \in X$.

Before we introduce the most crucial definition of this article, let us recall the well known definition of Lukasiewicz conjunction $A * B$ on the power set I^X, given by $A * B = \max(0, A + B - 1_X)$, for $A, B \in I^X$, i.e., for any $x \in X$,

$$(A * B)(x) = \begin{cases} A(x) + B(x) - 1, & \text{if } A(x) + B(x) > 1, \\ 0, & \text{otherwise.} \end{cases}$$

Incidentally it may be noted that the binary operation '$*$' on I^X is a t-norm(for detailed information one may refer to [7, 8, 9]).

Definition 2.3. Let (X, τ) be an fts and $\mathcal{G}$ be a fuzzy grill on X. We define $\phi : I^X \to I^X$, denoted by $\phi_{\mathcal{G}}(A)$ or simply by $\phi(A)$ (where A is a fuzzy set in X) and called the fuzzy operator associated with the fuzzy grill $\mathcal{G}$ and the fuzzy topology τ to be the union of all fuzzy points x_α of X such that if $U \in Q(x_\alpha)$, then $A * U \in \mathcal{G}$.

Proposition 2.4. Let (X, τ) be an fts.

(i) If $\mathcal{G}$ is any fuzzy grill on X, then for any two fuzzy sets A and B in X, $A \leq B \Rightarrow \phi_{\mathcal{G}}(A) \leq \phi_{\mathcal{G}}(B)$, i.e., $\phi_{\mathcal{G}}$ is an increasing function.

(ii) If $\mathcal{G}_1$ and $\mathcal{G}_2$ are two fuzzy grills on X with $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then $\phi_{\mathcal{G}_1}(A) \leq \phi_{\mathcal{G}_2}(A)$, for any fuzzy set A in X.

(iii) For any fuzzy grill $\mathcal{G}$ on X and any fuzzy set A in X, if $A \notin \mathcal{G}$ then $\phi_{\mathcal{G}}(A) = 0_X \notin \mathcal{G}$.

Proof.(i) $x_\alpha \leq \phi_{\mathcal{G}}(A)$ implies that for all $U \in Q(x_\alpha)$, $A * U \in \mathcal{G}$. Now, $A \leq B \Rightarrow A * U \leq B * U$. Thus for all $U \in Q(x_\alpha)$, $B * U \in \mathcal{G}$, i.e., $x_\alpha \leq \phi_{\mathcal{G}}(B)$.

(ii) Let $x_\alpha \leq \phi_{\mathcal{G}_1}(A)$. Then for all $U \in Q(x_\alpha)$, $A * U \in \mathcal{G}_1 \subseteq \mathcal{G}_2 \Rightarrow x_\alpha \leq \phi_{\mathcal{G}_2}(A)$.

(iii) $x_\alpha \leq \phi_{\mathcal{G}}(A) \Rightarrow$ for all $U \in Q(x_\alpha)$, $A * U \in \mathcal{G}$. But $U \leq 1_X \Rightarrow A * U \leq A$. Hence $A \in \mathcal{G}$, a contradiction to our hypothesis. Thus $\phi_{\mathcal{G}}(A) = 0_X$. ■

Proposition 2.5. Let (X, τ) be an fts and $\mathcal{G}$ be a fuzzy grill on X. Then for all fuzzy sets A, B in X,

(i) $\phi(A \bigvee B) = \phi(A) \bigvee \phi(B)$

(ii) $\phi(\phi(A)) \leq \phi(A) = cl(\phi(A)) \leq cl(A)$

(iii) $\phi(A \bigvee G) = \phi(A)$, for every $G \notin \mathcal{G}$.

Proof. (i) Since $A \leq A \bigvee B$ and $B \leq A \bigvee B$, thus $\phi(A) \leq \phi(A \bigvee B)$ and $\phi(B) \leq \phi(A \bigvee B)$ [by Proposition 2.4(i)]. So,

$$\phi(A) \bigvee \phi(B) \leq \phi(A \bigvee B) \tag{1}$$

Conversely $x_\alpha \nleq \phi(A) \bigvee \phi(B) \Rightarrow$ there exist U, $V \in Q(x_\alpha)$ such that $A * U \notin \mathcal{G}$ and $B * V \notin \mathcal{G}$. Then $U \bigwedge V \in Q(x_\alpha)$ and considering different cases depending on the values of U, V, A and B, it is a routine affair to show that $(A \bigvee B) * (U \bigwedge V) \leq (A * U) \bigvee (B * V) \notin \mathcal{G}$. Thus $x_\alpha \nleq \phi(A \bigvee B)$ and consequently

$$\phi(A \bigvee B) \leq \phi(A) \bigvee \phi(B) \tag{2}$$

From (1) and (2), the result follows.

(ii) Let $x_\alpha \nleq cl(A)$. Then there exists an open q-nbd U of x_α in X such that $U\overline{q}A$, i.e., $A(y) + U(y) \leq 1$, for each $y \in X$ and hence $A * U = 0_X \notin \mathcal{G}$. Then $x_\alpha \nleq \phi(A)$ and hence

$$\phi(A) \leq cl(A) \tag{1}$$

Now we shall show that $cl[\phi(A)] \leq \phi(A)$. Let $x_\alpha \leq cl[\phi(A)]$ and $U \in Q(x_\alpha)$. Then $\phi(A)qU$, i.e., there exists $y \in X$ such that $\phi(A)(y) + U(y) > 1$. Let $[\phi(A)](y) = t$. Then $y_t \leq \phi(A)$ and $U \in Q(y_t)$ imply that $A * U \in \mathcal{G}$. Thus

$$x_\alpha \leq \phi(A) \tag{2}$$

From (1) and (2) we have $cl\phi(A) = \phi(A)$. Hence $\phi(\phi(A)) \leq cl(\phi(A)) = \phi(A) \leq cl(A)$.

(iii) If $G \notin \mathcal{G}$, then $\phi(G) = 0_X$ [by Proposition 2.4(iii)]. Thus $\phi(A \bigvee G) = \phi(A) \bigvee \phi(G)$ [by (i)] $= \phi(A)$. ■

Definition 2.6. Let $\mathcal{G}$ be a fuzzy grill on an fts X. Let us define a map $\psi : I^X \to I^X$ by $\psi(A) = A \bigvee \phi(A)$ for all fuzzy set A in X.

Now we have

Theorem 2.7. *The above defined* $'\psi'$ *satisfies the following conditions:*

(a) For every fuzzy set A *in* X, $A \leq \psi(A)$.

(b) $\psi(0_X) = 0_X$.

(c) $\psi(A \bigvee B) = \psi(A) \bigvee \psi(B)$.

(d) $\psi(\psi(A)) = \psi(A)$.

Proof. (a) By definition of $'\psi'$, we have $A \leq A \bigvee \phi(A) = \psi(A)$.

(b) Since $0_X \notin \mathcal{G}$, $\phi(0_X) = 0_X$ [by Proposition 2.4(iii)] $\Rightarrow \psi(0_X) = 0_X$.

(c) $\psi(A \bigvee B) = (A \bigvee B) \bigvee \phi(A \bigvee B) = (A \bigvee B) \bigvee \phi(A) \bigvee \phi(B)$ [by Proposition 2.5(i)] $= (A \bigvee \phi(A)) \bigvee (B \bigvee \phi(B)) = \psi(A) \bigvee \psi(B)$ [by Proposition 2.5(ii)].

(d) $\psi(\psi(A)) = \psi(A \bigvee \phi(A)) = A \bigvee \phi(A) \bigvee \phi(A \bigvee \phi(A)) = A \bigvee \phi(A) \bigvee \phi(A) \bigvee \phi(\phi(A))$ [by Proposition 2.5(i)] $= A \bigvee \phi(A) \bigvee \phi(\phi(A)) = A \bigvee \phi(A)$[by Proposition 2.5(ii)]. ■

We are now equipped enough to give the following definition:

Definition 2.8. In an fts (X, τ), corresponding to a fuzzy grill $\mathcal{G}$ there exists a unique fuzzy topology $\tau_{\mathcal{G}}$ (say) on X given by $\tau_{\mathcal{G}} = \{U \in I^X / \psi(1_X - U) = 1_X - U\}$, where for any $A \in I^X$, $\psi(A) = A \bigvee \phi(A) = \tau_{\mathcal{G}}$-$cl(A)$.

EXAMPLE 2.9. In an fts (X,τ), the trivial fuzzy grill is $I^X \setminus \{0_X\}$. If we take $\mathcal{G} = I^X \setminus \{0_X\}$, then by using Proposition 2.5(ii), we have for any non-empty fuzzy set A in X, $\phi(A) = cl(A)$ and $\psi(A) = \tau_{\mathcal{G}}\text{-}cl(A) = A \bigvee \phi(A) = cl(A)$. Thus in this case $\tau = \tau_{\mathcal{G}}$.

THEOREM 2.10. *In an fts (X,τ)*

(i) if $\mathcal{G}_1$ and $\mathcal{G}_2$ be two fuzzy grills with $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then $\tau_{\mathcal{G}_2} \subseteq \tau_{\mathcal{G}_1}$.

(ii) if $\mathcal{G}$ be a fuzzy grill and $B \notin \mathcal{G}$, then B is closed in $(X,\tau_{\mathcal{G}})$.

(iii) for any fuzzy set A and any fuzzy grill $\mathcal{G}$ on X, $\phi(A)$ is $\tau_{\mathcal{G}}$-closed.

Proof. (i) Let $U \in \tau_{\mathcal{G}_2}$. Then $\tau_{\mathcal{G}_2}\text{-}cl(1_X - U) = \psi_{\mathcal{G}_2}(1_X - U) = 1_X - U \Rightarrow 1_X - U = (1_X - U) \bigvee \phi_{\mathcal{G}_2}(1_X - U) \Rightarrow \phi_{\mathcal{G}_2}(1_X - U) \leq (1_X - U)$. Since $\mathcal{G}_1 \subseteq \mathcal{G}_2$, by Proposition 2.4(ii), $\phi_{\mathcal{G}_1}(1_X - U) \leq \phi_{\mathcal{G}_2}(1_X - U) \leq (1_X - U)$. Thus $1_X - U = \tau_{\mathcal{G}_1}\text{-}cl(1_X - U)$, i.e., $(1_X - U)$ is $\tau_{\mathcal{G}_1}$-closed and hence $U \in \tau_{\mathcal{G}_1}$. So $\tau_{\mathcal{G}_2} \subseteq \tau_{\mathcal{G}_1}$.

(ii) By Proposition 2.4(iii), if $B \notin \mathcal{G}$ then $\phi(B) = 0_X$. Then $\tau_{\mathcal{G}}\text{-}cl(B) = \psi(B) = B \bigvee \phi(B) = B$, proving that B is $\tau_{\mathcal{G}}$-closed.

(iii) We have by using Proposition 2.5(ii), $\tau_{\mathcal{G}}\text{-}cl(\phi(A)) = \psi(\phi(A)) = \phi(A) \bigvee \phi(\phi(A)) = \phi(A)$, which implies that $\phi(A)$ is $\tau_{\mathcal{G}}$-closed. ∎

REMARK 2.11. For any two fuzzy grills $\mathcal{G}_1$ and $\mathcal{G}_2$ in an fts (X,τ), it can be checked that $\mathcal{G}_1 \cup \mathcal{G}_2$ is also a fuzzy grill on X. But $\mathcal{G}_1 \cap \mathcal{G}_2$ may not be a fuzzy grill on X, as shown by the following example.

EXAMPLE 2.12. Let $X = \{a,b\}$ and $\tau = \{0_X, 1_X, A, B\}$, where $A(a) = 0.2$ and $A(b) = B(a) = B(b) = 0.5$. Then (X,τ) is an fts. Let $\mathcal{G}_1$ consist of 1_X and all fuzzy sets G_1 on X such that $0.3 \leq G_1(a) \leq 1$ and $0 \leq G_1(b) \leq 1$, and $\mathcal{G}_2$ consist of 1_X and all fuzzy sets G_2 on X such that $0 \leq G_2(a) \leq 1$ and $0.2 \leq G_2(b) \leq 1$. Then $\mathcal{G}_1$ and $\mathcal{G}_2$ are fuzzy grills on X. Now let B and C be two fuzzy sets in X such that $B(a) = 0.4, B(b) = 0$ and $C(a) = 0, C(b) = 0.3$. Then $B \in \mathcal{G}_1$ but $B \notin \mathcal{G}_2$. Also $C \notin \mathcal{G}_1$ but $C \in \mathcal{G}_2$. Let $A = B \bigvee C$. Then $A(a) = 0.4$ and $A(b) = 0.3$ and thus $A = B \bigvee C \in \mathcal{G}_1 \cap \mathcal{G}_2$, but neither $B \in \mathcal{G}_1 \cap \mathcal{G}_2$ nor $C \in \mathcal{G}_1 \cap \mathcal{G}_2$.

PROPOSITION 2.13. *For any two fuzzy grills $\mathcal{G}_1$ and $\mathcal{G}_2$ in an fts (X,τ), if we define $\mathcal{G}_1 \wedge \mathcal{G}_2 = \{G_1 \bigwedge G_2 / G_1 \in \mathcal{G}_1, G_2 \in \mathcal{G}_2$ and $(G_1 \leq G_2$ or $G_2 \leq G_1)\}$, then $\mathcal{G}_1 \wedge \mathcal{G}_2$ is a fuzzy grill on X.*

Proof. Since $0_X \notin \mathcal{G}_1, \mathcal{G}_2$, we have $0_X \notin \mathcal{G}_1 \wedge \mathcal{G}_2$. Let $A \in \mathcal{G}_1 \wedge \mathcal{G}_2$, then $A = G_1 \bigwedge G_2$ where $G_1 \in \mathcal{G}_1$, $G_2 \in \mathcal{G}_2$ and at least one is contained in the other. Let $G_1 \bigwedge G_2 = G_1$. Now for any $B \geq A$, $G_1 \bigwedge G_2 \leq B \Rightarrow G_1 \leq B \in \mathcal{G}_1$. Then $B = B \bigwedge 1_X \in \mathcal{G}_1 \wedge \mathcal{G}_2$, since $B \in \mathcal{G}_1$, $1_X \in \mathcal{G}_2$ and $B \leq 1_X$. Finally $A = B \bigvee C \in \mathcal{G}_1 \wedge \mathcal{G}_2 \Rightarrow A = G_1 \bigwedge G_2$ where $G_1 \in \mathcal{G}_1$, $G_2 \in \mathcal{G}_2$ and $(G_1 \leq G_2$ or $G_2 \leq G_1)\}$. Suppose, without any loss, $G_1 \leq G_2$. Then $A = B \bigvee C = G_2 \in \mathcal{G}_1 \wedge \mathcal{G}_2$ which implies $(B \in \mathcal{G}_1$ or $C \in \mathcal{G}_1)$ and $(B \in \mathcal{G}_2$ or $C \in \mathcal{G}_2)$. Since $B = 1_X \bigwedge B = B \bigwedge 1_X$, $C = 1_X \bigwedge C = C \bigwedge 1_X$ and $1_X \in \mathcal{G}_1 \bigwedge \mathcal{G}_2$, it follows that $B \in \mathcal{G}_1 \wedge \mathcal{G}_2$ or $C \in \mathcal{G}_1 \wedge \mathcal{G}_2$.

THEOREM 2.14. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two fuzzy grills on an fts (X,τ). Then for any fuzzy set A in X,*

(a) $\phi_{\mathcal{G}_1}(A) \bigvee \phi_{\mathcal{G}_2}(A) = \phi_{\mathcal{G}_1 \cup \mathcal{G}_2}(A)$.
(b) $\phi_{\mathcal{G}_1}(A) \bigwedge \phi_{\mathcal{G}_2}(A) \leq \phi_{\mathcal{G}_1 \wedge \mathcal{G}_2}(A)$.

Proof. (a) Clearly $\mathcal{G}_1, \mathcal{G}_2 \subseteq \mathcal{G}_1 \cup \mathcal{G}_2$. Then by Theorem 2.4 (i), $\phi_{\mathcal{G}_1}(A) \leq \phi_{\mathcal{G}_1 \cup \mathcal{G}_2}(A)$ and $\phi_{\mathcal{G}_1}(A) \leq \phi_{\mathcal{G}_1 \cup \mathcal{G}_2}(A)$, for any fuzzy set A in X. Thus

$$\phi_{\mathcal{G}_1}(A) \bigvee \phi_{\mathcal{G}_2}(A) \leq \phi_{\mathcal{G}_1 \cup \mathcal{G}_2}(A) \tag{i}$$

Conversely, $x_\alpha \nleq \phi_{\mathcal{G}_1}(A) \bigvee \phi_{\mathcal{G}_2}(A) \Rightarrow x_\alpha \nleq \phi_{\mathcal{G}_1}(A)$ and $x_\alpha \nleq \phi_{\mathcal{G}_2}(A)$. Now, $x_\alpha \nleq \phi_{\mathcal{G}_1}(A) \Rightarrow$ there exists some $U_1 \in \mathcal{Q}(x_\alpha)$ such that $A * U_1 \notin \mathcal{G}_1$; and $x_\alpha \nleq \phi_{\mathcal{G}_2}(A) \Rightarrow$ there exists some $U_2 \in \mathcal{Q}(x_\alpha)$ such that $A * U_2 \notin \mathcal{G}_2$. Then $A * (U_1 \bigwedge U_2) \notin \mathcal{G}_1 \cup \mathcal{G}_2$, where $U_1 \bigwedge U_2 \in \mathcal{Q}(x_\alpha)$ so that $x_\alpha \nleq \phi_{\mathcal{G}_1 \cup \mathcal{G}_2}(A)$. Thus

$$\phi_{\mathcal{G}_1 \cup \mathcal{G}_2}(A) \leq \phi_{\mathcal{G}_1}(A) \bigvee \phi_{\mathcal{G}_2}(A) \tag{ii}$$

From (i) and (ii), the result follows.

(b) Let $y_\beta \leq \phi_{\mathcal{G}_1}(A) \bigwedge \phi_{\mathcal{G}_2}(A)$. Then $y_\beta \leq \phi_{\mathcal{G}_1}(A)$ and $y_\beta \leq \phi_{\mathcal{G}_2}(A)$. Thus for each $U \in \mathcal{Q}(y_\beta)$, $A * U \in \mathcal{G}_1$ and $A * U \in \mathcal{G}_2$. Let $A * U = G$. Then $G \in \mathcal{G}_1$, $G \in \mathcal{G}_2$ and $G \leq G$ give $G = G \bigwedge G \in \mathcal{G}_1 \wedge \mathcal{G}_2$, i.e., $A * U \in \mathcal{G}_1 \wedge \mathcal{G}_2$ and consequently $y_\beta \leq \phi_{\mathcal{G}_1 \wedge \mathcal{G}_2}(A)$. ■

We cite an example to show that the reverse inclusion in Theorem 2.14(b) is not true in general.

EXAMPLE 2.15. Let $X = \{a, b\}$ and $\tau = \{0_X, 1_X, A, B\}$, where $A(a) = 0.5$ and $A(b) = 0.6$, $B(a) = B(b) = 0.3$. Then (X, τ) is an fts. Let $\mathcal{G}_1$ consist of all fuzzy sets G_1 in X such that $0.6 \leq G_1(a) \leq 1$ and $0 \leq G_1(b) \leq 1$, and $\mathcal{G}_2$ consist of all fuzzy sets G_2 in X such that $0 \leq G_2(a) \leq 1$ and $0.1 \leq G_2(b) \leq 1$. Then $\mathcal{G}_1$ and $\mathcal{G}_2$ are fuzzy grills on X. Then by Proposition 2.13, $\mathcal{G}_1 \wedge \mathcal{G}_2$ is also a fuzzy grill on X. Now we see that the fuzzy point $b_{0.4} \leq \phi_{\mathcal{G}_1 \wedge \mathcal{G}_2}(A)$ since for each $U \in \mathcal{Q}(b_{0.4})$, $U(b) > 0.6$. So $(A * U)(a) > 0.1$, $(A * U)(b) > 0.2 \Rightarrow (A * U) \in \mathcal{G}_2$ and hence $(A * U) \in \mathcal{G}_1 \wedge \mathcal{G}_2$. Now we take $V = 1_X$, then $(A * V)(a) = 0.5$ and $(A * V)(b) = 0.6$. In this case, $(A * V) \notin \mathcal{G}_1$ which implies $b_{0.4} \nleq \phi_{\mathcal{G}_1}(A)$. Thus we have $b_{0.4} \nleq \phi_{\mathcal{G}_1}(A) \bigwedge \phi_{\mathcal{G}_2}(A)$ and hence $\phi_{\mathcal{G}_1 \wedge \mathcal{G}_2}(A) \nleq \phi_{\mathcal{G}_1}(A) \bigwedge \phi_{\mathcal{G}_2}(A)$.

THEOREM 2.16. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two fuzzy grills on an fts (X, τ). Then $\tau_{\mathcal{G}_1 \cup \mathcal{G}_2} = \tau_{\mathcal{G}_1} \cap \tau_{\mathcal{G}_2} = \tau_{\mathcal{G}_1 \wedge \mathcal{G}_2}$.*

Proof. (a) Since $\mathcal{G}_1$, $\mathcal{G}_2 \subseteq \mathcal{G}_1 \cup \mathcal{G}_2$, then by Theorem 2.10(i),

$$\tau_{\mathcal{G}_1 \cup \mathcal{G}_2} \subseteq \tau_{\mathcal{G}_1} \cap \tau_{\mathcal{G}_2} \tag{i}$$

Conversely let $V \in \tau_{\mathcal{G}_1} \cap \tau_{\mathcal{G}_2}$. Then $V \in \tau_{\mathcal{G}_1}$ and $V \in \tau_{\mathcal{G}_2}$. Thus $\phi_{\mathcal{G}_1}(1 - V) \leq 1 - V$ and $\phi_{\mathcal{G}_2}(1 - V) \leq 1 - V$ [since $(1 - V)$ is closed in $\tau_{\mathcal{G}_1}$ and $\tau_{\mathcal{G}_2}$]. Then $\phi_{\mathcal{G}_1}(1 - V) \bigvee \phi_{\mathcal{G}_2}(1 - V) \leq 1 - V \Rightarrow \phi_{\mathcal{G}_1 \cup \mathcal{G}_2}(1 - V) \leq (1 - V)$ [by Theorem 2.14(a)] $\Rightarrow V \in \tau_{\mathcal{G}_1 \cup \mathcal{G}_2}$. So

$$\tau_{\mathcal{G}_1} \cap \tau_{\mathcal{G}_2} \subseteq \tau_{\mathcal{G}_1 \cup \mathcal{G}_2} \tag{ii}$$

From (i) and (ii), $\tau_{\mathcal{G}_1 \cup \mathcal{G}_2} = \tau_{\mathcal{G}_1} \cap \tau_{\mathcal{G}_2}$. Now $\mathcal{G}_1 \wedge \mathcal{G}_2 \subseteq \mathcal{G}_1 \cup \mathcal{G}_2 \Rightarrow \tau_{\mathcal{G}_1 \cup \mathcal{G}_2} \subseteq \tau_{\mathcal{G}_1 \wedge \mathcal{G}_2}$. Also since $\mathcal{G}_1, \mathcal{G}_2 \subseteq \mathcal{G}_1 \wedge \mathcal{G}_2$, then $\tau_{\mathcal{G}_1 \wedge \mathcal{G}_2} \subseteq \tau_{\mathcal{G}_1}, \tau_{\mathcal{G}_2}$ (by Theorem 2.10(i)) and so

$\tau_{\mathcal{G}_1 \wedge \mathcal{G}_2} \subseteq \tau_{\mathcal{G}_1} \cap \tau_{\mathcal{G}_2}$. Thus we get $\tau_{\mathcal{G}_1} \cap \tau_{\mathcal{G}_2} = \tau_{\mathcal{G}_1 \cup \mathcal{G}_2} \subseteq \tau_{\mathcal{G}_1 \wedge \mathcal{G}_2} \subseteq \tau_{\mathcal{G}_1} \cap \tau_{\mathcal{G}_2}$ and hence $\tau_{\mathcal{G}_1 \cup \mathcal{G}_2} = \tau_{\mathcal{G}_1} \cap \tau_{\mathcal{G}_2} = \tau_{\mathcal{G}_1 \wedge \mathcal{G}_2}$.

We have already given the definition of the fuzzy derived set A^d of a fuzzy set A in an fts X. In the context of the present study, we now have:

Theorem 2.17. *In an fts (X, τ) and corresponding to a fuzzy grill $\mathcal{G}$ on X, $A^{d_\mathcal{G}} \leq A^d$ and $A^{d_\mathcal{G}} \leq \phi(A)$, for all fuzzy set A in X, where $A^{d_\mathcal{G}}$ denotes the fuzzy derived set of A in $(X, \tau_\mathcal{G})$.*

Proof. Let $x_\alpha \leq A^{d_\mathcal{G}}$. Since $\tau \subseteq \tau_\mathcal{G}$, every q-nbd of x_α in $(X, \tau_\mathcal{G})$ is quasi-coincident with A implies that every q-nbd of x_α in (X, τ) is quasi-coincident with A. Thus $x_\alpha \leq A^d$ so that $A^{d_\mathcal{G}} \leq A^d$. Again for any fuzzy point x_α in X, $x_\alpha \leq A^{d_\mathcal{G}}$ implies that $x_\alpha \leq \tau_\mathcal{G}\text{-}cl(A) = \psi(A) = A \bigvee \phi(A)$. Now if $x_\alpha \leq A$, then corresponding to each $U \in Q(x_\alpha)$ in $\tau_\mathcal{G}$, there exists $y \in X$ such that $x \neq y$ and $A(y) + U(y) > 1$. Thus x_α is a $\tau_\mathcal{G}$-accumulation point of the fuzzy set A^* such that

$$A^*(z) = \begin{cases} A(z), & \text{if } z \neq x \\ \alpha_1, & \text{if } z = x, \text{ where } 0 < \alpha_1 < \alpha. \end{cases}$$

Obviously $A^* \leq A$ so that $\phi(A^*) \leq \phi(A)$ and also $x_\alpha \nleq A^*$. Then $x_\alpha \leq \phi(A^*)$ since $x_\alpha \leq \tau_\mathcal{G}\text{-}cl(A^*)$, i.e., $x_\alpha \leq \phi(A^*) \leq \phi(A)$ so that $x_\alpha \leq \phi(A)$. Thus $x_\alpha \leq A^{d_\mathcal{G}} \Rightarrow x_\alpha \leq \phi(A)$ and hence $A^{d_\mathcal{G}} \leq \phi(A)$.

Theorem 2.18. *Let (X, τ) be an fts and $\mathcal{G}$ be a fuzzy grill on X. Then*

(a) for any fuzzy set $G \notin \mathcal{G}$ in X, $G^{d_\mathcal{G}} = 0_X$ and hence G is $\tau_\mathcal{G}$-closed.

(b) for any fuzzy set A in X, $\phi(A) = cl(A - H)$ for some fuzzy set $H \notin \mathcal{G}$.

Proof. (a) Since $G \notin \mathcal{G}$ then $\phi(G) = 0_X$ and hence by Theorem 2.17, $G^{d_\mathcal{G}} = 0_X$ and hence $\tau_\mathcal{G}\text{-}clG = G$.

(b) We first show that for any fuzzy set $G \notin \mathcal{G}$, one has

$$\phi(A) \leq cl(A - G) \tag{1}$$

In fact, for any fuzzy set $G \notin \mathcal{G}$, let $x_\alpha \nleq cl(A - G)$. Then there exists $V \in \mathcal{Q}(x_\alpha)$ such that $V \overline{q} (A - G)$, i.e.,

$$V(y) + (A - G)(y) \leq 1, \quad \text{for all } y \leq X \tag{2}$$

Let $y \in X$. If $A(y) \leq G(y)$ then $V(y) + (A - G)(y) = V(y) \leq 1$. Thus $A(y) + V(y) \leq 1 + G(y) \Rightarrow A(y) + V(y) - 1 \leq G(y)$. If $A(y) > G(y)$ then $V(y) + (A - G)(y) = V(y) + A(y) - G(y) \leq 1$ [by virtue of (2)]. Thus $V(y) + A(y) - 1 \leq G(y)$. So we have $V * A \leq G \notin \mathcal{G}$, i.e., $V * A \notin \mathcal{G}$. Then $x_\alpha \nleq \phi(A)$ and hence (1) is established.

On the other hand, let $x_\alpha \nleq \phi(A)$. Then there exists a $U \in \mathcal{Q}(x_\alpha)$ such that $A * U = H(\text{say}) \notin \mathcal{G}$. We claim that $(A - H) + U \leq 1_X$. Let $y \in X$. If $A(y) + U(y) > 1$, then $(A + U - 1_X)(y) = A(y) + U(y) - 1 = H(y)$. Then $A(y) + U(y) - H(y) = 1$, i.e.,

$$A(y) - H(y) + U(y) = 1 \tag{3}$$

If $A(y)+U(y)\le 1$ then $(A+U-1_X)(y)=0\Rightarrow H(y)=0$. Thus

$$A(y)+U(y)-H(y)\le 1 \quad \text{[since } H(y)=0] \tag{4}$$

From (3) and (4), we have $(A-H)+U\le 1_X$ and then $U\overline{q}(A-H)$. Also $U\in\mathcal{Q}(x_\alpha)\Rightarrow x_\alpha\nleq cl(A-H)$.Thus

$$cl(A-H)\le\phi(A) \tag{5}$$

From (1) and (5), $\phi(A)=cl(A-H)$. ∎

We now want to find a suitable fuzzy open base for the fuzzy topology $\tau_{\mathcal{G}}$. For this we require the following:

RESULT 2.19. *For any two fuzzy sets A and B in an fts (X,τ), $A-(A-B)\le B$.*

Proof. Here two cases will arise. For any $y\in X$,

Case-I: if $A(y)\le B(y)$, then $(A-B)(y)=0$ and hence $[A-(A-B)](y)=A(y)\le B(y)$.

Case-II: if $A(y)>B(y)$, then $[A-(A-B)](y)=A(y)-[A(y)-B(y)]=B(y)$. In both the cases we have $[A-(A-B)](y)\le B(y)$. Thus $A-(A-B)\le B$. ∎

THEOREM 2.20. *Let (X,τ) be an fts and $\mathcal{G}$ be a fuzzy grill on X. Then $\mathcal{B}(\mathcal{G},\tau)=\{V-A : V\in\tau$ and $A\notin\mathcal{G}\}$ is a fuzzy open base for $\tau_{\mathcal{G}}$.*

Proof. We first show that $\mathcal{B}(\mathcal{G},\tau)$ is a sub-collection of $\tau_{\mathcal{G}}$. Let $U\in\mathcal{B}(\mathcal{G},\tau)$. Then $U=V-A$ such that $V\in\tau$ and $A\notin\mathcal{G}$. We want to show that $\psi(1-U)=1-U$ for which it suffices to show that $\phi(1-U)\le 1-U$. If possible, let there exist a fuzzy point x_α such that

$$x_\alpha\le\phi(1-U) \tag{i}$$

but

$$x_\alpha\nleq(1-U) \tag{ii}$$

Now (i) implies that for each $W\in\mathcal{Q}(x_\alpha)$, $W+(1-U)-1\in\mathcal{G}$. It is easy to check that

$$W+(1-U)-1=W-U \tag{a}$$

Thus $W-U\in\mathcal{G}$. i.e.,

$$W-(V-A)\in\mathcal{G} \tag{iii}$$

Again by (ii), $x_\alpha\nleq(1-U)\Rightarrow\alpha>1-U(x)=1-(V-A)(x)\Rightarrow\alpha+V(x)>1+A(x)\ge 1$ [in fact, $V(x)\le A(x)$ is not possible since in that case $(V-A)(x)=0$ and so $1-(V-A)(x)=1-0<\alpha$. Thus $V(x)>A(x)$ and $(V-A)(x)=V(x)-A(x)$. Then $\alpha>1-(V-A)(x)=1-V(x)+A(x)\Rightarrow\alpha+V(x)>1+A(x)$].

Thus $V\in\mathcal{Q}(x_\alpha)$ and by (iii) above, $V-(V-A)\in\mathcal{G}$. Now $[V-(V-A)]\le A$(by Result 2.19) $\Rightarrow A\in\mathcal{G}$, which contradicts our hypothesis that $A\notin\mathcal{G}$. Thus $\phi(1-U)\le 1-U$. Hence $\mathcal{B}(\mathcal{G},\tau)$ is a sub-collection of $\tau_{\mathcal{G}}$. Next, let x_α be any fuzzy point in $(X,\tau_{\mathcal{G}})$ and U be an open q-nbd of x_α in $(X,\tau_{\mathcal{G}})$. Then by definition of q-nbd, there exists a $B \in \tau_{\mathcal{G}}$ such that $x_\alpha q B$ and $B \leq U$. Now $(1-B)$ is $\tau_{\mathcal{G}}$-closed and $\psi(1-B) = (1-B)$ which implies that $\phi(1-B) \leq 1-B$. Thus $x_\alpha \nleq \phi(1-B)$ which implies that there exists an open q-nbd V of x_α in (X,τ) such that $(1-B) * V \notin \mathcal{G}$. Then by (a), we have $(V-B) \notin \mathcal{G}$.

We now want to show that $x_\alpha q[V-(V-B)]$. Since $x_\alpha q B$ and $x_\alpha q V$, we have $\alpha + B(x) > 1$ and $\alpha + V(x) > 1$. Then if $V(x) \leq B(X)$, we have $\alpha + V(x) - (V-B)(x) = \alpha + B(x) > 1$, and if $V(x) > B(X)$, then also $\alpha + V(x) - (V-B)(x) = \alpha + V(x) - [V(x) - B(x)] = \alpha + V(x) - V(x) + B(x) = \alpha + B(x) > 1$. Thus we have for each fuzzy point x_α in $(X,\tau_{\mathcal{G}})$ and for each open q-nbd V of x_α in $(X,\tau_{\mathcal{G}})$, there exists a member $[V-(V-B)] \in \mathcal{B}(\mathcal{G},\tau)$ such that $x_\alpha q[V-(V-B)] \leq U$. Thus we say that $\mathcal{B}(\mathcal{G},\tau)$ is a base for $\tau_{\mathcal{G}}$. ∎

Corollary 2.21. *For any fuzzy grill $\mathcal{G}$ on an fts (X,τ), $\tau \subseteq \mathcal{B}(\mathcal{G},\tau) \subseteq \tau_{\mathcal{G}}$.*

Example 2.22. Let (X,τ) be an fts. If we take $\mathcal{G} = I^X \setminus \{0_X\}$ then $\tau_{\mathcal{G}} = \tau$. In fact, for any fuzzy basic open set $B = V - A$ (with $V \in \tau$ and $A \notin \mathcal{G}$) in $\tau_{\mathcal{G}}$, we have $A = 0_X$, so that $B = V \in \tau$. Hence by virtue of Corollary 2.21, $\tau = \mathcal{B}(\mathcal{G},\tau) = \tau_{\mathcal{G}}$.

3. Fuzzy topology suitable for a fuzzy grill

In this section, we now want to impose some suitability condition on the concerned grill $\mathcal{G}$ which makes $\tau_{\mathcal{G}}$ more well-behaved and compatible with the original fuzzy topology of the space X.

Definition 3.1. Let $\mathcal{G}$ be a fuzzy grill on an fts (X,τ). Then τ is said to be suitable for the fuzzy grill $\mathcal{G}$, if for every fuzzy set A in X : if corresponding to each fuzzy point $x_\alpha \leq A$, there exists a $U \in \mathcal{Q}(x_\alpha)$ such that $A * U \notin \mathcal{G}$, then $A \notin \mathcal{G}$.

Definition 3.2. For every fuzzy set A in X, we define the fuzzy set $\widetilde{A}$ by $\widetilde{A} = \{x_\alpha \leq A / x_\alpha \nleq \phi(A)\}$.

From the definition of $\widetilde{A}$, it is clear that $\widetilde{A} \bigwedge \phi(A) = 0_X$.

The following result gives us an expression for any fuzzy set A in terms of the operator ϕ.

Proposition 3.3. *For any fuzzy set A in X, $A = \widetilde{A} \bigvee (A \bigwedge \phi(A))$.*

Proof. By definition, $\widetilde{A} = \{x_\alpha \leq A / x_\alpha \nleq \phi(A)\}$. Thus $\widetilde{A} \leq A$. Also,

$$(A \bigwedge \phi(A)) \leq A \Rightarrow \widetilde{A} \bigvee (A \bigwedge \phi(A)) \leq A \tag{1}$$

On the other hand, let $x_\alpha \leq A$. If $x_\alpha \leq \phi(A)$, then $x_\alpha \leq A \bigwedge \phi(A)$, as otherwise $x_\alpha \leq \widetilde{A}$. In any case, $x_\alpha \leq \widetilde{A} \bigvee (A \bigwedge \phi(A))$, i.e.,

$$A \leq \widetilde{A} \bigvee (A \bigwedge \phi(A)) \tag{2}$$

From (1) and (2) the result follows. ∎

Proposition 3.4. *$\widetilde{A} \bigwedge \phi(\widetilde{A}) = 0_X$, for any fuzzy set A in X.*

Proof. If possible, let $x_\alpha \leq \widetilde{A} \bigwedge \phi(\widetilde{A})$. Then $x_\alpha \leq \widetilde{A}$ and $x_\alpha \leq \phi(\widetilde{A})$. Now,

$$x_\alpha \leq \widetilde{A} \Rightarrow x_\alpha \leq A \text{ but } x_\alpha \nleq \phi(A) \tag{1}$$

Also $x_\alpha \leq \phi(\widetilde{A}) \Rightarrow$ for all $U \in \mathcal{Q}(x_\alpha)$, $\widetilde{A} * U \in \mathcal{G}$. But $\widetilde{A} \leq A \Rightarrow A * U \in \mathcal{G}$, for all $U \in \mathcal{Q}(x_\alpha)$, i.e., $x_\alpha \leq \phi(A)$ which contradicts (1). ■

Theorem 3.5. *For a fuzzy grill $\mathcal{G}$ on a space (X, τ), the following conditions are equivalent:*

(a) τ is suitable for the grill $\mathcal{G}$.

(b) For any given fuzzy set A in X, $A \bigwedge \phi(A) = 0_X \Rightarrow A \notin \mathcal{G}$.

(c) $\widetilde{A} \notin \mathcal{G}$, for every fuzzy set A in X.

(d) For every fuzzy set A in X, if A contains no non-empty fuzzy set B with $B \leq \phi(B)$, then $A \notin \mathcal{G}$.

(e) For any $\tau_{\mathcal{G}}$-closed fuzzy set A in X, $\widetilde{A} \notin \mathcal{G}$.

Proof. (a) $\Rightarrow$ (b): Let for any given fuzzy set A in X, $A \bigwedge \phi(A) = 0_X$. Thus for each fuzzy point $x_\alpha \leq A$, we have $x_\alpha \nleq \phi(A)$. By definition of $\phi(A)$, there exists some $U_{x_\alpha} \in \mathcal{Q}(x_\alpha)$ such that $A * U_{x_\alpha} \notin \mathcal{G}$. Since τ is suitable for $\mathcal{G}$, we get $A \notin \mathcal{G}$.

(b) $\Rightarrow$ (c): Let $x_\lambda \leq \widetilde{A}$, i.e., $x_\lambda \leq A$ but $x_\lambda \nleq \phi(A)$. Then $x_\lambda \nleq \phi(\widetilde{A})$ as $\widetilde{A} \leq A$ [by Proposition 2.4(i)]. Thus $\phi(\widetilde{A})(x) = \lambda_0$ (say) $< \lambda$. If $\lambda_0 \neq 0$, then x_{λ_0} is a fuzzy point such that $x_{\lambda_0} \leq \phi(\widetilde{A})$ and $\phi(\widetilde{A}) \leq \phi(A)$. Thus we have

$$x_{\lambda_0} \leq \phi(A) \tag{1}$$

But $\lambda_0 < \lambda$ and $x_\lambda \leq \widetilde{A} \Rightarrow x_{\lambda_0} \leq \widetilde{A}$ and so $x_{\lambda_0} \nleq \phi(A)$ which contradicts (1). Hence $\lambda_0 = \phi(\widetilde{A})(x) = 0$. Thus for each $x \in supp(\widetilde{A})$, $\phi(\widetilde{A})(x) = 0$. So we have $\widetilde{A} \bigwedge \phi(\widetilde{A}) = 0_X$, for any fuzzy set A in X. Hence by condition (b), $\widetilde{A} \notin \mathcal{G}$.

(c) $\Rightarrow$ (d): Let A be any fuzzy set in X which contains no non-empty fuzzy set B with $B \leq \phi(B)$. By Proposition 3.3, $A = \widetilde{A} \bigvee (A \bigwedge \phi(A))$. Then $\phi(A) = \phi(\widetilde{A} \bigvee (A \bigwedge \phi(A))) = \phi(\widetilde{A}) \bigvee \phi(A \bigwedge \phi(A))$ [by Proposition 2.5 (i)]. Now by condition (c), $\widetilde{A} \notin \mathcal{G}$ so that $\phi(\widetilde{A}) = 0_X$ [by Proposition 2.4(iii)]. Thus $\phi(A \bigwedge \phi(A)) = \phi(A)$. But $A \bigwedge \phi(A) \leq A$ so that $A \bigwedge \phi(A) \leq \phi(A \bigwedge \phi(A))$. From the hypothesis we have $A \bigwedge \phi(A) = 0_X$ so that $A = \widetilde{A} \notin \mathcal{G}$.

(d) $\Rightarrow$ (c): Let A be any fuzzy set in X. Now $\widetilde{A} = \{x_\alpha \leq A / x_\alpha \nleq \phi(A)\}$. We claim that $\widetilde{A}$ does not contain any non-empty fuzzy set B such that $B \leq \phi(B)$. If possible, let B be a non-empty fuzzy set contained in $\widetilde{A}$ such that $B \leq \phi(B)$. Let $x_\alpha \leq B$, then $x_\alpha \leq \widetilde{A} \Rightarrow x_\alpha \leq A$ but

$$x_\alpha \nleq \phi(A) \tag{1}$$

Also $x_\alpha \leq B \leq \widetilde{A} \leq A$ implies that $\phi(B) \leq \phi(A)$ [by Proposition 2.4(i)]. But $x_\alpha \leq B \leq \phi(B) \leq \phi(A) \Rightarrow x_\alpha \leq \phi(A)$, contradicting (1). Thus our claim is justified and hence by (d), $\widetilde{A} \notin \mathcal{G}$.

(c)$\Rightarrow$(e): Obvious.

(e)$\Rightarrow$(a): Let A be any fuzzy set in X with the property that for every fuzzy point $x_\alpha \leq A$, there exists a $U \in \mathcal{Q}(x_\alpha)$ such that $A * U \notin \mathcal{G}$. Then $x_\alpha \not\leq \phi(A)$. Let $B = A \bigvee \phi(A)$. Then $\phi(B) = \phi(A \bigvee \phi(A)) = \phi(A) \bigvee \phi(\phi(A)) = \phi(A)$[by Proposition 2.5(ii)]. So $\tau_{\mathcal{G}}\text{-}cl(B) = \psi(B) = B \bigvee \phi(B) = A \bigvee \phi(A) \bigvee \phi(A) = B$. Hence B is $\tau_{\mathcal{G}}$-closed. Then by (e),

$$\widetilde{B} \notin \mathcal{G} \tag{1}$$

Now let $y_\alpha \leq \widetilde{B}$. Then $y_\alpha \leq B$ but $y_\alpha \not\leq \phi(B) = \phi(A)$. As $B = A \bigvee \phi(A)$, $y_\alpha \leq A$, i.e.,

$$\widetilde{B} \leq A \tag{2}$$

Now by hypothesis, if $x_\alpha \leq A$ then $x_\alpha \not\leq \phi(A) = \phi(B)$. Thus $x_\alpha \leq B$ but $x_\alpha \not\leq \phi(B)$ which gives $x_\alpha \leq \widetilde{B}$. Thus,

$$A \leq \widetilde{B} \tag{3}$$

From (2) and (3), $\widetilde{B} = A \notin \mathcal{G}$ [by (1)]. ∎

Theorem 3.6. *Let (X, τ) be an fts and $\mathcal{G}$ be a fuzzy grill on X. Then the following statements are equivalent and each is a necessary condition for the fuzzy topology τ to be suitable for the fuzzy grill $\mathcal{G}$.*

(a) For each fuzzy set A in X, $A \bigwedge \phi(A) = 0_X \Rightarrow \phi(A) = 0_X$.

(b) For each fuzzy set A in X, $\phi(\widetilde{A}) = 0_X$.

(c) For each fuzzy set A in X, $\phi(A \bigwedge \phi(A)) = \phi(A)$.

Proof. Here we shall show only the equivalence of (a) , (b) and (c). The rest follows from Theorem 3.5 and Proposition 2.4.

$(a) \Rightarrow (b)$: Let $A \in I^X$. By Proposition 3.4, we have $\widetilde{A} \bigwedge \phi(\widetilde{A}) = 0_X$. Then by (a), $\phi(\widetilde{A}) = 0_X$.

$(b) \Rightarrow (c)$: By Proposition 3.3, $A = \widetilde{A} \bigvee (A \bigwedge \phi(A))$, for any fuzzy set A in X. Then $\phi(A) = \phi(\widetilde{A} \bigvee (A \bigwedge \phi(A)) = \phi(\widetilde{A}) \bigvee \phi(A \bigwedge \phi(A)) = \phi(A \bigwedge \phi(A))$ [by (b)].

$(c) \Rightarrow (a)$: Let A be a fuzzy set in X and $A \bigwedge \phi(A) = 0_X$. Then by (c), $\phi(A) = \phi(A \bigwedge \phi(A)) = \phi(0_X) = 0_X$ [by Proposition 2.4(iii)]. ∎

Theorem 3.7. *If the topology τ of an fts X is suitable for a fuzzy grill $\mathcal{G}$ on X, then ϕ is an idempotent operator, i.e., $\phi(\phi(A)) = \phi(A)$, for any fuzzy set A in X.*

Proof. By Theorem 3.6 above, we have $\phi(A) = \phi(A \bigwedge \phi(A))$. Now by Proposition 2.5(i),

$$\phi(A) = \phi(A \bigwedge \phi(A)) \leq \phi(\phi(A)) \tag{1}$$

Also from Proposition 2.5(ii),

$$\phi(\phi(A)) \leq \phi(A) \tag{2}$$

From (1) and (2), the result follows.

THEOREM 3.8. *Let (X, τ) be an fts and $\mathcal{G}$ be a fuzzy grill on X such that τ is suitable for $\mathcal{G}$. Then any fuzzy set A in X is $\tau_{\mathcal{G}}$-closed if and only if it can be expressed as a union of a fuzzy set which is closed in (X, τ) and a fuzzy set which is not in $\mathcal{G}$.*

Proof. Let A be a $\tau_{\mathcal{G}}$-closed fuzzy set in X. Then $A = \phi(A) \bigvee A \Rightarrow \phi(A) \leq A$. Then by Proposition 3.3, $A = \widetilde{A} \bigvee (\phi(A))$. Since τ is suitable for $\mathcal{G}$, by Theorem 3.5, $\widetilde{A} \notin \mathcal{G}$ and by Proposition 2.5(ii), $\phi(A)$ is τ-closed.

Conversely, let $A = B \bigvee C$, where B is a τ-closed fuzzy set and C is a fuzzy set such that $C \notin \mathcal{G}$. Then $\phi(A) = \phi(B \bigvee C) = \phi(B) \bigvee \phi(C)$ [by Proposition 2.5(i)] $= \phi(B)$ [since $C \notin \mathcal{G} \Rightarrow \phi(C) = 0_X$] $\leq cl(B)$[by Proposition 2.5(ii)] $= B \leq A$, i.e., $A = A \bigvee \phi(A)$ and hence A is $\tau_{\mathcal{G}}$-closed. ■

EXAMPLE 3.9. Let (X, τ) be an fts and $\mathcal{G}$ be a fuzzy grill on X such that τ is suitable for $\mathcal{G}$. Let $\sigma = \{0_X, 1_X\}$ denote the fuzzy indiscrete topology on X, then 1_X is the only σ-open q-nbd of every fuzzy point x_α in X. Then for any fuzzy set A in X, $x_\alpha \leq \phi_{\mathcal{G}}(A) \Leftrightarrow A * 1_X \in \mathcal{G} \Leftrightarrow A \in \mathcal{G}$. Thus for every fuzzy set $A \in \mathcal{G}$, $\phi_{\mathcal{G}}(A) = 1_X$ and for every fuzzy set $A \notin \mathcal{G}$, $\phi_{\mathcal{G}}(A) = 0_X$. Then $\psi(A) = A \bigvee \phi_{\mathcal{G}}(A) = 1_X$ if $A \in \mathcal{G}$ and $\psi(A) = A$, if $A \notin \mathcal{G}$. Thus $\sigma_{\mathcal{G}} = \{U/(1-U) \notin \mathcal{G}\}$. Clearly $\sigma_{\mathcal{G}} \subseteq \tau_{\mathcal{G}}$ since for any $V \in \sigma_{\mathcal{G}}$, $1 - V \notin \mathcal{G}$ so that $1 - V$ is $\tau_{\mathcal{G}}$-closed(by Theorem 2.18(i)). So $V \in \tau_{\mathcal{G}}$.

Now for any fts (X, τ), we claim that $\tau_{\mathcal{G}} = \tau \cup \sigma_{\mathcal{G}}$. Indeed, $\tau \subseteq \tau_{\mathcal{G}}$ and $\sigma_{\mathcal{G}} \subseteq \tau_{\mathcal{G}}$ imply that $\tau \cup \sigma_{\mathcal{G}} \subseteq \tau_{\mathcal{G}}$. Conversely, let $U \in \tau_{\mathcal{G}}$. Then by Theorem 3.8, $1 - U = F \bigvee B$, where F is τ-closed and $B \notin \mathcal{G}$. Thus $U = (1 - F) \bigwedge (1 - B)$, where $(1 - F)$ is τ-open and $(1 - B)$ is $\sigma_{\mathcal{G}}$-open. Hence $U \in \tau \cup \sigma_{\mathcal{G}}$.

THEOREM 3.10. *If $\mathcal{G}$ be a fuzzy grill on an fts (X, τ) with τ is suitable for $\mathcal{G}$, then $\mathcal{B}(\mathcal{G}, \tau)$ is a fuzzy topology on X and hence $\mathcal{B}(\mathcal{G}, \tau) = \tau_{\mathcal{G}}$.*

Proof. We already have proved in Corollary 2.21,

$$\mathcal{B}(\mathcal{G}, \tau) \subseteq \tau_{\mathcal{G}} \tag{1}$$

On the other hand let A be $\tau_{\mathcal{G}}$-closed. Then $\tau_{\mathcal{G}}\text{-}cl(A) = A = A \bigvee \phi(A) \Rightarrow \phi(A) \leq A$. Thus $A = \widetilde{A} \bigvee (A \bigwedge \phi(A))$ [by Proposition 3.3] $= \widetilde{A} \bigvee \phi(A)$ [since $\phi(A) \leq A$].

Now by definition of $\widetilde{A}$, we have $\phi(A) \bigwedge \widetilde{A} = 0_X$. i.e.,

$$min\{\phi(A)(x), \widetilde{A}(x)\} = 0, \text{ for each } x \in X \tag{2}$$

Since τ is suitable for $\mathcal{G}$, by Theorem 3.5, $\widetilde{A} \notin \mathcal{G}$. Thus $A = \widetilde{A} \bigvee \phi(A)$ where $\phi(A)$ is τ-closed [by Proposition 2.5] and $\widetilde{A} \notin \mathcal{G}$. By (2), we can write $A = \phi(A) + \widetilde{A}$.

Then $1-A = 1-[\phi(A)+\widetilde{A}] = [1-\phi(A)]-\widetilde{A}$ [easily verifiable], where $[1-\phi(A)] \in \tau$ and $\widetilde{A} \notin \mathcal{G}$. Thus $(1-A) \in \mathcal{B}(\mathcal{G}, \tau)$. Hence in view of (1), $\mathcal{B}(\mathcal{G}, \tau) = \tau_{\mathcal{G}}$. ∎

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