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COMPREHENSIVE FAMILY OF k-UNIFORMLY HARMONIC STARLIKE FUNCTIONS

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Abstract. In this paper, we introduce a comprehensive family of k-uniformly harmonic univalent functions which contains various well-known classes of harmonic univalent functions as well as many new ones. Coefficient bounds, distortion bounds, extreme points, convolution conditions, and convex combination are determined for functions in this family. Further results on integral transforms are discussed. Consequently, many of our results are either extensions or new approaches to those corresponding previously known results.

1. Introduction and definitions

A continuous complex valued function $f = u+iv$ defined in a simply connected complex domain $\mathcal{D}$ is said to be harmonic in $\mathcal{D}$ if both u and v are real harmonic in $\mathcal{D}$. In any simply connected domain we can write $f = h + \overline{g}$, where h and g are analytic in $\mathcal{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense preserving in $\mathcal{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathcal{D}$.

Let $\mathcal{H}$ denote the family of functions $f = h+\overline{g}$ that are harmonic univalent and sense preserving in the unit disk $\mathcal{U} = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in \mathcal{H}$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \tag{1.1}$$

The harmonic function $f = h + \overline{g}$ for $g \equiv 0$ reduces to an analytic function $f = h$. Also let $\mathcal{TH}$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f = h + \overline{g}$, where the analytic functions h and g as

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \qquad g(z) = \sum_{n=1}^{\infty} |b_n| z^n \tag{1.2}$$

In 1984 Clunie and Sheil-Small [1] investigated the class $\mathcal{H}$ as well as its geometric subclasses and obtained some coefficient bounds. Since then, there has been

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several papers related on $\mathcal{H}$ and its subclasses. Jahangiri [3], Silverman [6], Silverman and Silvia [7] studied the harmonic starlike functions. Recently, Rosy et al. [5] defined the class $\mathcal{G}_{\mathcal{H}}(\alpha)$ consisting of functions $f = h + \overline{g}$ such that h and g are of the form (1.1) which satisfy the condition

$$\operatorname{Re}\left\{(1+e^{i\phi})\frac{zf'(z)}{z'f(z)} - e^{i\phi}\right\} > \alpha \tag{1.3}$$

where α $(0 \le \alpha < 1)$, $z' = \frac{\partial}{\partial\theta}$ $(z = re^{i\theta})$ and ϕ, θ are real. Also Rosy et al. [5] proved that if $f = h + \overline{g}$ is given by (1.1) and if

$$\sum_{n=1}^{\infty}\left(\frac{2n-1-\alpha}{1-\alpha}|a_n| + \frac{2n+1+\alpha}{1-\alpha}|b_n|\right) \le 2, \quad 0 \le \alpha < 1, \quad a_1 = 1, \tag{1.4}$$

then f is harmonic, univalent, and starlike of order α in $\mathcal{U}$. This condition is proved to be also necessary if $f \in \mathcal{TG}_{\mathcal{H}}(\alpha)$.

Very recently, Frasin [2] introduced and studied a generalized comprehensive class of harmonic univalent functions $f = h + \overline{g}$ denoted by $\mathcal{S}_{\mathcal{H}}(\Phi, \Psi; \alpha)$. and satisfying the criteria

$$\operatorname{Re}\left\{\frac{h(z) * \Phi(z) - \overline{g(z) * \Psi(z)}}{h(z) + \overline{g(z)}}\right\} > \alpha \tag{1.5}$$

where α $(0 \le \alpha < 1)$, $\Phi(z) = z + \sum_{n=2}^{\infty}\lambda_n z^n$ and $\Psi(z) = z + \sum_{n=2}^{\infty}\mu_n z^n$ are analytic in $\mathcal{U}$ with the conditions $\lambda_n \ge 0$, $\mu_n \ge 0$. Also Frasin [2] proved that if $f = h + \overline{g}$ is given by (1.1) and if

$$\sum_{n=1}^{\infty}\left(\frac{\lambda_n - \alpha}{1-\alpha}|a_n| + \frac{\mu_n + \alpha}{1-\alpha}|b_n|\right) \le 2, \quad a_1 = 1, \tag{1.6}$$

This condition is proved to be also necessary if $f \in \mathcal{TS}_{\mathcal{H}}(\Phi, \Psi; \alpha)$.

Motivated by Frasin [2] and Rosy et al. [5] we define a new comprehensive class of k-uniformly harmonic univalent functions $\mathcal{S}_{\mathcal{H}}(\Phi, \Psi; k, \alpha)$, the subclass of $\mathcal{H}$ consisting of functions $f = h + \overline{g} \in \mathcal{H}$ that satisfy the condition

$$\operatorname{Re}\left\{(1+ke^{i\phi})\frac{h(z) * \Phi(z) - \overline{g(z) * \Psi(z)}}{h(z) + \overline{g(z)}} - ke^{i\phi}\right\} > \alpha \tag{1.7}$$

where $\alpha(0 \le \alpha < 1)$, $\Phi(z) = z + \sum_{n=2}^{\infty}\lambda_n z^n$ and $\Psi(z) = z + \sum_{n=2}^{\infty}\mu_n z^n$ are analytic in $\mathcal{U}$ with the conditions $\lambda_n \ge 0$, $\mu_n \ge 0, k \ge 0$ and ϕ is real. The operator "$*$" stands for the Hadamard product or convolution of two power series. We further let $\mathcal{TS}_{\mathcal{H}}(\Phi, \Psi; k, \alpha)$ denote the subclass of $\mathcal{S}_{\mathcal{H}}(\Phi, \Psi; k, \alpha)$ consisting of functions $f = h + \overline{g} \in \mathcal{H}$ such that h and g are of the form (1.2).

The families $\mathcal{TS}_{\mathcal{H}}(\Phi, \Psi; 0, \alpha)$ and $\mathcal{TS}_{\mathcal{H}}(\Phi, \Psi; 1, \alpha)$ are of special interest because it contains various classes of well-known harmonic univalent functions as well as many new ones. For example $\mathcal{TS}_{\mathcal{H}}(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; 0, \alpha) \equiv \mathcal{T}_{\mathcal{H}}(\alpha)$ [3] and $\mathcal{TS}_{\mathcal{H}}(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; 1, \alpha) \equiv \mathcal{G}_{\mathcal{H}}(\alpha)$ [5].

In this paper, we obtain coefficient bounds, distortion bounds, extreme points, convolution conditions, and convex combination for functions in $\mathcal{TS}_{\mathcal{H}}(\Phi, \Psi; k, \alpha)$. Further results on integral transforms are also given .

2. Coefficients bounds

We begin with a sufficient condition for functions in $\mathcal{S}_{\mathcal{H}}(\Phi,\Psi;k,\alpha)$.

THEOREM 2.1. *Let the function $f=h+\overline{g}$ be such that h and g are given by* (1.1). *Furthermore, let*

$$\sum_{n=1}^{\infty}\left(\frac{\lambda_n(1+k)-(k+\alpha)}{1-\alpha}\,|a_n|+\frac{\mu_n(1+k)+(k+\alpha)}{1-\alpha}\,|b_n|\right)\le 2, \tag{2.1}$$

where $a_1=1$, $0\le\alpha<1$, $n(1-\alpha)\le\lambda_n(1+k)-(k+\alpha)$ and $n(1-\alpha)\le\mu_n(1+k)+(k+\alpha)$ for $n\ge 2$. Then f is harmonic univalent in $\mathcal{U}$, and $f\in\mathcal{S}_{\mathcal{H}}(\Phi,\Psi;k,\alpha)$.

Proof. First we note that f is locally univalent and sense-preserving in $\mathcal{U}$. This holds because

$$\begin{aligned}|h'(z)|&\ge 1-\sum_{n=2}^{\infty}n\,|a_n|\,r^{n-1}>1-\sum_{n=2}^{\infty}n\,|a_n|\ge 1-\sum_{n=2}^{\infty}\frac{\lambda_n(1+k)-(k+\alpha)}{1-\alpha}\,|a_n|\\&\ge\sum_{n=1}^{\infty}\frac{\mu_n(1+k)-(k+\alpha)}{1-\alpha}\,|b_n|\ge\sum_{n=2}^{\infty}n\,|b_n|>\sum_{n=2}^{\infty}n\,|b_n|\,r^{n-1}\ge|g'(z)|\,.\end{aligned}$$

To show that f is univalent in $\mathcal{U}$, suppose $z_1,z_2\in\mathcal{U}$ so that $z_1\ne z_2$, then

$$\begin{aligned}\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right|&\ge 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right|=1-\left|\frac{\sum_{n=1}^{\infty}b_n(z_1^n-z_2^n)}{(z_1-z_2)+\sum_{n=2}^{\infty}a_n(z_1^n-z_2^n)}\right|\\&>1-\left|\frac{\sum_{n=1}^{\infty}n\,|b_n|}{1-\sum_{n=2}^{\infty}n\,|a_n|}\right|\ge 1-\frac{\sum_{n=1}^{\infty}\frac{\mu_n(1+k)+(k+\alpha)}{1-\alpha}\,|b_n|}{1-\sum_{n=2}^{\infty}\frac{\lambda_n(1+k)-(k+\alpha)}{1-\alpha}\,|a_n|}\ge 0.\end{aligned}$$

Now, we show that $f\in\mathcal{S}_{\mathcal{H}}(\Phi,\Psi;k,\alpha)$. Using the fact that $\operatorname{Re}w\ge\alpha$ if and only if $|1-\alpha+w|\ge|1+\alpha-w|$, it suffices to show that

$$|A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)|\ge 0, \tag{2.2}$$

where $A(z)=(1+ke^{i\phi})(h(z)*\Phi(z))-\overline{g(z)*\Psi(z)}-ke^{i\phi}(h(z)+\overline{g(z)})$ and $B(z)=h(z)+\overline{g}(z)$.

Substituting for $A(z)$ and $B(z)$ in (2.2) and making use of (2.1) we obtain

$$\begin{aligned}&|A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)|\\&=\Big|(1-\alpha-ke^{i\phi})h(z)+(1+ke^{i\phi})(h(z)*\Phi(z))+(1-\alpha-ke^{i\phi})\overline{g(z)}\\&\quad-(1+ke^{i\phi})\overline{g(z)*\Psi(z)}\Big|-\Big|(1+\alpha+ke^{i\phi})h(z)-(1+ke^{i\phi})(h(z)*\Phi(z))\\&\quad+(1+\alpha+ke^{i\phi})\overline{g(z)}+(1+ke^{i\phi})\overline{g(z)*\Psi(z)}\Big|\end{aligned}$$

$$\geq (2-\alpha)\,|z| - \sum_{n=2}^{\infty}(\lambda_n(k+1)+1-\alpha-k)\,|a_n|\,|z|^n$$
$$- \sum_{n=1}^{\infty}(\mu_n(k+1)+k+\alpha-1)\,|b_n|\,|z|^n - \alpha\,|z|$$
$$- \sum_{n=2}^{\infty}(\lambda_n(k+1)-1-\alpha-k)\,|a_n|\,|z|^n - \sum_{n=1}^{\infty}(\mu_n(k+1)+k+\alpha+1)\,|b_n|\,|z|^n$$
$$= 2(1-\alpha)\,|z|\left\{1 - \sum_{n=2}^{\infty}\frac{\lambda_n(1+k)-(k+\alpha)}{1-\alpha}\,|a_n|\,|z|^{n-1}\right.$$
$$\left. - \sum_{n=1}^{\infty}\frac{\mu_n(1+k)+(k+\alpha)}{1-\alpha}\,|b_n|\,|z|^{n-1}\right\}$$
$$\geq 2(1-\alpha)\,|z|\left\{1 - \sum_{n=2}^{\infty}\frac{\lambda_n(1+k)-(k+\alpha)}{1-\alpha}\,|a_n|\right.$$
$$\left. - \sum_{n=1}^{\infty}\frac{\mu_n(1+k)+(k+\alpha)}{1-\alpha}\,|b_n|\right\} \geq 0.$$

The coefficient bound (2.1) is sharp for the function

$$f(z) = z + \sum_{n=2}^{\infty}\frac{1-\alpha}{\lambda_n(1+k)-(k+\alpha)}x_n z^n + \sum_{n=1}^{\infty}\frac{1-\alpha}{\mu_n(1+k)-(k+\alpha)}\overline{y_n z^n}, \qquad (2.3)$$

where $\sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n| = 1$.

The functions of the form (2.3) are in $\mathcal{S}_{\mathcal{H}}(\Phi,\Psi;k,\alpha)$ because

$$\sum_{n=1}^{\infty}\left(\frac{\lambda_n(1+k)-(k+\alpha)}{1-\alpha}\,|a_n| + \frac{\mu_n(1+k)+(k+\alpha)}{1-\alpha}\,|b_n|\right)$$
$$= 1 + \sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n| \leq 2. \quad \blacksquare$$

We next show that the above sufficient condition is also necessary for functions in $\mathcal{TS}_{\mathcal{H}}(\Phi,\Psi;k,\alpha)$.

Theorem 2.2. *Let the function $f = h + \overline{g}$ be such that h and g are given by (1.2). Then $f \in \mathcal{TS}_{\mathcal{H}}(\Phi,\Psi;k,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty}\left(\frac{\lambda_n(1+k)-(k+\alpha)}{1-\alpha}\,|a_n| + \frac{\mu_n(1+k)+(k+\alpha)}{1-\alpha}\,|b_n|\right) \leq 2, \qquad (2.4)$$

where $a_1 = 1$, $0 \leq \alpha < 1$, $n(1-\alpha) \leq \lambda_n(1+k)-(k+\alpha)$ and $n(1-\alpha) \leq \mu_n(1+k)+(k+\alpha)$ for $n \geq 2$.

Proof. The "if" part, follows from Theorem 2.1. To prove the "only if" part, let $f \in \mathcal{TS}_{\mathcal{H}}(\Phi,\Psi;k,\alpha)$; then from (1.7) we have

$$\operatorname{Re}\left\{(1+ke^{i\phi})\frac{h(z)*\Phi(z) - \overline{g(z)*\Psi(z)}}{h(z)+\overline{g(z)}} - (ke^{i\phi}+\alpha)\right\}$$
$$= \operatorname{Re}\left\{\left[(1-\alpha)z - \sum_{n=2}^{\infty}(\lambda_n(1+ke^{i\phi}) - (ke^{i\phi}+\alpha))\,|a_n|\,z^n\right.\right.$$
$$\left.\left. - \sum_{n=1}^{\infty}(\mu_n(1+ke^{i\phi}) + (ke^{i\phi}+\alpha))\,|b_n|\,\overline{z}^n\right] \Big/ \left[z - \sum_{n=2}^{\infty}|a_n|\,z^n + \sum_{n=1}^{\infty}|b_n|\,\overline{z}^n\right]\right\} > 0.$$

If we choose z to be real and $z \to 1^-$, and since $\operatorname{Re}(e^{i\phi}) \geq |e^{i\phi}| = 1$ we get

$$\frac{(1-\alpha) - \sum_{n=2}^{\infty} (\lambda_n(1+k) - (k+\alpha))\,|a_n| - \sum_{n=1}^{\infty} (\mu_n(1+k) + (k+\alpha))\,|b_n|}{1 - \sum_{n=2}^{\infty} |a_n| + \sum_{n=1}^{\infty} |b_n|} \geq 0,$$

or, equivalently,

$$\sum_{n=2}^{\infty} (\lambda_n(1+k) - (k+\alpha))\,|a_n| + \sum_{n=1}^{\infty} (\mu_n(1+k) + (k+\alpha))\,|b_n| \leq 1-\alpha$$

which is the required condition (2.4). ∎

Corollary 2.3. *Let the function $f = h + \overline{g}$ be such that h and g are given by (1.2). Then $f \in \mathcal{TS_H}(\Phi, \Psi; \alpha)$ is starlike of order $\frac{k+\alpha}{1+k}$.*

Proof. The proof is straightforward, substituting $\frac{k+\alpha}{1+k}$ for α in (1.6)to obtain

$$\sum_{n=1}^{\infty} \left(\frac{\lambda_n - \frac{k+\alpha}{1+k}}{1 - \frac{k+\alpha}{1+k}}\,|a_n| + \frac{\mu_n - \frac{k+\alpha}{1+k}}{1 - \frac{k+\alpha}{1+k}}\,|b_n| \right) \leq 2, \tag{2.5}$$

where $a_1 = 1$, $0 \leq \alpha < 1$, $n(1-\alpha) \leq \lambda_n(1+k) - (k+\alpha)$ and $n(1-\alpha) \leq \mu_n(1+k) + (k+\alpha)$ for $n \geq 2$, which is equivalent to the required condition (2.1). ∎

Taking different choices of $\Phi(z)$ and $\Psi(z)$ and suitable choice of k, in Theorem 2.2, we obtain the following corollaries:

Corollary 2.4. *Let the function $f = h + \overline{g}$ be such that h and g are given by* (1.2). *Then $f \in \mathcal{TS_H}(\Phi, \Psi; \alpha)$ if and only if*

$$\sum_{n=1}^{\infty} \left(\frac{\lambda_n - \alpha}{1-\alpha}\,|a_n| + \frac{\mu_n + \alpha}{1-\alpha}\,|b_n| \right) \leq 2, \tag{2.6}$$

where $a_1 = 1$, $0 \leq \alpha < 1$ $n(1-\alpha) \leq \lambda_n - \alpha$ and $n(1-\alpha) \leq \mu_n + \alpha$ for $n \geq 2$.

Corollary 2.5. *Let the function $f = h + \overline{g}$ be such that h and g are given by* (1.2). *Then $f \in \mathcal{TS_H}(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; k, \alpha)$ if and only if*

$$\sum_{n=1}^{\infty} \left(\frac{n(1+k) - k - \alpha}{1-\alpha}\,|a_n| + \frac{n(1+k) + k + \alpha}{1-\alpha}\,|b_n| \right) \leq 2, \tag{2.7}$$

where $a_1 = 1$, $0 \leq \alpha < 1$, $k \geq 0$.

Corollary 2.6. [5] *Let the function $f = h + \overline{g}$ be such that h and g are given by* (1.2). *Then $f \in \mathcal{TS_H}(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; 1, \alpha)$ if and only if*

$$\sum_{n=1}^{\infty} \left(\frac{2n - 1 - \alpha}{1-\alpha}\,|a_n| + \frac{2n + 1 + \alpha}{1-\alpha}\,|b_n| \right) \leq 2, \tag{2.8}$$

where $a_1 = 1$, $0 \leq \alpha < 1$.

Corollary 2.7. [3] *Let the function $f = h + \overline{g}$ be such that h and g are given by* (1.2). *Then $f \in \mathcal{TS}_{\mathcal{H}}(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; 0, \alpha)$ if and only if*

$$\sum_{n=1}^{\infty} \left(\frac{n-\alpha}{1-\alpha} |a_n| + \frac{n+\alpha}{1-\alpha} |b_n| \right) \leq 2, \tag{2.9}$$

where $a_1 = 1$, $0 \leq \alpha < 1$.

Corollary 2.8. [4] *Let the function $f = h + \overline{g}$ be such that h and g are given by* (1.2). *Then $f \in \mathcal{TS}_{\mathcal{H}}(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; k, \alpha)$ if and only if*

$$\sum_{n=1}^{\infty} \left(\frac{n^2(k+1)-\alpha-k}{1-\alpha} |a_n| + \frac{n(k+1)+k+\alpha}{1-\alpha} |b_n| \right) \leq 2, \tag{2.10}$$

where $a_1 = 1$, $0 \leq \alpha < 1$.

Corollary 2.9. *Let the function $f = h + \overline{g}$ be such that h and g are given by* (1.2). *Then $f \in \mathcal{TS}_{\mathcal{H}}(\frac{z+(1-2\beta)z^2}{(1-z)^{3-2\beta}}, \frac{z}{(1-z)^{2-2\beta}}; k, \alpha)$ if and only if*

$$\sum_{n=1}^{\infty} \left(\frac{nC(\beta,n)(k+1)-k-\alpha}{1-\alpha} |a_n| + \frac{C(\beta,n)(k+1)+k+\alpha}{1-\alpha} |b_n| \right) \leq 2, \tag{2.11}$$

where $a_1 = 1$, $0 \leq \alpha < 1$ and $C(\beta,n) = \prod_{i=2}^{n}(i-2\beta)/(n-1)$, $0 \leq \beta \leq 1$.

Remark 2.10. We note that Corollary 2.4 corrects the results obtained by Frasin [2, Theorems 2.1 and 2.2].

By Theorem 2.2 and using the techniques of [2, 3, 5, 6], we can state the following theorems without proof for functions in $\mathcal{TS}_{\mathcal{H}}(\Phi, \Psi; k, \alpha)$.

Theorem 2.11. (Distortion bounds) *Let $f \in \mathcal{TS}_{\mathcal{H}}(\Phi, \Psi; k, \alpha)$ and $\lambda_2 \leq \lambda_n$, $\lambda_2 \leq \mu_n$ for $n \geq 2$. Then we have*

$$|f(z)| \leq (1+|b_1|)r + \left(\frac{2(1-\alpha)-\lambda_1(1+k)+k+\alpha}{\lambda_2(1+k)-(k+\alpha)} - \frac{\mu_1(k+1)+k+\alpha}{\lambda_2(1+k)-(k+\alpha)} |b_1| \right) r^2, \tag{2.12}$$

$|z| = r < 1$, and

$$|f(z)| \geq (1+|b_1|)r - \left(\frac{2(1-\alpha)-\lambda_1(1+k)+k+\alpha}{\lambda_2(1+k)-(k+\alpha)} - \frac{\mu_1(k+1)+k+\alpha}{\lambda_2(1+k)-(k+\alpha)} |b_1| \right) r^2, \tag{2.13}$$

$|z| = r < 1$.

The following covering result follows from the left hand inequality in Theorem 2.11.

Corollary 2.12. *Let $f \in \mathcal{TS}_{\mathcal{H}}(\Phi, \Psi; k, \alpha)$ and $\lambda_2 \leq \lambda_n$, $\lambda_2 \leq \mu_n$ for $n \geq 2$. Then we have*

$$\left\{ w : |w| < \frac{(k+1)(\lambda_2+\lambda_1-2) + [(\mu_1-\lambda_2)(k+1)+2(k+\alpha)]\,|b_1|}{\lambda_2(1+k)-(k+\alpha)} \right\} \subset f(\mathcal{U}).$$

Theorem 2.13. (Extreme Points) *Let*

$$h_1(z) = z, \quad h_n(z) = z - \frac{1-\alpha}{\lambda_n(1+k)-(k+\alpha)} z^n \quad (n \geq 2)$$
$$g_n(z) = z + \frac{1-\alpha}{\mu_n(1+k)+(k+\alpha)} \overline{z}^n \quad (n \geq 1).$$

Then $f \in \mathcal{TS}_{\mathcal{H}}(\Phi,\Psi;\alpha)$ *if and only if it can be expressed as*

$$f(z) = \sum_{n=1}^{\infty} (x_n h_n + y_n g_n), \tag{2.14}$$

where $x_n \geq 0$, $y_n \geq 0$, $\sum_{n=1}^{\infty}(x_n + y_n) = 1$. *In particular, the extreme points of* $\mathcal{TS}_{\mathcal{H}}(\Phi,\Psi;k,\alpha)$ *are* $\{h_n\}$ *and* $\{g_n\}$.

3. Convolution and convex combinations

In this section, we show that the class $\mathcal{TS}_{\mathcal{H}}(\Phi,\Psi;k,\alpha)$ is invariant under convolution and convex combinations of its members.

For harmonic functions $f(z) = z - \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n$ and $F(z) = z - \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} B_n \overline{z}^n$ we define the convolution of two harmonic functions f and F as

$$(f * F)(z) = f(z) * F(z) = z - \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} b_n B_n \overline{z}^n.$$

Theorem 3.1. *If* $f \in \mathcal{TS}_{\mathcal{H}}(\Phi,\Psi;k,\alpha)$ *and* $F \in \mathcal{TS}_{\mathcal{H}}(\Phi,\Psi;k,\alpha)$ *then* $f * F \in \mathcal{TS}_{\mathcal{H}}(\Phi,\Psi;k,\alpha)$.

Proof. Let $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| \overline{z}^n$ and $F(z) = z - \sum_{n=2}^{\infty} |A_n| z^n + \sum_{n=1}^{\infty} |B_n| \overline{z}^n$ be in $\mathcal{TS}_{\mathcal{H}}(\Phi,\Psi;k,\alpha)$. Then by Theorem 2.2, we have

$$\sum_{n=1}^{\infty} \left(\frac{\lambda_n(1+k)-(k+\alpha)}{1-\alpha} |a_n| + \frac{\mu_n(1+k)+(k+\alpha)}{1-\alpha} |b_n| \right) \leq 2, \tag{3.1}$$

and

$$\sum_{n=1}^{\infty} \left(\frac{\lambda_n(1+k)-(k+\alpha)}{1-\alpha} |A_n| + \frac{\mu_n(1+k)+(k+\alpha)}{1-\alpha} |B_n| \right) \leq 2. \tag{3.2}$$

So for the coefficients of $f * F$ we can write

$$\sum_{n=1}^{\infty} \left(\frac{\lambda_n(1+k)-(k+\alpha)}{1-\alpha} |a_n A_n| + \frac{\mu_n(1+k)+(k+\alpha)}{1-\alpha} |b_n B_n| \right)$$
$$\leq \sum_{n=1}^{\infty} \left(\frac{\lambda_n(1+k)-(k+\alpha)}{1-\alpha} |a_n| + \frac{\mu_n(1+k)+(k+\alpha)}{1-\alpha} |b_n| \right) \leq 2.$$

Thus $f * F \in \mathcal{TS}_{\mathcal{H}}(\Phi,\Psi;k,\alpha)$. ∎

Now we show that $\mathcal{TS}_{\mathcal{H}}(\Phi,\Psi;k,\alpha)$ is closed under convex combinations of its members.

Corollary 3.2. *The class $\mathcal{TS}_{\mathcal{H}}(\Phi,\Psi;k,\alpha)$ is closed under convex combinations.*

Proof. For $i=1,2,3,\dots$ suppose that $f_i(z)\in\mathcal{TS}_{\mathcal{H}}(\Phi,\Psi;\alpha)$ where f_i is given by

$$f_i(z)=z-\sum_{n=2}^{\infty}|a_{i_n}|\,z^n+\sum_{n=1}^{\infty}|b_{i_n}|\,\overline{z}^n.$$

Then by (2.4),

$$\sum_{n=1}^{\infty}\left(\frac{\lambda_n(1+k)-(k+\alpha)}{1-\alpha}\,|a_{i_n}|+\frac{\mu_n(1+k)+(k+\alpha)}{1-\alpha}\,|b_{i_n}|\right)\le 2. \tag{3.3}$$

For $\sum_{i=1}^{\infty}t_i=1$, $0\le t_i\le 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty}t_if_i(z)=z-\sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty}t_i\,|a_{i_n}|\right)z^n+\sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty}t_i\,|b_{i_n}|\right)\overline{z}^n.$$

Then by (3.3),

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left[\frac{\lambda_n(1+k)-(k+\alpha)}{1-\alpha}\left|\sum_{i=1}^{\infty}t_i\,|a_{i_n}|\right|+\frac{\mu_n(1+k)-(k+\alpha)}{1-\alpha}\left|\sum_{i=1}^{\infty}t_i\,|b_{i_n}|\right|\right]\\
&=\sum_{n=1}^{\infty}t_i\left\{\sum_{n=1}^{\infty}\left[\frac{\lambda_n(1+k)-(k+\alpha)}{1-\alpha}\,|a_{i_n}|+\frac{\mu_n(1+k)-(k+\alpha)}{1-\alpha}\,|b_{i_n}|\right]\right\}\\
&\le 2\sum_{n=1}^{\infty}t_i=2
\end{aligned}$$

and so by Theorem 2.2, we have $\sum_{i=1}^{\infty}t_if_i(z)\in\mathcal{TS}_{\mathcal{H}}(\Phi,\Psi;k,\alpha)$. ■

Theorem 3.3. *If $f\in\mathcal{TS}_{\mathcal{H}}(\Phi,\Psi;k,\alpha)$ then f is convex in the disc*

$$|z|\le\min_n\left\{\frac{(1-|b_1|)(1-\alpha)}{n[2(1-\alpha)-(1+k)(\lambda_1+\mu_1)+(k+\alpha)(1-|b_1|)]}\right\}^{1/(n-1)},$$

$n=2,3,\dots$.

Proof. Let $f\in\mathcal{TS}_{\mathcal{H}}(\Phi,\Psi;k,\alpha)$ and let $r, 0<r<1$ be fixed. Then $r^{-1}f(rz)\in\mathcal{TS}_{\mathcal{H}}(\Phi,\Psi;k,\alpha)$ and we have

$$\begin{aligned}
\sum_{n=2}^{\infty}n^2(|a_n|+|b_n|)r^{n-1}&=\sum_{n=2}^{\infty}n(|a_n|+|b_n|)(nr^{n-1})\\
&\le\sum_{n=2}^{\infty}\left(\frac{\lambda_n(1+k)-(k+\alpha)}{1-\alpha}\,|a_n|+\frac{\mu_n(1+k)+(k+\alpha)}{1-\alpha}\,|b_n|\right)nr^{n-1}\\
&\le 1-|b_1|
\end{aligned}$$

provided

$$nr^{n-1} \leq \frac{(1-|b_1|)(1-\alpha)}{2(1-\alpha)-\lambda_1(1+k)+(k+\alpha)-\mu_1(1+k)-(k+\alpha)\,|b_1|}$$

which is true if

$$r \leq \min_n \left\{ \frac{(1-|b_1|)(1-\alpha)}{n[2(1-\alpha)-\lambda_1(1+k)+(k+\alpha)-\mu_1(1+k)-(k+\alpha)\,|b_1|]} \right\}^{1/(n-1)},$$

$n = 2, 3, \ldots.$ ■

4. Integral operator

Now, we will examine the closure properties of the class $\mathcal{TS_H}(\Phi, \Psi; k, \alpha)$ under the generalized Bernardi-Libera-Livingston integral operator $L_c(f)$ which is defined by

$$L_c(f) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\, dt, c > -1.$$

Theorem 4.1. *Let* $f(z) \in \mathcal{TS_H}(\Phi, \Psi; k, \alpha)$. *Then* $L_c(f(z)) \in \mathcal{TS_H}(\Phi, \Psi; k, \alpha)$

Proof. From the representation of $L_c(f(z))$, it follows that

$$\begin{aligned} L_c(f) &= \frac{c+1}{z^c} \int_0^z t^{c-1} \left[h(t) + \overline{g(t)} \right] dt. \\ &= \frac{c+1}{z^c} \left(\int_0^z t^{c-1} \left(t - \sum_{n=2}^{\infty} a_n t^n \right) dt + \overline{\int_0^z t^{c-1} \left(\sum_{n=1}^{\infty} b_n t^n \right) dt} \right) \\ &= z - \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} B_n z^n \end{aligned}$$

where $A_n = \frac{c+1}{c+n}\, a_n; B_n = \frac{c+1}{c+n}\, b_n$. Therefore,

$$\begin{aligned} &\sum_{n=1}^{\infty} \left[\frac{\lambda_n(1+k)-(k+\alpha)}{1-\alpha} \frac{c+1}{c+n} |a_n| + \frac{\mu_n(1+k)+(k+\alpha)}{1-\alpha} \frac{c+1}{c+n} |b_n| \right] \\ &\qquad \leq \sum_{n=1}^{\infty} \left[\frac{\lambda_n(1+k)-(k+\alpha)}{1-\alpha} |a_n| + \frac{\mu_n(1+k)+(k+\alpha)}{1-\alpha} |b_n| \right] \leq 2 \end{aligned}$$

Since $f(z) \in \mathcal{TS_H}(\Phi, \Psi; k, \alpha)$, therefore by Theorem 2.2,
$L_c(f(z)) \in \mathcal{TS_H}(\Phi, \Psi; k, \alpha)$. ■

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