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GRÜSS-TYPE INEQUALITIES FOR POSITIVE LINEAR OPERATORS WITH SECOND ORDER MODULI

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Abstract. We prove two Grüss-type inequalities for positive linear operator approximation, i.e., inequalities explaining the non-multiplicativity of such mappings. Instead of the least concave majorant of the first order modulus of continuity, we employ second order moduli of smoothness and show in the case of the classical Bernstein operators that in certain cases this leads to better results than those obtained earlier.

1. Introduction

In a recent paper Acu, Gonska and Rasa [1] studied the non-multiplicativity of linear positive operators $H : C[a,b] \to C[a,b]$ which reproduce constant functions. For fixed $x \in [a,b]$ we consider the positive linear functionals $L(f) = H(f;x)$. Below we study the differences

$$D(f,g;x) := H(fg;x) - H(f;x) \cdot H(g;x).$$

We first cite an earlier inequality in which $\tilde{\omega}$, the least concave majorant of the first order modulus of continuity, appears in the upper bound. For the definition of $\tilde{\omega}$ see [1], for example. Throughout the paper we will use the function $e_1(t) := t$, $t \in [a,b]$. The following result was obtained in [1], see Theorem 4 there.

THEOREM A. *If $f, g \in C[a,b]$ and $x \in [a,b]$ is fixed, then*

$$|D(f,g;x)| \le \frac{1}{4} \cdot \tilde{\omega}\Big(f; 2\sqrt{2H((e_1-x)^2;x)}\Big) \cdot \tilde{\omega}\Big(g; 2\sqrt{2H((e_1-x)^2;x)}\Big). \quad (1.1)$$

We note here that the constant $\sqrt{2}$ in the arguments of $\tilde{\omega}(f)$ and $\tilde{\omega}(g)$ in Theorem A can be removed.

If we choose $H = B_n$, the n-th Bernstein operator, then inequality (1.1) implies

$$|B_n(fg;x) - B_n(f;x) \cdot B_n(g;x)| \le \frac{1}{4} \cdot \tilde{\omega}\Big(f; 2\sqrt{\frac{2x(1-x)}{n}}\Big) \cdot \tilde{\omega}\Big(g; 2\sqrt{\frac{2x(1-x)}{n}}\Big), \quad (1.2)$$

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for $f, g \in C[0,1]$. Our goal is to modify the result in Theorem A for linear positive operators which reproduce linear functions. Instead of $\tilde{\omega}$ we measure the non-multiplicativity of H in terms of the second order modulus of continuity and the second order Ditizian-Totik modulus of smoothness. For their properties see the monograph [4]. The key for our new estimates are the following two general results. The first one can be found in Păltănea's book [2], see Corollary 2.2.1 there.

Theorem B. *If $L : C[a,b] \to C[a,b]$ is a positive linear operator reproducing linear functions then for $f \in C[a,b]$, $x \in [a,b]$ and each $0 < h$ the following holds:*

$$|(Lf)(x) - f(x)| \leq \left[1 + \frac{1}{2h^2} \cdot (L(e_1 - x)^2; x)\right] \cdot \omega_2(f; h). \tag{1.3}$$

The next result was proved by Gavrea et al. in [3].

Theorem C. *Under the conditions of Theorem B, if $[a,b] = [0,1], x \in (0,1)$, $h \in (0, \frac{1}{\sqrt{2}}]$ and $\varphi(x) = \sqrt{x(1-x)}$, then we have*

$$|(Lf)(x) - f(x)| \leq \left[\frac{3}{2} + \frac{3}{2} \cdot \frac{(L(e_1 - x)^2; x)}{(h\varphi(x))^2}\right] \cdot \omega_2^{\varphi}(f; h). \tag{1.4}$$

In Section 2 we prove our new estimates. In Section 3 we give applications to some classical linear positive operators and show the advantages of our estimates if compared to Theorem A.

2. Main results

Our first main result states the following:

Theorem 1. *If $f, g \in C[a,b]$, $x \in [a,b]$ is fixed and $H : C[a,b] \to C[a,b]$ is a positive linear operator reproducing linear functions, then the following holds:*

$$|D(f,g;x)| \leq \frac{3}{2} M(f) \cdot M(g),$$

$$M(f) := \sqrt{\omega_2(f^2; \sqrt{H((e_1 - x)^2; x)}) + 2\|f\| \cdot \omega_2(f; \sqrt{H((e_1 - x)^2; x)})}, \tag{2.1}$$

and $M(g)$ is defined analogously.

Proof. It was proved in [1]—see inequality (8) there—that

$$|D(f,g;x)| \leq \sqrt{D(f,f;x)} \cdot \sqrt{D(g,g;x)}.$$

We proceed as follows:

$$\begin{aligned} D(f,f;x) &= H(f^2;x) - f^2(x) + f^2(x) - (H(f;x))^2 \\ &= H(f^2;x) - f^2(x) + [f(x) - H(f;x)] \cdot [f(x) + H(f;x)]. \end{aligned}$$

Hence

$$\begin{aligned} |D(f,f;x)| &\leq |H(f^2;x) - f^2(x)| + |f(x) - H(f;x)| \cdot (\|f\| + \|Hf\|) \\ &\leq |H(f^2;x) - f^2(x)| + |f(x) - H(f;x)| \cdot \|f\|(1 + \|H\|). \end{aligned}$$

Due to positivity and preservation of constant functions we have $\|H\| = 1$.

If $H((e_1-x)^2;x)=0$, then $H(f;x)=f(x)$ for all $f\in C[a,b]$. Hence both sides in (2.1) equal 0.

Otherwise we apply (1.3) with $h:=\sqrt{H((e_1-x)^2;x)}>0$ to get

$$|D(f,f;x)|\le \frac{3}{2}\cdot\omega_2(f^2;\sqrt{H((e_1-x)^2;x)})+\frac{3}{2}\cdot 2\|f\|\cdot\omega_2(f;\sqrt{H((e_1-x)^2;x)}). \tag{2.2}$$

The same holds for the function $g(x)$. The proof of Theorem 1 is complete. ∎

If $x\in(0,1)$, $\varphi(x)=\sqrt{x(1-x)}$ we now set

$$h:=\frac{\sqrt{H((e_1-x)^2;x)}}{\sqrt{2}\cdot\varphi(x)}\ge 0.$$

Our next result is the following

Theorem 2. *If $f,g\in C[0,1]$, $x\in[0,1]$ is fixed and $H:C[0,1]\to C[0,1]$ is a positive linear operator reproducing linear functions, then the following holds*

$$|D(f,g;x)|\le\frac{9}{2}\sqrt{A(f)}\cdot\sqrt{A(g)}, \tag{2.3}$$

where $A(f):=\omega_2^{\varphi}(f^2;h)+2\|f\|\cdot\omega_2^{\varphi}(f;h)$, *and h is defined as above.*

Proof. Again, if $H((e_1-x)^2;x)=0$, then $h=0$, and the statement of (2.3) is trivial since H interpolates at x. Otherwise, we use the same decomposition as in the proof of Theorem 1 and (1.4). Note that

$$\frac{H((e_1-x)^2;x)}{\varphi^2(x)}\le 1$$

due to the properties of H. Hence $h\le\frac{1}{\sqrt{2}}$, Theorem C is applicable, and this concludes the proof. ∎

3. Applications

3.1. Bernstein operator

If $H:=B_n$ where B_n is the Bernstein operator given by

$$B_n(f;x)=\sum_{k=0}^{n}f(\frac{k}{n})\binom{n}{k}x^k(1-x)^{n-k},\quad x\in[0,1],\ n=1,2,\ldots,$$

then it is known that $B_n((e_1-x)^2;x)=\frac{x(1-x)}{n}$. From Theorem 1 we obtain, writing D_n instead of D now,

Corollary 1. *If $f,g\in C[0,1]$, then for all $x\in[0,1]$ and $n\ge 1$ one has*

$$|D_n(f,g;x)|\le\frac{3}{2}\cdot\sqrt{B(f;x)}\cdot\sqrt{B(g;x)}, \tag{3.1}$$

where

$$B(f;x):=\omega_2(f^2;\sqrt{\frac{x(1-x)}{n}})+2\|f\|\cdot\omega_2(f;\sqrt{\frac{x(1-x)}{n}}).$$

Analogously from Theorem 2 we arrive at

Corollary 2. *If $f, g \in C[0,1]$, then for all $x \in [0,1]$ we have*

$$|D_n(f,g;x)| \le \frac{9}{2} \cdot \sqrt{C(f)} \cdot \sqrt{C(g)}, \tag{3.2}$$

where

$$C(f) := \omega_2^{\varphi}(f^2; \frac{1}{\sqrt{2n}}) + 2\|f\| \cdot \omega_2^{\varphi}(f; \frac{1)}{\sqrt{2n}}).$$

If one of the functions f *or* g is a constant, then in both sides of (2.1), (2.2), (3.1) and (3.2) we have 0, so we have equality in non-trivial cases. The same is fulfilled in Theorem A, too. This observation is important, because it is possible to obtain upper bounds for $|D_n(f,g;x)|$ in terms of ω_2 or ω_2^{φ}, different from those in Theorems 1 and 2, which are not equal to 0 if one of the functions f or g is a constant. For example, we may proceed as follows, suppressing x for the moment:

$$\begin{aligned} D(f,g) &= H(fg) - H(f)H(g) \\ &= H(fg) - fg + fg - fH(g) + fH(g) - H(f)H(g) \\ &= [H(fg) - fg] + f[g - H(g)] + H(g)[f - H(f)]. \end{aligned}$$

If $H = B_n$ is again the Bernstein operator, then

$$\begin{aligned} |D_n(f,g;x)| \le \frac{3}{2} \cdot \omega_2(f \cdot g; \sqrt{\frac{x(1-x)}{n}}) + |f(x)| \cdot \frac{3}{2} \cdot \omega_2(g; \sqrt{\frac{x(1-x)}{n}}) + \\ + |B_n(g)(x)| \cdot \frac{3}{2} \cdot \omega_2(f; \sqrt{\frac{x(1-x)}{n}}). \end{aligned} \tag{3.3}$$

It is clear that, if $f =$ const or $g =$ const, then the left side is 0, but this is not necessarily true for the right hand side.

Example 1. Let $f(x) = x \ln x$, $x \in (0,1]$, $f(0) = 0$, and $g(x) = x$. We will show that the estimate (3.2) is better than (1.1). Let $x = \frac{1}{2}$. It is easy to calculate that

$$\tilde{\omega}\Big(f; \sqrt{\frac{x(1-x)}{n}}\Big) \approx \frac{\ln n}{\sqrt{n}}, \quad \text{and} \quad \tilde{\omega}\Big(g; \sqrt{\frac{x(1-x)}{n}}\Big) = \frac{1}{2\sqrt{n}}.$$

Now from (1.2) it follows that $|D_n(f,g;\frac{1}{2})| = O(\frac{\ln n}{n})$, $n \to \infty$. On the other hand,

$$\omega_2^{\varphi}(g; \frac{1}{\sqrt{2n}}) = 0,\ \omega_2^{\varphi}(g^2; \frac{1}{\sqrt{2n}}) = O(\frac{1}{n}).$$

To calculate the second order modulus of smoothness of f we apply the following estimate, valid for all functions f such that $\|\varphi^2 \cdot f''\|_{C[0,1]} < \infty$:

$$\omega_2^{\varphi}(f; \frac{1}{\sqrt{2n}}) \le C \cdot \frac{1}{2n} \cdot \|\varphi^2 \cdot f''\|_{C[0,1]}$$

for some positive absolute constant C, independent of n and f. Therefore

$$\omega_2^{\varphi}(f; \frac{1}{\sqrt{2n}}) \le C \cdot \frac{1}{2n}.$$

Also $(f^2(x))'' = 2.[\ln^2 x + 3\ln x + 1]$. Hence

$$\omega_2^{\varphi}(f^2; \frac{1}{\sqrt{2n}}) \le C \cdot \frac{1}{2n} \cdot \|\varphi^2.(f^2)''\|_{C[0,1]} = O(\frac{1}{n}).$$

We conclude that

$$|D_n(f, g; \frac{1}{2})| = O(\frac{1}{n}),\ n \to \infty,$$

and the last bound is better than that obtained in terms of the least concave majorant of ω_1. If x is close to the ends, for example $x = \frac{1}{n}$, then the estimate with $\tilde{\omega}$ is better than those involving ω_2 and ω_2^{φ}. In this case on the right side of (1.2) we have $O(\frac{\ln n}{n^2})$, on the right side of (2.1) only

$$O\Big(\frac{\sqrt{\ln n}}{n^{\frac{3}{2}}}\Big) \quad n \to \infty,$$

and in the right side of (2.2) only $O(\frac{1}{n})$. So when $x = \frac{1}{n}$ the better estimate is the one with $\tilde{\omega}$.

3.2 Schoenberg's variation—diminishing spline operator

To find a Grüss-type inequality for Schoenberg operators we need an upper bound for the second moment. For the case of equidistant knots we denote the Schoenberg operator by $S_{n,k}$. Consider the knot sequence $\Delta_n = \{x_i\}_{-k}^{n+k}$, $n \ge 1$, $k \ge 1$, with equidistant "interior knots", namely

$$\Delta_n : x_{-k} = \cdots = x_0 = 0 < x_1 < x_2 < \cdots < x_n = \cdots = x_{n+k} = 1$$

and $x_i = \frac{i}{n}$ for $0 \le i \le n$. For a bounded real-valued function f defined over the interval $[0, 1]$ the variation-diminishing spline operator of degree k w.r.t. Δ_n is given by

$$S_{n,k}(f, x) = \sum_{j=-k}^{n-1} f(\xi_{j,k}) \cdot N_{j,k}(x)$$

for $0 \le x < 1$ and $S_{n,k}(f, 1) = \lim_{y \to 1,\, y<1} S_{n,k}(f, y)$ with the nodes (Greville abscissas)

$$\xi_{j,k} := \frac{x_{j+1} + \cdots + x_{j+k}}{k},\ -k \le j \le n-1,$$

and the normalized $B-$ splines as fundamental functions

$$N_{j,k}(x) := (x_{j+k+1} - x_j)\dot{[}x_j, x_{j+1}, \ldots, x_{j+k+1}](\cdot - x)_+^k.$$

A main result in [5] states that for $n \ge 1$, $k \ge 1$, $x \in [0, 1]$ one has

$$S_{n,k}((e_1 - x)^2; x) \le 1 \cdot \frac{\min\{2x(1-x), \frac{k}{n}\}}{n + k - 1}. \tag{3.4}$$

We set

$$d(n, k, x) := \sqrt{\frac{\min\{2x(1-x), \frac{k}{n}\}}{n + k - 1}}.$$

From Theorem A we obtain

Corollary 3. *If $f, g \in C[0,1]$ and $x \in [0,1]$ is fixed, then the following inequality is true:*

$$|D(f,g)| \leq \frac{1}{4} \cdot \tilde{\omega}(f; 2\sqrt{2}d(n,k,x)) \cdot \tilde{\omega}(g; 2\sqrt{2}d(n,k,x)), \tag{3.5}$$

where $D(f,g) = S_{n,k}(f \cdot g; x) - S_{n,k}(f; x) \cdot S_{n,k}(g; x)$.

From Theorem 1 we get

Corollary 4. *If $f, g \in C[0,1]$ and $x \in [0,1]$ is fixed, then the following inequality holds:*

$$|D(f,g)| \leq \frac{3}{2} \cdot \sqrt{A_1(f)} \cdot \sqrt{A_1(g)}, \tag{3.6}$$

where $A_1(f) := \omega_2(f^2; d(n,k,x)) + 2\|f\| \cdot \omega_2(f; d(n,k,x))$.

If we set $h(n,k) := \sqrt{\frac{1}{n+k-1}}$, then Theorem 2 leads to

Corollary 5

$$|D(f,g)| \leq \frac{9}{2}\sqrt{A_2(f)} \cdot \sqrt{A_2(g)}, \tag{3.7}$$

where

$$A_2(f) := \omega_2^{\varphi}(f^2; h(n,k)) + 2\|f\| \cdot \omega_2^{\varphi}(f; h(n,k)).$$

In the last three corollaries, for $k = 1$ and $n \geq 1$, we obtain three different upper bounds which show how non-multiplicative the piecewise linear interpolant at equidistant points is.

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