МАТЕМАТИЧКИ ВЕСНИК
64, 1 (2012), 61–71
March 2012

оригинални научни рад
research paper

ADDITIONAL CHARACTERIZATIONS OF THE T_2 AND WEAKER SEPARATION AXIOMS

Charles Dorsett

Abstract. Within this paper, the weaker separation axioms of T_0, T_1, R_0, T_2, and R_1 are further characterized using mathematical induction, closed sets, convergence, and T_0-identification spaces. The results are used to further investigate general topological spaces, to further investigate constant nets and sequences, and finite nets and sequences in topological spaces.

1. Introduction

In 1978 [2] several weaker separation axioms were further characterized using convergence and other properties. Later, in 2007 [3], weaker separation axioms were further characterized by using mathematical induction. The results within those papers motivated the work within this paper.

Within this paper, all spaces are topological spaces.

2. New characterizations of T_0, T_1, and R_0 spaces.

Within the definition of the T_0 separation axiom, two distinct elements are used. Could the two distinct elements be extended to finitely many distinct elements in a similar manner? This question was resolved in a positive manner in the 2007 paper [3]. Let S_0 be the property of a space where for any n distinct elements in the space, $n \geq 2$, j of the elements can be separated from the remaining elements by an open set, $j < n$? Is there a T_0 space that is not S_0? Below this question and similar questions for other separation axioms are resolved and the results are used to further characterize the separation axioms using finitely many distinct elements, closed sets, and convergence.

For a space, a straightforward proof shows that a constant net or sequence converges to exactly each element of the closure of the constant. Under what condition would it converge to only the constant? If the net or sequence is finite, i.e., the net or sequence takes on only finitely many element values, and convergent,

2010 AMS Subject Classification: 54D10, 54A20.
Keywords and phrases: T_0-space; T_1-space; T_2-space; R_0-space; R_1-space.

under what conditions would the net or sequence eventually be a constant? Within this paper, these questions are also investigated and resolved using the results within this paper cited above.

THEOREM 2.1. *A space (X,T) is T_0 iff for finitely many distinct elements $x_1, \cdots, x_n$, $n \geq 2$, in X, there exists a closed set C containing all but one of the elements.*

Proof. Suppose (X,T) is T_0. Let $x_1, \cdots, x_n$, $n \geq 2$, be distinct elements of X. Then there exists an open set O containing only one of the n distinct elements [3] and $C = X \setminus O$ is closed and contains all but one of the distinct elements.

Conversely, suppose that for distinct elements $x_1, \cdots, x_n$, $n \geq 2$, there exists a closed set C containing all but one of the distinct elements. Then for distinct elements $x_1, \cdots, x_n$, $n \geq 2$, there exists an open set that contains only one of the distinct elements, which implies (X,T) is T_0 [3]. ■

THEOREM 2.2. *Let (X,T) be a space. Then (a) (X,T) is T_0 iff (b) for distinct elements $x_1, \cdots, x_n$, $n \geq 2$, in X, for each $j \in N$, $j < n$, there exists an open set containing exactly j of the distinct elements.*

Proof. (a) implies (b): By definition, the statement is true for $n = 2$. Assume the statement is true for $n < k$, $k > 2$.

Let $x_1, \cdots, x_k$ be k distinct elements of X. Let $j < k$. Consider the case that $j < k - 1$. Then $x_1, \cdots, x_{k-1}$ are distinct elements of X and there exists an open set U containing exactly j of the distinct elements. If $x_k \notin U$, then U is open and contains exactly j of the distinct elements $x_1, \cdots, x_k$. Thus, consider the case that $x_k \in U$. Then $\{x_i \mid x_i \in U\}$ is a set of $j + 1$ distinct elements of X, where $j + 1 < k$, and there exists an open set V containing exactly j of those distinct elements. Then $U \cap V$ is an open set containing exactly j of the distinct elements $x_1, \cdots, x_k$.

Consider the case that $j = k - 1$. From above, there exists an open set U containing exactly $k - 2$ of the distinct elements $x_1, \cdots, x_{k-1}$. If $x_k \in U$, then U is an open set containing exactly $k - 1$ of the distinct elements $x_1, \cdots, x_k$. Thus consider the case that $x_k \notin U$. Let $m < k$ such that $x_m \notin U$. Then x_m, x_k are 2 distinct elements of X and there exists an open set V containing only one of x_m, x_k. Then $U \cup V$ is an open set containing exactly $k-1$ of the distinct elements $x_1, \cdots, x_k$.

Hence, by mathematical induction, the statement is true for each natural number n.

The proof that (b) implies (a) is immediate letting $n = 2$. ■

THEOREM 2.3. *Let (X,T) be a space. Then (a) (X,T) is T_0 iff (b) for distinct elements $x_1, \cdots, x_n$, $n \geq 2$, in X, for each $j \in N$, $j < n$, there exists a closed set containing exactly j of the distinct elements.*

The proof is straightforward using Theorem 2.2 and is omitted.

Convergence of nets and sequences are used to define and characterize many properties in mathematics. Can T_0 spaces be characterized using convergence of nets and/or sequences? For example, what is the relationship between T_0 spaces and spaces where for distinct elements x and y, there exists a constant sequence converging to only one of x and y? Below this question is answered and similar questions for other weak separation axioms are investigated.

THEOREM 2.4. *Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is T_0, (b) for distinct elements x and y in X, there exists a constant sequence in X converging to only one of x and y, (c) for distinct elements x and y in X, there exists a constant net in X converging to only one of x and y, (d) for distinct elements x and y in X, there exists a net in X converging to only one of x and y, and (e) for distinct elements x and y in X, there exists a sequence in X converging to only one of x and y.*

Proof. (a) implies (b): Let x and y be distinct elements in X. Then there exists an open set O containing only one of x and y, say O contains only x. For each natural number n, let $x_n = y$. Then $\{x_n\}_{n\in N}$ is a constant sequence in X converging to y but not x.

Clearly, (b) implies (c) and (c) implies (d).

(d) implies (e): Suppose (X,T) is not T_0. Let x and y be distinct elements in X such that every open set containing one of x and y contains both x and y. For each $n \in N$, let $x_n = x$. Then the net $\{x_n\}_{n\in N}$ converges to both x and y, which is a contradiction. Thus (X,T) is T_0 and, from above, there exists a sequence in X converging to only one of x and y.

(e) implies (a): Let x and y be distinct elements in X. Then there exists a sequence in X converging to only one of x and y, which implies there is a net in X converging to only one of x and y and, by the argument above, (X,T) is T_0. ■

THEOREM 2.5. *Let (X,T) be a space. Then (a) (X,T) is T_1 iff (b) for distinct elements x and y in X, there exists a closed set containing y and not x.*

The proof is straightforward and is omitted.

The results above for T_0 spaces raised similar questions for T_1 and T_2 spaces, which are resolved below.

THEOREM 2.6. *Let (X,T) be a space. Then (a) (X,T) is T_1 iff (b) for distinct elements $x_1,\cdots,x_n$, $n \geq 2$, in X, for each nonempty, proper subset D of $F = \{x_i \mid i = 1,\cdots,n\}$, there exists a closed set C such that $C \cap F = D$.*

Proof. (a) implies (b): Let $x_1,\cdots,x_n$, $n \geq 2$, be distinct elements in X and let D be a nonempty, proper subset of $F = \{x_i \mid i = 1,\cdots,n\}$. Since singleton sets are closed, D is closed and $D \cap F = D$.

The proof of the converse is straightforward using $n = 2$ and Theorem 2.5 and is omitted. ■

Theorem 2.7. *Let* (X,T) *be a space. Then (a)* (X,T) *is* T_1 *iff (b) for distinct elements* $x_1, \cdots, x_n$, $n \geq 2$, *in* X, *for each nonempty, proper subset* D *of* $F = \{x_i \mid i = 1, \cdots, n\}$, *there exists an open set* O *such that* $O \cap F = D$.

The proof is straightforward using Theorem 2.6 and is omitted.

Theorem 2.8. *Let* (X,T) *be a space. Then the following are equivalent: (a)* (X,T) *is* T_1, *(b) each constant net in* X *converges to only the constant, (c) each constant sequence in* X *converges only to the constant, (d) for distinct elements* x *and* y *in* X, *there exists a constant sequence in* X *converging to* y *and not to* x, *(e) for distinct elements* x *and* y *in* X, *there exists a constant net in* X *converging to* y *and not to* x, *(f) for distinct elements* x *and* y *in* X, *there exists a net in* X *converging to* y *and not to* x, *and (g) for distinct elements* x *and* y *in* X, *there exists a sequence in* X *converging to* y *and not to* x.

Proof. (a) implies (b): Let $x \in X$. For each $n \in N$, let $x_n = x$. Then $\{x_n\}_{n \in N}$ is a net in X converges to x. Let $y \in X$, $y \neq x$. Then there exists an open set O containing y and not x, which implies $\{x_n\}_{n \in N}$ does not converge to y.

Clearly (b) implies (c), (c) implies (d), (d) implies (e), and (e) implies (f).

(f) implies (g): Let x and y be distinct elements of X. Let $\{x_\alpha\}_{\alpha \in A}$ be a net in X converging to y and not to x. If every open set containing x contains y, then $\{x_\alpha\}_{\alpha \in A}$ converges to x, which is a contradiction. Thus there exists an open set containing x and not y. Hence (X,T) is T_1, which implies (d), which implies (g).

Clearly (g) implies (f) and by the argument above, (X,T) is T_1. ■

Theorem 2.9. *Let* (X,T) *be a space. Then the following are equivalent: (a)* (X,T) *is* T_1, *(b) every convergent finite net in* X *is eventually exactly one of the net values, (c) every convergent finite net in* X *is eventually constant, (d) every convergent finite sequence in* X *is eventually constant, and (e) every convergent finite sequence in* X *is eventually exactly one of the sequence values.*

Proof: (a) implies (b): Let $\{x_\alpha\}_{\alpha \in A}$ be a convergent finite net in X. Let $\{x_i \mid i = 1, \cdots, n\}$ be the distinct net values. Let $x \in X$ such that the net converges to x. If $x \neq x_i$ for some $i \in \{1, \cdots, n\}$, then there exists an open set O such that O contains only x of $x, x_1, \cdots, x_n$, but the net is not eventually in O, which is a contradiction. Thus $x = x_i$ for some $i \in \{1, \cdots, n\}$.

Clearly (b) implies (c) and (c) implies (d).

(d) implies (e): Suppose (X,T) is not T_1. Let $x \in X$ such that $Cl(\{x\}) \neq \{x\}$. Let $y \in Cl(\{x\}$ such that $y \neq x$. For each odd natural number n, let $x_n = x$ and for each even natural number n, let $x_n = y$. Then $\{x_n\}_{n \in N}$ is a finite sequence converging to y that is not eventually constant, which is a contradiction. Thus (X,T) is T_1 and (b) is true, which implies (e).

Clearly (e) implies (d) and , by the argument above, (X,T) is T_1. ■

In 1943 [6] T_1 spaces were generalized to R_0 spaces.

DEFINITION 2.1. A space (X,T) is R_0 iff for each closed set C and each $x \notin C$, $Cl(\{x\}) \cap C = \phi$.

In past studies of R_0 spaces, T_0-identification spaces have proven to be a useful tool.

DEFINITION 2.2. Let R be the equivalence relation on the space (X,T) defined by xRy iff $Cl(\{x\}) = Cl(\{y\})$. Then the T_0-identification space of (X,T) is $(X_0, Q(X_0))$, where X_0 is the set of equivalence classes of R and $Q(X_0)$ is the decomposition topology on X_0 [7]. For each $x \in X$, let C_x denote the equivalence class containing x and let $P_X : (X,T) \to (X_0, Q(X_0))$ be the natural map.

As established below, T_0-identification spaces continue to be a useful tool.

THEOREM 2.10. *Let (X,T) be a space. Then (a) (X,T) is R_0 iff (b) for elements $x_1, \cdots, x_n$, $n \geq 2$, in X such that $Cl(\{x_i\}) = Cl(\{x_j\})$ iff $i = j$, for each nonempty, proper subset D of $F = \{x_i \mid i = 1, \cdots, n\}$, there exists an open set O such that $O \cap F = D$.*

Proof: (a) implies (b): Let $x_1, \cdots, x_n$, $n \geq 2$, be elements of X such that $Cl(\{x_i\}) = Cl(\{x_j\})$ iff $i = j$ and let D be a nonempty proper subset of $F = \{x_i \mid i = 1, \cdots, n\}$. Since (X,T) is R_0, then $(X_0, Q(X_0))$ is T_1 [4]. Since for each $i, j \in \{1, \cdots, n\}$, $i \neq j$, $Cl(\{x_i\}) \neq Cl(\{x_j\})$, $C_{x_1}, \cdots, C_{x_n}$ are distinct elements in X_0. Then $\mathcal{D} = \{C_{x_i} \mid x_i \in D\}$ is a nonempty proper subset of $\mathcal{F} = \{C_{x_i} \mid i = 1, \cdots, n\}$ and there exists $\mathcal{O} \in Q(X_0)$ such that $\mathcal{O} \cap \mathcal{F} = \mathcal{D}$. Thus $O = P_X^{-1}(\mathcal{O})$ is open in X such that $O \cap F = D$.

(b) implies (a): Let C be closed in X and let $x \notin C$. Let $y \in C$. Then $Cl(\{x\}) \neq Cl(\{y\})$ and there exists an open set O such that $y \in O$ and $x \notin O$. Hence $y \notin Cl(\{x\})$ and $Cl(\{x\}) \subset X \setminus C$. Thus (X,T) is R_0. ■

THEOREM 2.11. *Let (X,T) be a space. Then (a) (X,T) is R_0 iff (b) for elements $x_1, \cdots, x_n$, $n \geq 2$, in X such that $Cl(\{x_i\}) = Cl(\{x_j\})$ iff $i = j$, for each nonempty, proper subset D of $F = \{x_i \mid i = 1, \cdots, n\}$, there exists a closed set C such that $C \cap F = D$.*

The proof is straightforward using Theorem 2.9 and is omitted.

Within the paper [2] it was shown that a space (X,T) is R_0 iff for each $x \in X$, $Cl(\{x\})$ is the intersection of all open sets containing x, which can be used to give the following characterization of T_1 spaces.

COROLLARY 2.1. *A space (X,T) is T_1 iff each element of X is the intersection of all open sets containing the element.*

DEFINITION 2.3. Let (X,T) be a space and let $\{x_\alpha\}_{\alpha \in A}$ be a net in X. Then $lim\{x_\alpha\}_{\alpha \in A} = \{y \in X \mid \{x_\alpha\}_{\alpha \in A}$ converges to y$\}$.

THEOREM 2.12 *Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is R_0, (b) for each constant net $\{x_\alpha\}_{\alpha \in A}$ in X, $lim\{x_\alpha\}_{\alpha \in A} = C_x$, where*

$x = x_\alpha$, $\alpha \in A$, (c) for each constant sequence $\{x_n\}_{n\in N}$ in X, $lim\{x_n\}_{n\in N} = C_x$, where $x = x_n$, $n \in N$, (d) for elements x and y in X such that $Cl(\{x\}) \neq Cl(\{y\})$, there exists a constant sequence converging to y and not to x, (e) for x and y in X such that $Cl(\{x\}) \neq Cl(\{y\})$, there exists a constant net in X converging to y and not to x, (f) for elements x and y in X such that $Cl(\{x\}) \neq Cl(\{y\})$, there exists a net in X converging to y and not to x, (g) for elements x and y in X such that $Cl(\{x\}) \neq Cl(\{y\})$, there exists a sequence in X converging to y and not to x, (h) for each convergent finite net $\{x_\alpha\}_{\alpha\in A}$ in X such that for the distinct net values $x_1, \cdots, x_n$, $Cl(\{x_i\}) = Cl(\{x_j\})$ iff $i = j$, the net is eventually exactly one of the net values, (i) for each convergent finite net $\{x_\alpha\}_{\alpha\in A}$ in X such that for the distinct net values $x_1, \cdots, x_n$, $Cl(\{x_i\}) = Cl(\{x_j\})$ iff $i = j$, the net is eventually constant, (j) for each convergent finite sequence in X with distinct sequence values $x_1, \cdots, x_n$ such that $Cl(\{x_i\}) = Cl(\{x_j\})$ iff $i = j$, the sequence is eventually constant, and (k) for each convergent finite sequence in X with sequence values $x_1, \cdots, x_n$ such that $Cl(\{x_i\}) = Cl(\{x_j\})$ iff $i = j$, the sequence is eventually exactly one of the sequence values.

Proof. (a) implies (b): Let $\{x_\alpha\}_{\alpha\in A}$ be a constant net in X with $x = x_\alpha$, $\alpha \in A$. Then $\{C_{x_\alpha}\}_{\alpha\in A}$ is a constant net in X_0 with $C_x = C_{x_\alpha}$, $\alpha \in A$. Since (X,T) is R_0, $(X_0, Q(X_0))$ is T_1 and $\{C_{x_\alpha}\}_{\alpha\in A}$ converges only to C_x. If $y \in lim\{x_\alpha\}_{\alpha\in A}$, then $\{C_{x_\alpha}\}_{\alpha\in A}$ converges to C_y in X_0 and $y \in C_y = C_x$. If $z \in C_x$, then $Cl(\{z\}) = Cl(\{x\})$ and every open set containing z contains x, which implies $z \in lim\{x_\alpha\}_{\alpha\in A}$. Thus $lim\{x_\alpha\}_{\alpha\in A} = C_x$.

Clearly (b) implies (c).

(c) implies (d): Let x and y be elements in X such that $Cl(\{x\}) \neq Cl(\{y\})$. Then $C_x \neq C_y$. For each $n \in N$, let $y_n = y$. Then $L = lim\{y_n\}_{n\in N} = C_y$. Thus $y \in L$ and $x \notin L$.

Clearly (d) implies (e) and (e) implies (f).

(f) implies (g): Suppose (X,T) is not R_0. Let C be a closed set and $y \notin C$ such that $D = Cl(\{y\}) \cap C \neq \phi$. Let $x \in D$. Then $Cl(\{x\}) \neq Cl(\{y\})$ and there exists a net $\{y_\alpha\}_{\alpha\in A}$ in X converging to y and not to x, but every open set containing x contains y, which implies $\{y_\alpha\}_{\alpha\in A}$ converges to x and is a contradiction. Thus (X,T) is R_0 and by (d) above, (g) is satisfied.

(g) implies (h): Clearly (g) implies (f) and, by the argument above, (X,T) is R_0. Then $(X_0, Q(X_0))$ is T_1. Let $\{x_\alpha\}_{\alpha\in A}$ be a convergent finite net in X such that for the distinct net values $x_1, \cdots, x_n$, $Cl(\{x_i\}) = Cl(\{x_j\})$ iff $i = j$. Then $C_{x_1}, \cdots, C_{x_n}$ are the distinct elements in X_0 of the net $\{C_{x_\alpha}\}_{\alpha\in A}$ and is eventually C_{x_i} for some $i \in \{1, \cdots, n\}$. Thus the net in X is eventually x_i.

Clearly (h) implies (i) and (i) implies (j).

(j) implies (k): Let $\{x_n\}_{n\in N}$ be a convergent finite sequence in X with distinct elements $x_{n_1}, \cdots, x_{n_p}$ such that $Cl(\{x_{n_i}\}) = Cl(\{x_{n_j}\})$ iff $i = j$. Let $x \in X$ such that the sequence is eventually x. Then $x \in Cl(\{x_{n_i}\})$ for some $i \in \{1, \cdots, p\}$,for suppose not. Then the sequence is eventually in $X \setminus \cup_{i=1}^{p} Cl(\{x_{n_i}\})$, which is a contradiction. Let $i \in \{1, \cdots, p\}$ such that $x \in Cl(\{x_{n_i}\})$. Then the sequence

$\{x_i\}_{i\in N}$ is eventually in $X \setminus \cup_{j\neq i} Cl(\{x_{n_j}\})$, which implies the sequence is eventually x_{n_i}.

(k) implies (a): Suppose (X,T) is not R_0. Let C be closed in X and let $x \notin C$ such that $Cl(\{x\} \cap C \neq \phi$. Let $y \in C \cap Cl(\{x\})$. For each odd natural number, let $x_i = x$ and for each even number, let $x_i = y$. Then $\{x_i\}_{i\in N}$ is finite and converges to y, but is not eventually x or y, which is a contradiction. Hence, (X,T) is R_0. ■

The results above will be combined with the fact that a space (X,T) is R_0 iff $C_x = Cl(\{x\})$ for each $x \in X$ [4] to obtain the next characterization of R_0 spaces.

THEOREM 2.13. *Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is R_0, (b) for each convergent finite net $\{x_\alpha\}_{\alpha\in A}$ with distinct net values $x_1,\cdots,x_n$, the net is eventually in C_{x_i} for some $i \in \{1,\cdots,n\}$, and (c) for each convergent finite sequence $\{x_n\}_{n\in N}$ with distinct sequence values $x_{n_1},\cdots,x_{n_p}$, the sequence is eventually in $C_{x_{n_i}}$ for some $i \in \{1,\cdots,p\}$.*

Proof. (a) implies (b): Let $\{x_\alpha\}_{\alpha\in A}$ be a convergent finite net in X with distinct net values $x_1,\cdots,x_n$. Let $\{n_i \mid i = 1,\cdots,p\} \subset \{1,\cdots,n\}$ such that $\cup_{i=1}^n Cl(\{x_i\}) = \cup_{j=1}^p Cl(\{x_{n_j}\})$ and $Cl(\{x_{n_i}\}) = Cl(\{x_{n_j}\}$ iff $i = j$. Then $\{C_{x_\alpha}\}_{\alpha\in A}$ is a finite net in the T_1 space $(X_0, Q(X_0))$ with distinct net values $C_{x_{n_i}} = Cl(\{x_{n_i}\})$, $i = 1,\cdots,p\}$. Let $x \in X$ such that the net in X converges to x. Let $i \in \{1,\cdots,p\}$ such that $x \in Cl(\{x_{n_i}\})$. Let $\mathcal{O} \in Q(X_0)$ such that $C_{x_{n_i}} \in \mathcal{O}$. Then $x \in P_X^{-1}(\mathcal{O}) = O \in T$ and x is eventually in O, which implies $C_x = C_{x_{n_i}} \in \mathcal{O}$. Thus $\{C_{x_\alpha}\}_{\alpha\in A}$ converges to $C_{x_{n_i}}$ and, by the arguments above, is eventually $C_{x_{n_i}}$. Hence $\{x_\alpha\}_{\alpha\in A}$ is eventually in $C_{x_{n_i}}$.

Clearly (b) implies (c).

(c) implies (a): Suppose (X,T) is not R_0. Let C be closed in X and let $x \notin C$ such that $C \cap Cl(\{x\}) \neq \phi$. Let $y \in C \cap Cl(\{x\})$ such that $y \neq x$. Then $C_x \neq C_y$. For each odd natural number, let $x_n = x$ and for each even natural number n, let $x_n = y$. Then the sequence $\{x_n\}_{n\in N}$ is finite and converges to y, but the sequence is not eventually in C_x or C_y, which is a contradiction. Thus (X,T) is R_0. ■

THEOREM 2.14. *Let (X,T) be a space and let $x_1,\cdots,x_n$, $n \geq 2$, be elements of X such that $Cl(\{x_i\}) = Cl(\{x_j\})$ iff $i = j$. Then for each $j \in N$, $j < n$, there exists an open set containing exactly j of the elements $x_1,\cdots,x_n$.*

Proof. Let $j \in N$, $j < n$. Since $Cl(\{x_l\}) = Cl(\{x_k\})$ iff $l = k$, $C_{x_1},\cdots,C_{x_n}$ are distinct elements of the T_0 space $(X_0, Q(X_0))$ [7] and there exists an open set $\mathcal{O}$ in X_0 containing exactly j of the distinct elements $C_{x_1},\cdots,C_{x_n}$. Then $P_X^{-1}(\mathcal{O})$ is open in X and contains exactly j of the elements $x_1,\cdots,x_n$. ■

In a similar manner, the results for T_0 spaces given in Theorem 2.4 can be extended to all spaces.

3. New characterizations of T_2 and R_1 spaces.

In 1961 [1] T_2 spaces were generalized to R_1 spaces.

Definition 3.1. A space (X,T) is R_1 iff for x and y in X such that $Cl(\{x\}) \neq Cl(\{y\})$, there exist disjoint open sets U and V such that $Cl(\{x\}) \subset U$ and $Cl(\{y\}) \subset V$.

Below a characterization of R_1 spaces is used to further characterize T_2 spaces.

Theorem 3.1. *Let (X,T) be a space. Then (a) (X,T) is T_2 iff (b) for distinct elements x_1 and x_2 in X, there exist closed sets C_1 and C_2 such that C_i contains only x_i of x_1 and x_2, $i = 1, 2$, and $X = C_1 \cup C_2$.*

Proof. (a) implies (b): Let x_1 and x_2 be distinct elements of X. Then $Cl(\{x_i\}) = \{x_i\}$, $i = 1, 2$. Since (X,T) is T_2, (X,T) is R_1 and there exist closed sets C_i, $i = 1, 2$, such that C_i contains only x_i of x_1 and x_2, $i = 1, 2$, and $X = C_1 \cup C_2$ [2].

(b) implies (a): Let x_1 and x_2 be distinct elements of X. Let C_i, $i = 1, 2$, be closed sets containing only x_i of x_1 and x_2 such that $X = C_1 \cup C_2$. Then $x_1 \in O_1 = X \setminus C_2 \in T$, $x_2 \in O_2 = X \setminus C_1 \in T$, and $O_1 \cap O_2 = \phi$. Thus (X,T) is T_2. ∎

Theorem 3.2. *Let (X,T) be a space. Then (a) (X,T) is T_2 iff (b) for distinct elements $x_1, \cdots, x_n$, $n \geq 2$, there exist closed sets C_i, $i = 1, \cdots, n$, containing only x_i of $x_1, \cdots, x_n$ with $X = \bigcup_{i=1}^{n} C_i$.*

Proof. (a) implies (b): By Theorem 3.1, the statement is true for $n = 2$. Assume the statement is true for $n = k$, $k \geq 2$. Let $x_1, \cdots, x_k, x_{k+1}$ be distinct elements of X. Let K_i, $i = 1, \cdots, k$, be closed sets containing only x_i of $x_1, \cdots, x_k$ with $X = \bigcup_{i=1}^{k} K_i$. Let $N_{k+1} = \{l \in \{1, \cdots, k\} \mid x_{k+1} \in K_l\} \neq \phi$. Then for each $l \in N_{k+1}$, x_l and x_{k+1} are distinct elements in the T_2 space (K_l, T_{K_l}). For each $l \in N_{k+1}$, let M_l and P_l be closed sets in K_l such that M_l contains only x_l of x_l and x_{k+1}, P_l contains only x_{k+1} of x_l and x_{k+1}, and $K_l = M_l \cup P_l$. For each $i \in \{1, \cdots, k\}$, $i \notin N_{k+1}$, let $C_i = K_i$, for each $i \in N_{k+1}$, let $C_i = M_i$, and let $C_{k+1} = \bigcup_{i \in N_{k+1}} P_i$. Then C_i is a closed set containing only x_i of $x_1, \cdots, x_{k+1}$, $i = 1, \cdots, k+1$, and $X = \bigcup_{1=1}^{k+1} C_i$. Thus, by mathematical induction, the statement is true for each natural number n.

The proof that (b) implies (a) is straightforward using $n = 2$ and Theorem 3.1 and is omitted. ∎

Theorem 3.3. *Let (X,T) be a space. Then (a) (X,T) is T_2 iff (b) for distinct elements $x_1, \cdots, x_n$, $n \geq 2$, for each decomposition $\mathcal{D} = \{D_i \mid i = 1, \cdots, j\}$, $j \geq 2$, of $\{1, \cdots, n\}$, there exist closed sets C_l, $l = 1, \cdots, j$, such that $\{x_i \mid i \in D_l\} \subset C_l$, $l = 1, \cdots, j$, and $X = \bigcup_{l=1}^{j} C_l$.*

Proof. (a) implies (b): Let $x_1, \cdots, x_n$, $n \geq 2$, be distinct elements of X. Let K_i, $i = 1, \cdots, n$, be closed sets containing only x_i of $x_1, \cdots, x_n$ with $X = \bigcup_{i=1}^{n} K_i$.

Let $\mathcal{D} = \{D_i \mid i = 1, \cdots, j\}$ be a decomposition of $\{1, \cdots, n\}$, $j \geq 2$. For each $l \in \{1, \cdots, j\}$, let $D_l = \bigcup_{i \in D_l} K_i$. Then D_l, $l = 1, \cdots, j$, are closed sets satisfying the required properties.

The proof of (b) implies (a) is straightforward using $n = 2$ and $\mathcal{D} = \{\{1\}, \{2\}\}$ and is omitted. ∎

Theorem 3.4. *Let (X, T) be a space. Then (a) (X, T) is T_2 iff (b) for distinct elements $x_1, \cdots, x_n$, $n \geq 2$, for each decomposition $\mathcal{D} = \{D_i \mid i = 1, \cdots, j\}$, $j \geq 2$, of $\{1, \cdots, n\}$, there exist disjoint open sets O_l, $l = 1, \cdots, j$, such that $\{x_i \mid i \in D_l\} \subset O_l$, $l = 1, \cdots, j$.*

Proof. (a) implies (b): Let $x_1, \cdots, x_n$, $n \geq 2$, be distinct elements of X. Then there exist disjoint open sets U_i, $i = 1, \cdots, n$, containing only x_i of $x_1, \cdots, x_n$[ ]. Let $\mathcal{D} = \{D_i \mid i = 1, \cdots, j\}$ be a decomposition of $\{1, \cdots, n\}$, $j \geq 2$. For each $l \in \{1, \cdots, j\}$, let $O_l = \bigcup_{i \in D_l} U_i$. Then the open sets O_l, $l = 1, \cdots, j$, are disjoint open sets satisfying the required properties.

The proof of (b) implies (a) is straightforward using $n = 2$ and $\mathcal{D} = \{\{1\}, \{2\}\}$ and is omitted. ∎

Theorem 3.5. *Let (X, T) be a space. Then (a) (X, T) is T_2 iff (b) for each $x \in X$, $\{x\} = \bigcap_{x \in O \in T} Cl(O)$.*

The straightforward proof is omitted.

Theorem 3.6. *Let (X, T) be a space. Then (a) (X, T) is T_2 iff (b) for distinct elements x and y in X, for nets $\{x_\alpha\}_{\alpha \in A}$ and $\{y_\beta\}_{\beta \in B}$ in X converging to x and y respectively, there exists an $\alpha_0 \in A$ and a $\beta_0 \in B$ such that $\{x_\alpha \mid \alpha \geq \alpha_0\} \cap \{x_\beta \mid \beta \geq \beta_0\} = \phi$.*

Proof. The proof that (a) implies (b) is straightforward and is omitted.

(b) implies (a): Suppose (X, T) is not T_2. Let x and y be distinct elements of X such that every open set containing x intersects every open set containing y. Let $\mathcal{A} = \{U \cap V \mid x \in U \in T \text{ and } y \in V \in T\}$. Define $\geq$ on $\mathcal{A}$ by $A \geq B$ iff $A \subset B$. For each $A \in \mathcal{A}$, let $x_A \in A$ and let $y_A = x_A$. Then $\{x_A\}_{A \in \mathcal{A}}$ and $\{y_A\}_{A \in \mathcal{A}}$ are nets in X that converge to x and y respectively, but do not satisfy the required property. Hence (X, T) is T_2. ∎

Theorem 3.7. *Let (X, T) be a space. Then (a) (X, T) is R_1, (b) for elements $x, y \in X$ such that $Cl(\{x\}) \neq Cl(\{y\})$, there exist disjoint sets U and V such that $x \in U$ and $y \in V$, and (c) for elements $x_1, \cdots, x_n$, $n \geq 2$, such that $Cl(\{x_i\}) = Cl(\{x_j\})$ iff $i = j$, there exist closed sets C_i, $i = 1, \cdots, n$, containing only x_i of $x_1, \cdots, x_n$ with $X = \bigcup_{i=1}^{n} C_i$.*

Proof. Clearly (a) implies (b).

(b) implies (c): Let $O \in T$. Let $a \in O$. If $b \notin O$, $Cl(\{a\}) \neq Cl(\{b\})$ and there exist disjoint open sets A and B such that $a \in A$ and $b \in B$, which implies

$Cl(\{a\}) \subset O$. Let $x, y \in X$ such that $Cl(\{x\}) \neq Cl(\{y\})$. Let U and V be disjoint open sets such that $x \in U$ and $y \in V$. Then, by the argument above, $Cl(\{x\}) \subset U$ and $Cl(\{y\}) \subset V$. Hence (X,T) is R_1. Let $x_1, \cdots, x_n$, $n \geq 2$, be elements of X such that $Cl(\{x_i\}) = Cl(\{x_j\})$ iff $i = j$. Then C_{x_i}, $i = 1, \cdots, n$, are distinct elements of X_0. Since (X,T) is R_1, $(X_0, Q(X_0))$ is T_2 [5] and there exist closed sets $\mathcal{C}_i$, $i = 1, \cdots, n$, in X_0 containing only C_{x_i} of $C_{x_1}, \cdots, C_{x_n}$ with $X_0 = \bigcup_{i=1}^n \mathcal{C}_i$. Then $C_i = P_X^{-1}(\mathcal{C}_i)$, $i = 1, \cdots, n$, are closed sets in X containing only x_i of $x_1, \cdots, x_n$ with $X = \bigcup_{i=1}^n C_i$.

(c) implies (a): Let $x_1, x_2 \in X$ such that $Cl(\{x_1\}) \neq Cl(\{x_2\})$. Let C_1, C_2 be closed sets such that $x_1 \in C_1$, $x_2 \in C_2$, and $X = C_1 \cup C_2$. Then $x_1 \in U = X \setminus C_2$ and $x_2 \in V = X \setminus C_1$, where U and V are disjoint open sets. Thus, by the argument above, (X,T) is R_1. ■

THEOREM 3.8. *Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is R_1 iff (b) for elements $x_1, \cdots, x_n$, $n \geq 2$, such that $Cl(\{x_i\}) = Cl(\{x_j\})$ iff $i = j$, for each decomposition $\mathcal{D} = \{D_i \mid i = 1, \cdots, j\}$, $j \geq 2$, there exist closed sets C_l, $l = 1, \cdots, j$, such that $\{x_i \mid i \in D_l\} \subset C_l$, $l = 1, \cdots, j$, and $X = \bigcup_{l=1}^j C_l$.*

Proof. (a) implies (b): Let $x_1, \cdots, x_n$ be elements of X, $n \geq 2$, such that $Cl(\{x_i\}) = Cl(\{x_j\})$ iff $i = j$. Let $\mathcal{D} = \{D_i \mid i = 1, \cdots, j\}$, $j \geq 2$, be a decomposition of $\{1, \cdots, n\}$. Since (X,T) is R_1, $(X_0, Q(X_0))$ is T_2. Then C_{x_i}, $i = 1, \cdots, n$, are distinct elements of X_0. Let $\mathcal{C}_l$, $l = 1, \cdots, j$, be closed sets in X_0 such that $\{C_{x_i} \mid i \in D_l\} \subset \mathcal{C}_l$, $l = 1, \cdots, j$, and $X_0 = \bigcup_{l=1}^j \mathcal{C}_l$. Then $C_l = P_X^{-1}(\mathcal{C}_l)$, $l = 1, \cdots, j$, are closed sets in X such $\{x_i \mid i \in D_l\} \subset C_l$, $l = 1, \cdots, j$, and $X = \bigcup_{l=1}^j C_l$.

(b) implies (a): Let $x_1, \cdots, x_n$, $n \geq 2$, be elements of X such that $Cl(\{x_i\}) = Cl(\{x_j\})$ iff $i = j$. Let $\mathcal{D} = \{\{i\} \mid i = 1, \cdots, n\}$. Let C_i, $i = 1, \cdots, n$, be closed sets such that $\{x_i\} \subset C_i$ and $X = \bigcup_{i=1}^n C_i$. Thus, by theorem 3.7, (X,T) is R_1. ■

THEOREM 3.9. *Let (X,T) be a space. Then (a) (X,T) is R_1 iff (b) for elements $x_1, \cdots, x_n$, $n \geq 2$, such that $Cl(\{x_i\}) = Cl(\{x_j\})$ iff $i = j$, for each decomposition $\mathcal{D} = \{D_i \mid i = 1, \cdots, j\}$, $j \geq 2$, of $\{1, \cdots, n\}$, there exist disjoint open sets O_l, $l = 1, \cdots, j$, such that $\{x_i \mid i \in D_l\} \subset O_l$, $l = 1, \cdots, j$.*

Proof. (a) implies (b): Let $x_1, \cdots, x_n$, $n \geq 2$, be elements of X such that $Cl(\{x_i\}) = Cl(\{x_j\})$ iff $i = j$. Let $\mathcal{D} = \{D_l \mid l = 1, \cdots, j\}$, $j \geq 2$, be a decomposition of $\{1, \cdots, n\}$. Since (X,T) is R_1, $(X_0, Q(X_0))$ is T_2. Then $C_{x_1}, \cdots, C_{x_n}$ are distinct elements of X_0. Let $\mathcal{O}_l$, $l = 1, \cdots, j$, be disjoint open sets in X_0 such that $\{C_{x_i} \mid i \in D_l\} \subset \mathcal{O}_l$, $l = 1, \cdots, j$. Then $O_l = P_X^{-1}(\mathcal{O}_l)$ are disjoint open sets in X such that $\{x_i \mid i \in D_l\} \subset O_l$, $l = 1, \cdots, j$.

(b) implies (a): Let $x_1, x_2 \in X$ such that $Cl(\{x_1\}) \neq Cl(\{x_2\})$ and let $\mathcal{D} = \{\{1\}, \{2\}\}$. Then there exist disjoint open sets O_1 and O_2 such that $\{x_1\} \subset O_1$ and $\{x_2\} \subset O_2$ and, by Theorem 3.7, (X,T) is R_1. ■

THEOREM 3.10. *Let (X,T) be a space. Then (a) (X,T) is R_1 iff (b) for each $x \in X$, $Cl(\{x\}) = \bigcap_{x \in O \in T} Cl(O)$.*

Proof. (a) implies (b): Let $x \in X$. Let $O \in T$ such that $x \in O$. By the proof in Theorem 3.7 (b) implies (c), $Cl(\{x\}) \subset O$. Thus $Cl(\{x\}) \subset \bigcap_{x\in O\in T} O \subset \bigcap_{x\in O\in T} Cl(O)$. Let $y \in \bigcap_{x\in O\in T} Cl(O)$. Then $y \in Cl(\{x\})$, for suppose not. Then $Cl(\{x\}) \neq Cl(\{y\})$. Let U and V be disjoint open sets such that $Cl(\{x\}) \subset U$ and $Cl(\{y\}) \subset V$. Then $y \in \bigcap_{x\in O\in T} Cl(O) \subset Cl(U) \subset (X \setminus V)$, which is a contradiction. Thus $\bigcap_{x\in O\in T} \subset Cl(\{x\})$ and $Cl(\{x\}) = \bigcap_{x\in O\in T} Cl(O)$.

(b) implies (a): Let $x, y \in X$ such that $Cl(\{x\}) \neq Cl(\{y\})$. Then $x \notin Cl(\{y\})$ or $y \notin Cl(\{x\})$, say $y \notin Cl(\{x\})$. Let $O \in T$ such that $x \in O$ and $y \notin Cl(O)$. Then O an $X \setminus Cl(O)$ are disjoint open sets containing x and y respectively and by Theorem 3.7. (X, T) is R_1. ■

THEOREM 3.11. *Let (X, T) be a space. Then (a) (X, T) is R_1 iff (b) for elements x and y in X such that $Cl(\{x\}) \neq Cl(\{y\})$, for nets $\{x_\alpha\}_{\alpha\in A}$ and $\{y_\beta\}_{\beta\in B}$ in X converging to x and y respectively, there exists an $\alpha_0 \in A$ and a $\beta_0 \in B$ such that $\{x_\alpha \mid \alpha \geq \alpha_0\} \cap \{y_\beta \mid \beta \geq \beta_0\} = \phi$.*

Proof (a) implies (b): Let $x, y \in X$ such that $Cl(\{x\}) \neq Cl(\{y\})$. Let $\{x_\alpha\}_{\alpha\in A}$ and $\{y_\beta\}_{\beta\in B}$ be nets in X converging to x and y respectively. Let U and V be disjoint open sets such $x \in U$ and $y \in V$. Let $\alpha_0 \in A$ and $\beta_0 \in B$ such that $\{x_\alpha \mid \alpha \geq \alpha_0\} \subset U$ and $\{y_\beta \mid \beta \geq \beta_0\} \subset V$. Thus (b) is satisfied.

(b) implies (a): Let C_x and C_y be distinct elements of X_0. Let $\{C_{x_\alpha}\}_{\alpha\in A}$ and $\{C_{y_\beta}\}_{\beta\in B}$ be nets in X_0 converging to C_x and C_y respectively. Then $\{x_\alpha\}_{\alpha\in A}$ is a net in X. Let $O \in T$ such that $x \in O$. Then $C_x \in P_X(O) \in Q(X_0)$ and the net $\{C_\alpha\}_{\alpha\in A}$ is eventually in $P_X(O)$. Hence $\{x_\alpha\}_{\alpha\in A}$ is eventually in $P_X^{-1}(P_X(O)) = O$. Thus $\{x_\alpha\}_{\alpha\in A}$ converges to x. Similarly, $\{y_\beta\}_{\beta\in B}$ converges to y. Since $C_x \neq C_y$, $Cl(\{x\}) \neq Cl(\{y\})$. Let $\alpha_0 \in A$ and $\beta_0 \in B$ such that $\{x_\alpha \mid \alpha \geq \alpha_0\} \cap \{y_\beta \mid \beta \geq \beta_0\} = \phi$. Then $\{C_{x_\alpha} \mid \alpha \geq \alpha_0\} \cap \{C_{y_\beta} \mid \beta \geq \beta_0\} = \phi$. Thus $(X_0, Q(X_0))$ is T_2 and (X, T) is R_1. ■

REFERENCES

[**1**] A. Davis, *Indexed systems of neighborhoods for general topological spaces*, Amer. Math. Monthly **68** (1961), 886–893.

[**2**] C. Dorsett, *R_0 and R_1 topological spaces*, Mat. Vesnik **5(15)(30)** (1978), 117–122.

[**3**] C. Dorsett, *New characterizations of separation axioms*, Bull. Calc. Math. Soc. **99** (2007), 37–44.

[**4**] C. Dorsett, *Characterizations of spaces using T_0-identification spaces*, Kyungpook Math. J. **17** (1977), 175–179.

[**5**] W. Dunham, *Weakly Hausdorff spaces*, Kyungpook Math. J. **15** (1975), 41–50.

[**6**] N. Shanin, *On separations in topological spaces*, Doklady Akad. Nauk SSSR **48** (1943), 110–113.

[**7**] S. Willard, *General Topology*, Reading Massachusetts, Addison Wesley Pub. Co., 1970.

(received 06.10.2010; in revised form 14.04.2011; available online 01.07.2011)

Texas A&M University – Commerce

E-mail: Charles_Dorsett@tamu-commerce.edu