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STABILITY OF SOME INTEGRAL DOMAINS ON A PULLBACK

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Abstract. Let I be a nonzero ideal of an integral domain T and let $\varphi\colon T \to T/I$ be the canonical surjection. If D is an integral domain contained in T/I, then $R = \varphi^{-1}(D)$ arises as a pullback of type $\square$ in the sense of Houston and Taylor such that $R \subseteq T$ is a domains extension. The stability of atomic domains, domains satisfying ACCP, HFDs, valuation domains, PVDs, AVDs, APVDs and PAVDs observed on all corners of pullback of type $\square$ under the assumption that the domain extension $R \subseteq T$ satisfies *Condition* 1 : For each $b \in T$ there exist $u \in \cup(T)$ and $a \in R$ such that $b = ua$.

1. Introduction and preliminaries

Following Cohn [13], an integral domain R is said to be *atomic* if each nonzero nonunit element of R is a product of a finite number of irreducible elements (atoms) of R. The illustrious examples of atomic domains are UFDs and Noetherian domains. An integral domain R satisfies the *ascending chain condition on principal ideals* ($ACCP$) if there does not exist any strict ascending chain of principal ideals of R. An integral domain R satisfies ACCP if and only if $R[\{X_\alpha\}]$ satisfies ACCP for any family of indeterminates $\{X_\alpha\}$ (cf. [1, p. 5]). However, the polynomial extension an atomic domain is not an atomic domain (see [20]). A domain satisfying $ACCP$ is an atomic domain but the converse does not hold (see [15, 27]).

By [1], an atomic domain R is a *bounded factorization domain* (BFD) if for each nonzero nonunit element x of R, there is a positive integer $N(x)$ such that whenever $x = x_1 \cdots x_n$, a product of irreducible elements of R, then $n \leq N(x)$. The best known examples of BFDs are Noetherian and Krull domains [1, Proposition 2.2]. Also, in general a BFD satisfies ACCP but the converse is not true (cf. [1, Example 2.1]).

Following Zaks [26], an atomic domain R is a *half-factorial domain* (HFD) if for each nonzero nonunit element x of R, if $x = x_1 \cdots x_m = y_1 \cdots y_n$ with each x_i, y_j irreducible in R, then $m = n$. Obviously a UFD is an HFD. A Krull domain R is an HFD if divisor class group $Cl(R) \cong 0$ or $Cl(R) \cong \mathbb{Z}_2$. An HFD is a BFD

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(see [1]). By [1, Page 11], if $R[Y]$ is an HFD, then certainly R is an HFD. However, $R[Y]$ need not be an HFD if R is an HFD. For example $R = \mathbb{R} + X\mathbb{C}[X]$ is an HFD, but $R[Y]$ is not an HFD, as $(X(1+iY))(X(1-iY)) = X^2(1+Y^2)$ are decompositions into atoms of different lengths (cf. [1, p. 11]).

By [1], an integral domain R is known as an *idf-domain* if each nonzero nonunit element of R has at most a finite number of non-associate irreducible divisors. UFDs are examples of idf-domains. But there are idf-domains which are not even atomic. Moreover, the Noetherian domain $\mathbb{R} + X\mathbb{C}[X]$ is an HFD but not an idf-domain (cf. [1, Example 4.1(a)]).

By [1], an atomic domain R is a *finite factorization domain* (FFD) if each nonzero nonunit element of R has a finite number of non-associate divisors. Hence it has only a finite number of factorizations up to order and associates. Further, an integral domain R is an FFD if and only if R is an atomic idf-domain (cf. [1, Theorem 5.1]).

Following Cohn [13], an element x of an integral domain R is said to be *primal* if x divides a product a_1a_2; $a_1, a_2 \in R$, then x can be written as $x = x_1x_2$ such that x_i divides a_i, $i = 1, 2$. An element whose divisors are primal elements is called completely primal. An integral domain R is a *pre-Schreier* if every nonzero element x of R is primal. An integrally closed pre-Schreier domain is known as *Schreier* domain. By [13], any GCD-domain (an integral domain in which every pair of elements has a greatest common divisor) is a Schreier domain but the converse is not true.

By [24], an element x of an integral domain R is said to be *rigid* if whenever r, $s \in R$ and r, s divide x, then s divides r or r divides s. An integral domain R is said to be a *semirigid* domain if every nonzero element of R can be expressed as a product of a finite number of rigid elements.

We recall from [25] that: Let R be an integral domain.

property-$*$: $(\cap_i(a_i))(\cap_j(b_j)) = \cap_{i,j}(a_ib_j)$ for all a_i, $b_j \in R$, where $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

property-$**$: $((a) \cap (b))((c) \cap (d)) = (ac) \cap (ad) \cap (bc) \cap (bd)$, where a, b, c, $d \in R^*$.

An integral domain R is called *$*$-domain* (respectively *$**$-domain*) if it satisfies property -$*$ (respectively property-$**$). An integral domain R is said to be a *locally $*$-domain* if for each maximal ideal M, R_M has property-$*$.

Condition 1. The whole study in [18, 22, 23] is based on a property for a unitary commutative ring (respectively domain) extension, known as *Condition* 1. In [18, 22, 23] the stability (ascent and descent) of UFDs, atomic domains, domains satisfying ACCP, FFDs, BFDs, HFDs, RBFDs, CK-domains, BVDs, CHFDs, idf-domains, a particular case of LHFDs, valuation domains, semirigid domains, PVDs and GCD-domains, Schreier domains, pre-Schreier domains, $*$-domains, $**$-domains, locally $*$-domains has been observed for a domain extension $R \subseteq T$ which satisfy *Condition* 1. In most of the situations the assumption that works is, the conductor ideal $R:T=\{x\in R: xT\subseteq R\}$, the largest common ideal of R and T, is maximal in R.

Condition 1 : "Let $R\subseteq T$ be a unitary commutative ring (respectively domain) extension. For each $b\in T$ there exists $u\in U(T)$ and $a\in R$ such that $b=ua$."

The followings are a few examples of unitary (commutative) ring extensions which satisfy *Condition* 1.

EXAMPLE 1. [18, Example 1] (a) If T is a field, then the unnitart ommutative ring extension $R\subseteq T$ satisfies *Condition* 1.

(b) If T is a fraction ring of the ring R, then the ring extension $R\subseteq T$ satisfies *Condition* 1. Hence *Condition* 1 generalizes the concept of localization.

(c) If the ring extensions $R\subseteq T$ and $T\subseteq W$ satisfy *Condition* 1, then so does the ring extension $R\subseteq W$.

(d) If the ring extension $R\subseteq T$ satisfies *Condition* 1, then the unitary commutative ring extensions $R+XT[X]\subseteq T[X]$ and $R+XT[[X]]\subseteq T[[X]]$ also satisfy *Condition* 1.

The following remark provides examples of domain extensions $R\subseteq T$ satisfying *Condition* 1, where the conductor ideal $R:T$ is a maximal ideal of R.

REMARK 1. (i) Let $F\subset K$ be any field extension, the domain extension $F+XK[X]\subseteq K[X]$ (respectively $F+XK[[X]]\subseteq K[[X]]$) satisfies *Condition* 1, where the conductor ideal $F+XK[X]:K[X]$ (respectively $F+XK[[X]]:K[[X]]$) is maximal ideal in $F+XK[X]$ (respectively in $F+XK[[X]]$).

(ii) Let $F\subset K$ be a field extension, where K is a root extension of F and $K(Y)$ is the quotient field of $K[Y]$; then $R=F+XK(Y)[[X]]\subseteq K+XK(Y)[[X]]=T$ satisfies *Condition* 1 and $R:T=XK(Y)[[X]]$ is the maximal ideal in R.

There are a number of examples of domain extensions $R\subseteq T$ satisfying *Condition* 1, where the conductor ideal $R:T$ is not a maximal ideal of R. The following remark shows a few of those.

REMARK 2. (i) Let V be a valuation domain such that its quotient field K is the countable union of an increasing family $\{V_i\}_{i\in I}$ of valuation overrings of V. Let L be a proper field extension of K with L^*/K^* infinite. The it follows by [3, Example 5.3] that:

(a) The domain extension $V_i+XL[[X]]\subseteq L[[X]]$ satisfies *Condition* 1 since the extension $V_i\subseteq L$ satisfies *Condition* 1. But $XL[[X]]$ is not a maximal ideal of $V_i+XL[[X]]$. Also note that $U(V_i+XL[[X]])\neq U(L[[X]])$.

(b) The domain extension $V_i+XL[[X]]\subseteq K+XL[[X]]$ satisfies *Condition* 1, but $XL[[X]]$ is not a maximal ideal in $V_i+XL[[X]]$. Also, $U(V_i+XL[[X]])\neq U(K+XL[[X]])$.

(ii) The domain extension $R=\mathbb{Z}_{(2)}+X\mathbb{R}[[X]]\subseteq\mathbb{Q}+X\mathbb{R}[[X]]=T$ satisfies *Condition* 1, but the conductor ideal $R:T$ is not a maximal ideal in R.

(iii) The domain extension $R=\mathbb{Z}_{(2)}+X\mathbb{R}[[X]]\subseteq\mathbb{R}[[X]]=E$ satisfies *Condition* 1, but the conductor ideal $R:E$ is not a maximal ideal in R.

Pullback. Pullback plays an important role in commutative ring theory as a great source of providing examples and counter examples. For a most recent survey article where some classes of commutative rings are characterized as a pullback see [8].

By [21, p. 51], a unitary (commutative) ring R together with ring homomorphisms $f : R \to A$ and $g : R \to B$ is called a pullback of the pair of homomorphisms $\alpha : A \to C$ and $\beta : B \to C$ if

(a) the diagram

$$\begin{array}{ccc} R & \xrightarrow{g} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow[\alpha]{} & C \end{array}$$

commutes.

(b) (Universal property) If there exits another ring $R^{'}$ with a pair of ring homomorphisms $f^{'} : R^{'} \to A$, $g^{'} : R^{'} \to B$ such that the diagram

$$\begin{array}{ccc} R' & \xrightarrow{g'} & B \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow[\alpha]{} & C \end{array}$$

commutes. Then there exists a unique ring homomorphism $\theta : R^{'} \to R$ such that $f \circ \theta = f^{'}$ and $g \circ \theta = g^{'}$.

A pullback is said to be weak pullback for which the "Universal property" does not hold.

Every pair of ring homomorphisms $\alpha : B \to A$ and $\beta : C \to A$ has a pullback (see [21, Exercise 2.46, p. 52]).

In the following we consolidate discussions of [21, p. 51,52 and Exercise 2.47] as a proposition.

Proposition 1. *Let A, B and C be unitary (commutative) rings such that $C \subseteq A$ and $f : B \to A$ is an onto ring homomorphism, then $L = f^{-1}(C)$ is a pullback of ring homomorphisms f and g, that is*

$$\begin{array}{ccc} L = f^{-1}(C) & \xrightarrow{\alpha} & C \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow[f]{} & A \end{array}$$

The pullback L in Proposition 1 is a substructure of B.

Pullback of type $\square$. Houston and Taylor [17] introduce a pullback of type $\square$ as: Let I be a nonzero ideal of an integral domain T, $\varphi : T \to T/I = E$ be the natural surjection and D be an integral domain contained in E. Then the integral domain $R = \varphi^{-1}(D)$ arises as a pullback of the following diagram

$$\begin{array}{ccc} R = \varphi^{-1}(D) & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/I = E \end{array}$$

Here it is noticed that in fact $R \subseteq T$ and $D \subseteq E$.

J. Boynton [11] introduces the pullback as: Let $R \subseteq T$ be any unitary (commutative) ring extension and $I = R : T$ is the nonzero conductor ideal of T into R. Setting $D = R/I$ and $E = T/I$, we obtain the natural surjections $n_1 : T \longrightarrow E$, $n_2 : R \longrightarrow D$ and the inclusions $i_1 : D \hookrightarrow E$, $i_2 : R \hookrightarrow T$. These maps yield a commutative diagram, called *a conductor square* $\square$, which defines R as a pullback of n_1 and i_1.

$$\begin{array}{ccc} R & \xrightarrow{i_2} & T \\ {\scriptstyle n_2}\downarrow & & \downarrow{\scriptstyle n_1} \\ D & \xrightarrow[i_1]{} & E \end{array}$$

Lemma 1. [11, Lemma 2.2] *For conductor square* $\square$*, if* $I = R : T$ *is a regular ideal, then* T *is an overring of* R*.*

Remark 3. (i) Every conductor square $\square$ is a pullback of type $\square$.

(ii) If in conductor square $\square$, $R \subseteq T$ is a domain extension, then T is always an overring of R.

Lemma 2. [17, Lemma 1.1] *In a pullback of type* $\square$*, if each maximal ideal of* R *contains* I*, then each maximal ideal of* T *contains* I*.*

Recall that in a Prufer domain if every finitely generated fractional ideal is invertible. Equivalently, an integral domain R is Prufer if R_P is a valuation domain for each $P \in Spec(R)$.

In this study we shall follow the lines of the following results of [17] and [12].

Theorem 1. [17, Theorem 1.3] *In a pullback of type* $\square$*, let* I *be a prime ideal in* T *and* $qf(D) = qf(E)$*. Then* R *is a Prufer domain (respectively a valuation domain) if and only if* D *and* T *are Prufer domains (respectively valuation domains).*

Corollary 1. [17, Corollary 1.4] *Consider a pullback diagram of type* $\square$ *in which* I *is a maximal ideal of* T*. Then* R *is a Prufer domain (respectively a valuation domain) if and only if* D *and* T *are Prufer domains (respectively valuation domains) and* E *is quotient field of* D*.*

The following is an example of Prufer pullback which is not a valuation domain.

Example 2. Every nonzero prime ideal is a maximal ideal in $T = \mathbb{Q}[X]$. Take $I = X\mathbb{Q}[X]$, so $T/I \cong \mathbb{Q}$. Further $\mathbb{Z} \cong \{a + I : a \in \mathbb{Z}\} = D \subset T/I$. Then $\varphi : \mathbb{Q}[X] \to \mathbb{Q}[X]/X\mathbb{Q}[X] \cong \mathbb{Q}$ is surjection. Consider $R = \varphi^{-1}(D) = \varphi^{-1}(\{a + I : a \in \mathbb{Z}\}) = \{h(x) \in \mathbb{Q}[X] : h(0) \in \mathbb{Z}\}$. This implies $R = \mathbb{Z} + X\mathbb{Q}[X] \subset \mathbb{Q}[X] = T$. Hence we obtain the following commutative diagram

$$\begin{array}{ccc} R = \varphi^{-1}(D) & \xrightarrow{\alpha} & D \\ \downarrow & & \downarrow \\ T & \xrightarrow[\varphi]{} & T/I = E \end{array}$$

R is a pullback of type $\square$ and $D \subseteq E$, whereas $qf(D) = E$ and I is a maximal ideal in $\mathbb{Q}[X]$. As $\mathbb{Z}$ and $\mathbb{Q}[X]$ are Prufer domains, so $\mathbb{Z} + X\mathbb{Q}[X]$ being Bezout, a Prufer domain (an example on [17, Corollary 1.4]). Further, also it is an example on [17, Theorem 1.3] since none of $\mathbb{Z} + X\mathbb{Q}[X], \mathbb{Q}[X]$ and $\mathbb{Z}$ is a valuation domain.

2. Relative stability of some domain's properties on corners of a pullback

The inclusions $L \subseteq B$, $C \subseteq A$ of Proposition 1 and inclusion $R \subseteq T$ (respectively $D \subseteq E$) in the pullback of type $\square$ (respectively conductor square $\square$) are the main motivation to consider *Condition* 1. In this new scenario the properties of the elements of the unitary commutative rings L, B, C, A (respectively integral domains R, T, D, E) are concern. In [18, 22, 23], there are inquiries for stability (ascent and descent) of some atomic and non atomic classes of integral domains for a domain extension $R \subseteq T$ which satisfy *Condition* 1. The main purpose of this study is to escort the inquiries of [18, 22, 23] and observe the stability of classes of atomic and non atomic domains on all corners of the conductor square $\square$ under the assumption that the domain extension $R \subseteq T$ satisfies *Condition* 1. However besides this we also added a few more results regarding stability (ascent and descent) of some atomic and non atomic classes of integral domains for a domain extension $R \subseteq T$ in continuation to [18, 22, 23].

2.1. Some indispensable facts. We begin by the following proposition.

Proposition 2. *Let $R \subseteq T$ be a domain extension such that I is an ideal in T (hence $J = I \cap R$ is an ideal in R) and $f : T \longrightarrow T/I$ is the canonical surjection. Then*

(1) *$R = f^{-1}(R/J)$ is a pullback of type $\square$.*

(2) *If T is integral over R, then T/I is integral over R/J.*

Proof. (1) Since I is a nonzero ideal of T and $R/J \subseteq T/I$. Also $T \to T/I$ is surjection, so the result follows by Proposition 1.

(2) It is [5, Proposition 5.6]. ∎

Remark 4. (i) Let I be a prime ideal in T. Then I is a maximal ideal in T if and only if J is a maximal ideal in R. Indeed, as $R \subseteq T$ and T is integral over R, so T/I is integral over R/J. Now by [5, Proposition 5.7] T/I is a field if and only if R/J is a field.

(ii) Let $R \subseteq T$ be a domain extension, then $q^{-1}(R) = R + XT[X]$ arises as a pullback of the following diagram (see [19]).

$$\begin{array}{ccc} q^{-1}(R) = R + XT[X] & \longrightarrow & R \\ \downarrow & & \downarrow \\ T[X] & \xrightarrow[q]{} & T \end{array}$$

(iii) The extension $q^{-1}(R) = R + XT[X] \subseteq T[X]$ satisfies *Condition* 1 if $R \subseteq T$ does.

(iv) By [9], if I is the common ideal of R and T, then T is an overring of R. Also it follows that $R \to R/I = D$ is the canonical surjection.

The following Theorem provides the necessary and sufficient condition for a vertical inclusion of a pullback of type $\square$ to satisfy *Condition* 1.

THEOREM 2. *In a pullback of type $\square$, let I be a nonzero common ideal for R, T, and $\varphi : T \to T/I$ be the canonical surjection. Then $R \subseteq T$ satisfies Condition 1 if and only if $D \subseteq E$ satisfies Condition 1.*

Proof. Suppose $R \subseteq T$ satisfies *Condition* 1. For $s + I \in T/I$, $s \in T$, $s = tr$, where $t \in U(T)$ and $r \in R$. This means $s + I = (t + I)(r + I)$, whereas $t + I \in U(T/I)$ and $r + I \in R/I$. Hence $D \subseteq E$ satisfies *Condition* 1.

The converse follows by [18, Proposition 1.2]. ■

REMARK 5. (i) In the pullback of type $\square$ of Example 2, we see that the extension $\mathbb{Z} \subset \mathbb{Q}$ satisfies *Condition* 1, so the extension $\mathbb{Z} + X\mathbb{Q}[X] \subset \mathbb{Q}[X]$ does and vice versa.

(ii) In a pullback of type $\square$, if I is a maximal ideal in T, then $E = T/I$ is a field and by [18, Example (ii)] $D \subseteq E$ satisfies *Condition* 1.

The following extends [18, Proposition 1.3] in the perspective of pullback of type $\square$.

PROPOSITION 3. *In a pullback of type $\square$, let $\varphi^{-1}(D) = R \subseteq T$ such that I is a common ideal of R and T. If for each $t \in T\backslash I$, there exists $i \in I$ with $t + i \in U(T)$. Then*

(1) I is a maximal ideal in T.

(2) The extension $R \subseteq T$ satisfies Condition 1.

Proof. (1) Let $0 \neq t \in T\backslash I$, then there exists $i \in I$ such that $t + i \in U(T)$. Thus $t + i + I \in U(T/I)$, that is $\varphi(t + i) \in U(T/I)$. So T/I is a field. Hence I is a maximal ideal in T.

(2) If $t \in I$, then $t = 1.t$, as $1 \in U(T)$. Let $t \in T \backslash I$, then there exists $i \in I$ such that $t+i \in U(T)$, by (1). So we may write $t = (t+i)(t+i)^{-1} t$ and obviously $(t+i)^{-1} t \in R$, as $(t+i)^{-1} t = 1 + j$, where $j \in I$. ■

Remark 6. In a pullback of type $\square$, if $R \subseteq T$ satisfies *Condition* 1, then $(I \cap R) T = I$. Indeed, as I is an ideal of T, so $(I \cap R) T \subseteq IT \subseteq I$. Conversely let $s \in I$, then by *Condition* 1, $s = rt$, where $t \in U(T)$ and $r \in R$. This implies $r \in I \cap R$ and so $s = rt \in (I \cap R) T$. Hence $I \subseteq (I \cap R) T$.

Remark 7. In a conductor square $\square$, if $I = R : T$ is a maximal in R such that the extension $R \subseteq T$ satisfies *Condition* 1, then I is a maximal in T. Indeed, let $s \in T \backslash I$, then $s = tr$, where $t \in U(T)$ and $r \in R$. Whereas $r \notin I$, because if $r \in I$, then $s = tr \in I$, which cause a contradiction. Now $\varphi(r)$ is unit in R/I.Thus $\varphi(s) = \varphi(t) \varphi(r)$ is unit in T/I. Hence I is a maximal in T.

2.2. Atomic generalizations of a UFD. The following extends a part of [18, Proposition 2.6].

Proposition 4. *In a conductor square $\square$, let the domain extension $R \subseteq T$ satisfies Condition 1 and $I = R : T$ is a maximal ideal in R. Then R is atomic (respectively has ACCP, BFD and an HFD) if and only if D and T are atomic (respectively have ACCP, BFD and an HFD).*

Proof. R is atomic (respectively has ACCP, BFD and an HFD) if and only if T is atomic (respectively has ACCP, BFD and an HFD) follows by [18, Proposition 2.6]. D being a field is an atomic domain, has ACCP, a BFD and an HFD. ■

Remark 8. In Proposition 4 E being a field is atomic, has ACCP, BFD and an HFD.

2.3. Valuation domain and its generalizations. By [16], an integral domain R with quotient field K is said to be a *pseudo-valuation domain* (PVD), if whenever P is a prime ideal in D and $xy \in P$, where $x, y \in K$, then $x \in P$ or $y \in P$ (i.e. in a PVD every prime ideal is strongly prime). Equivalently an integral domain R with quotient field K is said to be a PVD if for any nonzero element $x \in K$, either $x \in R$ or $ax^{-1} \in R$ for every non unit $a \in R$. A valuation domain is a PVD but the converse is not true, for example the PVD $\mathbb{R} + X\mathbb{C}[[X]]$, which is not a valuation domain.

By [2] an integral domain R is said to be an *almost valuation domain (AVD)* if for every nonzero $x \in K$, there exists an integer $n \geq 1$ (depending on x) with $x^n \in R$ or $x^{-n} \in R$. Equivalently the domain R is said to be an AVD if for each pair $a, b \in R$, there is a positive integer $n = n(a, b)$ such that $a^n \mid b^n$ or $b^n \mid a^n$. A valuation domain is an AVD but converse is not true. For example if F is a finite field, then $R = F + X^2 F[[X]]$ is a non valuation AVD (cf. [7, Example 3.8]).

By [6], an integral domain R is said to be an *almost pseudo valuation domain (APVD)* if and only if R is quasilocal with maximal ideal M such that for every nonzero element $x \in K$, either $x^n \in M$ for some integer $n \geq 1$ or $ax^{-1} \in M$ for every nonunit $a \in R$. Equivalently a prime ideal P of R is a *strongly primary ideal,* if $xy \in P$, where $x, y \in K$ implies that either $x^n \in P$ for some integer $n \geq 1$ or $y \in P$. If each prime ideal of R is strongly primary ideal, then R is *an* APVD. For example $R = \mathbb{Q} + X^4\mathbb{Q}[[X]]$ is an APVD (cf. [6, Example 3.9]) which is not a PVD.

By [7] a prime ideal P of an integral domain R is said to be a *pseudo-strongly prime ideal* if, whenever x, $y \in K$ and $xyP \subseteq P$, then there is an integer $m \geq 1$ such that either $x^m \in R$ or $y^m P \subseteq P$. If each prime ideal in an integral domain R is a *pseudo-strongly prime ideal,* then R is called a *pseudo-almost valuation domain (PAVD).* Equivalently an integral domain R is a PAVD if and only if for every nonzero element $x \in K$, there is a positive integer $n \geq 1$ such that either $x^n \in R$ or $ax^{-n} \in R$ for every nonunit $a \in R$. For example if F is a finite field, and $H = F[[X]]$, then $R = F + FX^2 + X^4F[[X]]$ is a PAVD (cf. [7, Example 3.8]).

In general

$$\begin{array}{ccccc} & & & & quasilocal \\ & & & & \Uparrow \\ AVD & & \Rightarrow & & PAVD \\ \Uparrow & & & & \Uparrow \\ VD & \Rightarrow & PVD & \Rightarrow & APVD \end{array}$$

but none of the above implications is reversible.

We readjust [18, Lemma 1.7] as follows.

LEMMA 3. [18, Lemma 1.7] *In a pullback of type* $\square$*, let* I *be the common ideal in* R *and* T*. Then* $R = \varphi^{-1}(\varphi(R))$*, where* $\varphi : T \to T/I$ *is the canonical surjection.*

Proof. Clearly $R \subseteq \varphi^{-1}(\varphi(R))$. Conversely, let $x \in \varphi^{-1}(\varphi(R))$, so $\varphi(x) \in \varphi(R)$ and therefore $\varphi(x) = \varphi(r)$ for some $r \in R$. This means $x - r \in I$ and therefore $x \in R$. Hence $\varphi^{-1}(\varphi(R)) \subseteq R$. ∎

Following Zafrullah [24], an element x of an integral domain R is said to be rigid if whenever r, $s \in R$ and r and s divides x, then s divides r or r divides s. The domain R is said to be semirigid domain if every nonzero element of R can be expressed as a product of a finite number of rigid elements.

The following is an improved form of [18, Theorem 2.10].

THEOREM 3. *In a conductor square* $\square$*, let* $R \subseteq T$ *satisfies Condition* 1 *and* $I = R : T$ *is the maximal ideal in* R*. If* R *is a semirigid-domain, then* T *is a semirigid-domain.*

Proof. Suppose R is a semirigid-domain. Let $x \in T$, so either $x \in I$ or $x \in T \backslash I$. The case $x \in I$ is trivial. If $x \in T \backslash I$, then by *Condition* 1, $x = ru$, where $r \in R$, $u \in U(T)$. But R is semirigid-domain, so $r = r_1 r_2 \cdots r_n$ is a product of rigid elements in R and therefore by [18, Theorem 2.8(b)] $x = (ur_1)r_2..r_n$ is the product of rigid elements in T. Hence T is a semirigid-domain. ∎

In the rest of the discussion we assume that $I = R : T$ is a prime ideal.

For the sake of a quick reference we state the following lemma.

Lemma 4. [14, Lemma 4.5(i)] *Let R be a PVD and P is its prime ideal. Then R/P is a PVD.*

Theorem 4. *Let $R \subseteq T$ be the domain extension which satisfies Condition 1. If R is a PVD, then T is a PVD.*

Proof. Let $a, b \in T$ such that $x = \frac{a}{b} \in qf(T)$ with $b \neq 0$. So $a = a_1a_2$, $b = b_1b_2$, where a_1, $b_1 \in R$ and a_2, $b_2 \in U(T)$. This implies $x_1 = \frac{a_1}{b_1} \in qf(R)$, where $b_1 \neq 0$. Since R is a PVD, therefore either $x_1 = \frac{a_1}{b_1} \in R$ or $rx_1^{-1} = r\frac{b_1}{a_1} \in R$, where r is nonzero nonunit in R. If $x_1 \in R$ and $x_2 = \frac{a_2}{b_2} \in U(T)$, then $x \in T$. If $rx_1^{-1} \in R$ and $x_2 = \frac{a_2}{b_2} \in U(T)$ (hence $x_2^{-1} = \frac{b_2}{a_2} \in U(T)$), then $rx^{-1} \in T$, whereas $r \notin U(T)$. ∎

Remark 9. In the proof of Theorem 4 if $r \in U(T)$, then T must be a valuation domain.

Remark 10. The converse of Theorem 4 does not hold. For example in the domain extension $\mathbb{Z} + X\mathbb{Q}[[X]] \subset \mathbb{Q}[[X]]$ which satisfies *Condition* 1, $\mathbb{Q}[[X]]$ is a DVR and hence a PVD, but $\mathbb{Z} + X\mathbb{Q}[[X]]$ is not a PVD.

In the following we extend [17, Theorem 1.3] for PVDs with the addition of *Condition* 1.

Theorem 5. *In a conductor square $\square$, let the domain extension $R \subseteq T$ satisfy Condition 1 such that $I = R : T$ is contained in the maximal ideal M of R and $qf(D) = qf(E)$. Then T and D are PVDs if and only if R is a PVD.*

Proof. Assume that T and D are PVDs. It is known that: M is a maximal ideal of R if and only if M/I is a maximal ideal of D. Let $x \in qf(R) = qf(T)$; then either $x \in T$ or $tx^{-1} \in T$, where $t \in T \backslash U(T)$.

If $x \in T \backslash R$, we have $x = x_1x_2$, where $x_1 \in R$ and $x_2 \in U(T)$. So $\varphi(x_1) \in D$, $\varphi(x_2) \in U(E)$. Since D is a PVD, therefore by [16, Theorem 1.5(3)], $\varphi(x_2)^{-1} M/I \subseteq M/I$, that is $\varphi(x_2)^{-1}(m + I) \in M/I, (m + I) \in M/I$. This implies $x_2^{-1}m \in M$, $\varphi(m) = (m + I) \in M/I$ for some $m \in M$. So $x_1x_1{}^{-1}x_2^{-1}m = x_1mx^{-1} = ax^{-1} \in M$, where $x_1m = a \in R \backslash U(R)$, which shows that M is strongly prime.

If $tx^{-1} \in T \backslash R$, then $tx^{-1} = ru$, where $r \in R$ and $u \in U(T)$. So $\varphi(r) \in D$, $\varphi(u) \in U(E)$. Since D is a PVD, therefore by [16, Theorem 1.5(3)] $\varphi(u)^{-1} M/I = \varphi\left(u^{-1}\right) M/I \subseteq M/I$ if and only if $u^{-1}M \subseteq M$. This implies $u^{-1}m = rr^{-1}u^{-1}m = rm\,(ru)^{-1} = r_1\left(tx^{-1}\right)^{-1} = r_1t^{-1}x \in M$, where m, $r_1 = rm \in M$. This implies M is a strongly prime ideal, as $t^{-1}x \in qf(R)$ and hence R is a PVD.

Conversely, by Theorem 4 T is a PVD whenever R is a PVD. By [14, Lemma 4.5(i)], if R is a PVD, then $D = R/I$ is a PVD. ∎

The following examples are through the $D + M$ construction as elaborated in [10, Theorem 2.1].

Remark 11. [4, Example 3.12] (i) In Theorem 4 there is no need to assume that $qf(D) = qf(E)$. For instance in the conductor square $\square$

$$\begin{array}{ccc} R = \mathbb{R}+X\mathbb{C}(t)[[X]] & \longrightarrow & \mathbb{R} = D \\ \downarrow & & \downarrow \\ T = \mathbb{C}(t)+X\mathbb{C}(t)[[X]] & \longrightarrow & \mathbb{C}(t) = E \end{array}$$

$I = R : T = XC(t)[[X]]$ and $R \subseteq T$ satisfies *Condition* 1. Indeed, let $f = f_1 + Xf_2(X) \in T$. In this pullback $qf(D) \neq qf(E)$ but R, T and D are HFDs.

(ii) Let K be the field. The following is a conductor square $\square$.

$$\begin{array}{ccc} R = K + XK(Y)[[X]] & \longrightarrow & K \\ \downarrow & & \downarrow \\ T = K(Y)[[X]] & \longrightarrow & K(Y) \end{array}$$

Whereas $I = R : T = XK(Y)[[X]]$ is a maximal ideal in R and $R \subseteq T$ satisfies *Condition* 1. R is a PVD but T and K are DVRs. However $qf(D) \neq qf(E)$.

Theorem 6. *Let $R \subseteq T$ be the domain extension which satisfies Condition* 1. *If R is an AVD, then T is an AVD.*

Proof. Let $x = \frac{a}{b} \in qf\,(T)$, $a, b \in T$. We may consider $a = a_1a_2, b = b_1b_2$, where $a_1, b_1 \in R$ and $a_2, b_2 \in U\,(T)$. Of course $\frac{a_1}{b_1} \in qf\,(R)$ and R is an AVD, so either $(\frac{a_1}{b_1})^n \in R$ or $(\frac{a_1}{b_1})^{-n} \in R$, where $n \geq 1$ be an integer. Similarly $u = \frac{a_2}{b_2} \in U\,(T)$ implies either $(\frac{a_1}{b_1}u)^n = x^n \in T$ or $(\frac{a_1}{b_1}u)^{-n} = x^{-n} \in T$. ■

Remark 12. [4, Example 3.12] Let F be a finite field, and $H = F\,(X)$ be the quotient field of $F\,[X]$. $R = F + Y^3H\,[[Y]]$ is not an AVD but $V = H + Y^3H\,[[Y]]$ is an AVD (cf. [7. Example 2.20]). Obviously $R \subseteq V$ satisfies *Condition* 1.

Proposition 5. *Let R be an AVD and P is a prime ideal of R. Then R/P is an AVD.*

Proof. R is a quasilocal domain if and only if for any $a, b \in R$ either $a \mid b^n$ or $b \mid a^n$ for some $n \geq 1$, by [7, Proposition 2.7]. Now for $x = a+P$, $y = b+P \in R/P$, suppose that $x \nmid y^n$ for some integers $n \geq 1$. This implies that $a \nmid b^n$ for some $n \geq 1$. Therefore $b \mid a^n$ for some integers $n \geq 1$. This implies $y \mid x^n$ for some integers $n \geq 1$. Thus R/P is quasilocal as well as AB-domain, by [2. Theorem 4.10]. Hence by [2, Theorem 5.6] R/P is an AVD. ■

Theorem 7. *In a conductor square $\square$, let the domain extension $R \subseteq T$ satisfies Condition* 1 *such that $I = R : T$ is the conductor ideal and $qf(D) = qf(E)$. Then T and D are AVDs if and only if R is an AVD.*

Proof. Assume that T and D are AVDs. Let $a \in qf(R) = qf\,(T)$, then either a^n or $a^{-n} \in T$.

(i) Consider $a^n \in T$, so by *Condition* 1, we have $a^n = a_1a_2$, where $a_1 \in R$ and $a_2 \in U(T)$. Then $\hat{a}_1 = \varphi(a_1) \in D$ and $\hat{a}_2 = \varphi(u_2) \in U(E)$. Since D is an

AVD, therefore $\hat{a}_2^p \in D$ or $\hat{a}_2^{-p} \in D$, where p is a positive integer. This implies $a_2^p = \varphi^{-1}(\hat{a}_2^p) \in R$ and hence $a_1^p a_2^p = a^{np} \in R$.

Now if $\hat{a}_2^{-p} \in D$, then $a_2^{-p} = \varphi^{-1}(\hat{a}_2^{-p}) \in R$. We claim $a_1^{-p} \notin R$, if not, then $a_1^{-p} \in R$, and we may have $a_1^{-p}a_2^{-p} = a^{-np} \in R \subset T$, a contradiction to the fact that $a^{-n} \notin T$. Thus $a^{-np} \notin R$ and $\hat{a}_2^{-p} \notin D$.

(ii) Now if $a^{-n} \in T$, then by *Condition* 1, we have $a^{-n} = a_1 a_2$, where $a_1 \in R$ and $a_2 \in U(T)$.This means $\hat{a}_1 = \varphi(a_1) \in D$ and $\hat{a}_2 = \varphi(a_2) \in U(E)$. As D is an AVD, so $\hat{a}_2^p \in D$ or $\hat{a}_2^{-p} \in D$, where p is a positive integer.This implies $a_2^p = \varphi^{-1}(\hat{a}_2^p) \in R$. This implies $a_1^p a_2^p = a^{-np} \in R$. If $\hat{a}_2^{-p} \in D$, then $a_2^{-p} = \varphi^{-1}(\hat{a}_2^{-p}) \in R$. We claim that $a_1^{-p} \notin R$; if not then $a_1^{-p}a_2^{-p} = a^{np} \in R \subset T$, which contradict to the fact that $a^n \notin T$. Thus $a^{np} \notin R$ and $\hat{a}_2^{-p} \notin D$.

Conversely, by Theorem 6, T is an AVD whenever R is an AVD. Hence it followed by Proposition 5 that D is an AVD. ∎

THEOREM 8. *Let the domain extension $R \subseteq T$ satisfies Condition 1 such that $I = R : T$ is contained in the maximal ideal M of R. If R is an APVD, then T is an APVD.*

Proof. Let $x = \frac{a}{b} \in qf(T)$, where $a, b \in T$. By *Condition* 1 $a = a_1 a_2, b = b_1 b_2$, where $a_1, b_1 \in R$ and $a_2, b_2 \in U(T)$. This implies $\frac{a_1}{b_1} \in qf(R)$. Since R is an APVD with maximal ideal M, then either $(\frac{a_1}{b_1})^n \in M$ for $n \geq 1$, or $r(\frac{a_1}{b_1})^{-1} \in M$, where $r \in R \backslash U(R)$ and say $u = \frac{a_2}{b_2} \in U(T)$. Let N be the maximal ideal of T such that $N \cap R = M$. Therefore either $(\frac{a_1}{b_1}u)^n \in N$ or $r(\frac{a_1}{b_1}u)^{-1} \in N$, where $r \in T \backslash U(T)$. ∎

REMARK 13. In the proof of Theorem 8 if $r \in U(T)$, then T must be a valuation domain.

EXAMPLE 3. [4, Example 3.12] Let F be a finite field and $H = F(X)$ is the quotient field of $F[X]$. $R = F + Y^2H[[Y]]$ is not an APVD but $T = F + FY + Y^2H[[Y]]$ is an APVD. Whereas $R \subseteq T$ does not satisfy *Condition* 1.

By [6], let S be a subset of an integral domain R with quotient field K, then $E(S) = \{x \in K : x^n \notin S$ for every integer $n \geq 1\}$.

PROPOSITION 6. *An integral domain R is an APVD if and only if for every $x \in E(R)$ such that $ax^{-1} \in R$ for every nonunit $a \in R$.*

Proof. Suppose that R is an APVD. Then R is a quasilocal by [6, Proposition 3.2]. Let M be the maximal ideal of R and $x \in E(R)$. Then by [6, Lemma 2.3] $x^{-1}M \subseteq M \subseteq R$. Conversely, assume that for every $x \in E(R)$ such that $ax^{-1} \in R$ for every nonunit $a \in R$. Let a, b be nonzero nonunit elements of R. Suppose that $a \nmid b^n$ in R for every $n \geq 1$. Then $x = b/a \in E(R)$. Hence, by hypothesis $cx^{-1} \in R$ for every nonunit c of R. In particular $a^2/b = ax^{-1} \in R$. Then $b \mid a^2$ in R. Thus by [7, Proposition 2.7], the prime ideals of R are linearly ordered. Hence R is quasilocal. Thus, by hypothesis, $ax^{-1} \in R$ for every $a \in M$. Since M is the only maximal ideal of R and $x \in E(R)$, we conclude that $ax^{-1} \in M$ for every

$a \in M$. By [6, Lemma 2.3]BH, M is a strongly prime ideal of R. Hence R is an APVD, by [6, Theorem 3.4(2)]BH. ■

In the following we restate Proposition 6.

PROPOSITION 7. *An integral domain R is an APVD if and only if for every $a, b \in R$ either $a^n \mid b^n$ in R for some $n \geq 1$ or $b \mid ca$ in R for every nonunit c of R.*

PROPOSITION 8. *Let R be an APVD and P is a prime ideal of R. Then R/P is an APVD.*

Proof. Let R be an APVD and P is a prime ideal of R. Set $D = R/P$ and let $x, y \in D$. Then $x = a + P$ and $y = b + P$ for some $a, b \in R$. Suppose that $x^n \nmid y^n$ in D for every positive integer $n \geq 1$. Then, $a^n \nmid b^n$ in R for every positive integer $n \geq 1$. Thus by Proposition 7, $b \mid ca$ in R for every nonunit c of R.Thus $y \mid zx$ for every nonunit z of D. Hence by Proposition 7, D is an APVD. ■

THEOREM 9. *In a conductor square $\square$, let the domain extension $R \subseteq T$ satisfy Condition 1 such that $I = R : T$ contained in the maximal ideal M of R and $qf(D) = qf(E)$. Then T and D are APVDs if and only if R is an APVD.*

Proof. Assume that T and D are APVDs. As $I \subseteq M$, so $M/I = \varphi(M)$ is maximal ideal of D. For $x \in E(R)$, we have the following possibilities:

(i) If $x \in T \backslash R$, then $x = x_1x_2$, where $x_1 \in R$, $x_2 \in U(T)$. So $\hat{x}_1 \in D$, $\hat{x}_2 \in U(E)$. By [6, Lemma 2.3]BH $(\hat{x}_2)^{-1}M/I \subseteq M/I$. This implies $(x_2)^{-1}M \subseteq M$, this means $x_1(x_1)^{-1}(x_2)^{-1}m = x_1(x_1x_2)^{-1}m = rx^{-1} \in M$, where $x_1m = r \in R\backslash U(R)$, $m \in M$.

(ii) If $x \in qf(T)\backslash T$, then either $x^n \in N$ or $tx^{-1} \in N$, $t \in T\backslash U(T)$, where N is maximal in T. (a) If $x^n \in N$ and $N \cap R = M$, the maximal ideal in R. Using *Condition* 1, $x^n = ru$, where $r \in R$ and $u \in U(T)$. This implies $\varphi(r) \in D$ and $\varphi(u) \in U(E)$. Either $\varphi(u)^t \in M/I$ or $d\varphi(u)^{-1} \in M/I$, $t > 0$ and $d \in D\backslash U(D)$.

If $\varphi(u)^t \in M/I$, so $\varphi(r)^t \varphi(u)^t = \varphi(ru)^t = \varphi(x^n)^t = \varphi(x^{nt}) \in M/I$. This implies $x^{nt} \in M$, a contradiction. Now, if $d\varphi(u)^{-1} \in M/I$, then there exists $m \in M$ such that $d = \varphi(m)$. This implies $\varphi(m)\varphi(u)^{-1} \in M/I$. This means $mu^{-1} = rr^{-1}mu^{-1} = m_1(ru)^{-1} = m_1x^{-n} \in M$, where $m_1 = rm \in M$.

(b) Finally; if $tx^{-1} \in N$. We have $tx^{-1} = ru$, $r \in R$ and $u \in U(T)$. This implies $\varphi(r) \in D$ and $\varphi(u) \in U(E)$. Then by [6, Lemma 2.3] $\varphi(u)^{-1} M/I = \varphi(u^{-1}) M/I \subseteq M/I$, this implies $u^{-1}M \subseteq M$, and $u^{-1}m = rr^{-1}u^{-1}m = r(ru)^{-1}m = r_1(ru)^{-1} = r_1(tx^{-1})^{-1} \in M$, where $m, r_1(= rm) \in M$. Thus M becomes strongly primary. Hence R is an APVD.

Conversely by Theorem 8, T is an APVD whenever R is an APVD.

By Proposition 8, D is an APVD. ■

EXAMPLE 4. [4, Example 3.12] Let $F \subset K$ be a field extension, where K is the root extension of F. The pullback

$$\begin{array}{ccc} R = F + XK(Y)[[X]] & \longrightarrow & F \\ \big\downarrow & & \big\downarrow \\ T = K + XK(Y)[[X]] & \longrightarrow & K \end{array}$$

is of type $\Box$, whereas $I = R : T = XK(Y)[[X]]$ and $R \subseteq T$ satisfies *Condition* 1. R is an APVD if and only if T *is a* PVD. Whereas $qf(D) = F \neq K = qf(E)$.

We state the following proposition from [7] for the sake of completeness.

Proposition 9. [7, Proposition 2.14] *Let R be a PAVD and P be a prime ideal of R. Then R/P is a PAVD.*

Theorem 10. *Let $R \subseteq T$ be a domain extension which satisfies Condition 1. If R is PAVD, then T is PAVD.*

Proof. Let $x = \frac{a}{b} \in qf\,(T)\,, a, b \in T$. By *Condition* 1 $a = a_1 a_2, b = b_1 b_2$, where $a_1, b_1 \in R$, $a_2, b_2 \in U\,(T)$. This implies $\frac{a_1}{b_1} \in qf\,(R)$ and so either $(\frac{a_1}{b_1})^n \in R$ or $r(\frac{a_1}{b_1})^{-n} \in R$, where $n > 0$, $r \in R \backslash U(R)$, $u = \frac{a_2}{b_2} \in U(T)$. Hence either $(\frac{a_1}{b_1}u)^n \in T$ or $t(\frac{a_1}{b_1}u)^{-n} \in T$, where $t = rq$, where $t, q \in T \backslash U(T)$. ∎

Example 5. In domain extension $\mathbb{C}[[X^2, X^5] \subseteq \mathbb{C}[[X^2, X^3]]$, $\mathbb{C}[[X^2, X^3]]$ is a PAVD but $\mathbb{C}[[X^2, X^5]]$ is not a PAVD. So descent does not hold.

Theorem 11. *In a conductor square $\Box$, let the domain extension $R \subseteq T$ satisfy Condition 1 such that $I = R : T$ contained in the maximal ideal M of R and $qf(D) = qf(E)$. Then T and D are PAVDs if and only if R is a PAVD.*

Proof. Assume that T and D are PAVDs. As $I \subseteq M$, so $M/I = \varphi(M)$ is a maximal ideal of D. For $x \in E(R)$, we have the following possibilities:

(i) If $x \in T \backslash R$, then $x = x_1 x_2$, where $x_1 \in R, x_2 \in U(T)$. This implies $\hat{x}_1 = \varphi(x_1) \in D$, $\hat{x}_2 = \varphi(x_2) \in U(E)$. Then by [7, Lemma 2.1] $(\hat{x}_2)^{-n} M/I \subseteq M/I$, and hence $\varphi^{-1}\left((\hat{x}_2)^{-n} M/I\right) \subseteq M$. This implies $x_2^{-n} m = x_1^n x_1^{-n} x_2^{-n} m = m_1 x^{-n} \in M$, where $m, m_1 \in M$. Thus M is a pseudo-strongly prime ideal.

(ii) If $x \in Q(T) \backslash T$, then $x^n \in T$ or $tx^{-n} \in T$, for $t \in T \backslash U(T)$ and $n > 0$. (a) If $x^n \in T$, then $x^n = x_1 x_2$; where $x_1 \in R$ and $x_2 \in U(T)$. This implies $\varphi(x_1) \in D$ and $\varphi(x_2) \in U(E)$. By [7, Lemma 2.1] $\varphi(x_2)^{-k} M/I \subseteq M/I$, for an integer $k \geq 0$ and hence $x_2^{-k} M \subseteq M$. This implies $x_2^{-k} r = x_1^k x_1^{-k} x_2^{-k} r = r_1 x^{-kn} \in M$, for $r, r_1 = x_1^k r \in M$. Hence M is a pseudo-strongly prime ideal.

(b) Finally, if $tx^{-n} \in T$, then $tx^{-n} = ru$, where $r \in R$ and $u \in U(T)$. This implies $\varphi\,(r) \in D$ and $\varphi\,(u) \in U(E)$. By [7, Lemma 2.1] $\varphi\,(u)^{-k} M/I \subseteq M/I$, for an integer $k > 0$ and hence $u^{-k} M \subseteq M$. This implies $u^{-k} m = r^k r^{-k} u^{-k} m = m_1 (tx^{-n})^{-k} \in M$, where $m, m_1 (= r^k m) \in M$. Thus M is a pseudo-strongly prime ideal. Hence R is a PAVD.

Conversely by Theorem 10, T is a PAVD whenever R is a PAVD. By [7, Proposition 2.14], if R is a PAVD, then $D = R/I$ is a PAVD. ∎

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