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SOME PROPERTIES OF GENERALIZED SZÀSZ TYPE OPERATORS OF TWO VARIABLES

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Abstract. A generalization of Szàsz type operators for two variables is constructed and the theorems on convergence and the degree of convergence are established. In addition, we consider the simultaneous approximation of these operators.

1. Introduction

For an analytic function $g(x) \equiv \sum_{n=0}^{\infty} a_n x^n$ $(g(1) \neq 0)$ consider the polynomials P_k defined by

$$g(u)e^{ux} = \sum_{k=0}^{\infty} P_k(x)u^k. \tag{1}$$

In [6], Jakimovski and Leviatan defined the operators $P_n : C_A[0,\infty) \to C[0,\infty)$ as follows,

$$P_n(f;x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} P_k(nx) f\Big(\frac{k}{n}\Big) \tag{2}$$

where $C_A[0,\infty) = \big\{ f \in C[0,\infty) : \ |f(x)| \leq \beta e^{Ax} \big\}$. When $g(x) \equiv 1$ in (1), we obtain classical Szàsz operators. In [9], Wood proved that this operator is positive in $[0,\infty)$ if and only if $a_n/g(1) \geq 0$ for $n \in \mathbb{N}$. In [6], Jakimovski and Leviatan established several new approximation results for the (2) operators. They proved that If $f \in C_A[0,\infty)$, then $P_n(f;x)$ converges uniformly to f any compact interval of the positive real axis. In [4], A. Ciupa gave a generalization of the (2) operators as follows:

$$P_{n,t}(f;x) = \frac{e^{-nt}}{g(1)} \sum_{k=0}^{\infty} p_k(nt) f\Big(x + \frac{k}{n}\Big). \tag{3}$$

and studied the approximation properties of these operators. In a recent paper [2] Atakut and Büyükyazıcı have studied a Stancu type generalization of $P_{n,t}(f;x)$ as

$$P_{n,t}^{\alpha,\beta}(f;x) = \frac{e^{-nt}}{g(1)} \sum_{k=0}^{\infty} p_k(nt) f\Big(x + \frac{k+\alpha}{n+\beta}\Big).$$

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Here, Inspired by (3), we introduce a similar generalization of Szàsz type operators for two variables as follows:

$$P_{n,m}^{t_1,t_2}(f;x,y)=\frac{e^{-(nt_1+mt_2)}}{g^2(1)}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}P_k(nt_1)P_j(mt_2)f\Big(x+\frac{k}{n},y+\frac{j}{m}\Big)\ ; \qquad (4)$$

where $g(.)$ and $P_k(.;g)=P_k(.)$ have same properties given by (1). It is obvious that the operator is positive on $[0,\infty)\times[0,\infty)$, if $a_n\geq 0$ $(n=0,1,2,\dots)$ for $0\leq t_1,t_2<\infty$. In this paper, we will give some approximation properties of these operators.

2. Basic results

In this section, we shall mention some definitions and certain lemmas to prove our main theorems.

Lemma 1. *Let*

$$g(u_1)g(u_2)e^{u_1x+u_2y}=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}P_k(x)P_j(y)u_1^ku_2^j\,. \qquad (5)$$

Then we have

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}P_k(nt_1)P_j(mt_2)=g^2(1)e^{nt_1+mt_2} \qquad (6)$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}kP_k(nt_1)P_j(mt_2)=\big(g'(1)g(1)+nt_1g^2(1)\big)\,e^{nt_1+mt_2} \qquad (7)$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}jP_k(nt_1)P_j(mt_2)=\big(g'(1)g(1)+mt_2g^2(1)\big)\,e^{nt_1+mt_2} \qquad (8)$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}k^2P_k(nt_1)P_j(mt_2)=(g''(1)g(1)+2nt_1g'(1)g(1)+n^2t_1^2g^2(1)$$
$$+g'(1)g(1)+nt_1g^2(1))e^{nt_1+mt_2} \qquad (9)$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}j^2P_k(nt_1)P_j(mt_2)=(g''(1)g(1)+2mt_2g'(1)g(1)+m^2t_2^2g^2(1)$$
$$+g'(1)g(1)+mt_2g^2(1))e^{nt_1+mt_2}. \qquad (10)$$

Proof. In (5), letting $u_1=u_2=1$ and $x=nt_1,\ y=mt_2$,the equality (6) is easily obtained. To prove (7), we take partial derivatives of the two sides of the equation (5) with respect to u_1

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}kP_k(x)P_j(y)u_1^{k-1}u_2^j=(g'(u_1)g(u_2)+xg(u_1)g(u_2))\,e^{u_1x+u_2y} \qquad (11)$$

substituting $u_1=u_2=1$ and $x=nt_1,\ y=mt_2$ in (11), we get

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}kP_k(nt_1)P_j(mt_2)=\big(g'(1)g(1)+nt_1g^2(1)\big)\,e^{nt_1+mt_2}.$$

We can make a similar way in the proof of (8). Now, we show the accuracy of the equation (9). Taking partial derivatives of (11) with respect to u_1, we have

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} k^2 P_k(x)P_j(y)u_1^{k-2}u_2^j = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} kP_k(x)P_j(y)u_1^{k-1}u_2^j + \left(g''(u_1)g(u_2) + 2xg'(u_1)g(u_2) + x^2 g(u_1)g(u_2)\right) e^{u_1x+u_2y}.$$

From (7), we get

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} k^2 P_k(x)P_j(y)u_1^{k-1}u_2^j = (g'(u_1)g(u_2) + xg(u_1)g(u_2) + g''(u_1)g(u_2) + 2xg'(u_1)g(u_2) + x^2 g(u_1)g(u_2))e^{u_1x+u_2y}.$$

Letting $u_1 = u_2 = 1$ and $x = nt_1,\ y = mt_2$ in last equation, we obtain the desired result. Finally, proof of equation (10) can be done similarly. ∎

Definition 2. Let $D = [0,\infty)\times[0,\infty)$ and A be an certain finite. We denote by $C_A(D) = \left\{f \in C(D) : |f(x,y)| \le \beta e^{A(x+y)}\right\}$ and for $i = 0,1,2,3,\ e_i$ the test functions defined by $e_0(x,y) = 1$, $e_1(x,y) = x$, $e_2(x,y) = y$ and $e_3(x,y) = x^2 + y^2$.

Definition 3. [3] Let $D_{ab} = [0,a]\times[0,b]$. For $f \in C(D_{ab})$ and $\delta > 0$, the Peetre K-functional is defined by

$$K(f;\delta) = \inf_{\varphi\in C^2(D_{ab})}\left\{\|f-\varphi\|_{C(D_{ab})} + \delta\,\|\varphi\|_{C^2(D_{ab})}\right\} \tag{12}$$

where $C^2(D_{ab})$ is the space of functions of φ such that $\varphi, \dfrac{\partial^i\varphi}{\partial x^i}, \dfrac{\partial^i\varphi}{\partial y^i}$ $(i = 1,2)$ belong to $C(D_{ab})$. The norm on the space $C^2(D_{ab})$ can be defined as

$$\|\varphi\|_{C^2(D_{ab})} = \|\varphi\|_{C(D_{ab})} + \sum_{i=1}^{2}\left(\left\|\frac{\partial^i\varphi}{\partial x^i}\right\|_{C(D_{ab})} + \left\|\frac{\partial^i\varphi}{\partial y^i}\right\|_{C(D_{ab})}\right).$$

Definition 4. Let $f \in C(D_{ab})$ be a continuous function and δ is a positive number. The full continuity modulus of the function $f(x,y)$ is

$$\omega(f;\delta) = \max\{|f(x_1,y_1) - f(x_2,y_2)| : x,y \in D_{ab}, \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} \le \delta\,\}$$

and its partial continuity moduli with respect to x and y are

$$\omega^{(1)}(f;\delta) = \max_{0\le y\le b}\ \max_{|x_1-x_2|\le\delta} |f(x_1,y) - f(x_2,y)| \tag{13}$$

$$\omega^{(2)}(f;\delta) = \max_{0\le x\le a}\ \max_{|y_1-y_2|\le\delta} |f(x,y_1) - f(x,y_2)|\,. \tag{14}$$

It is known that $\lim_{\delta\to0}\omega(f;\delta) = 0$ and for any $\lambda > 0$, $\omega(f;\lambda\delta) \le (\lambda+1)\omega(f;\delta)$. The same properties are satisfied by partial continuity moduli.

3. Main results

In this section, we shall prove the following main results.

THEOREM 5. *From Lemma 1, we have*

$$P_{n,m}^{t_1,t_2}(e_0;x,y)=1, \tag{15}$$

$$P_{n,m}^{t_1,t_2}(e_1;x,y)=x+t_1+\frac{1}{n}\frac{g'(1)}{g(1)}, \tag{16}$$

$$P_{n,m}^{t_1,t_2}(e_2;x,y)=y+t_2+\frac{1}{m}\frac{g'(1)}{g(1)}, \tag{17}$$

$$\begin{aligned}P_{n,m}^{t_1,t_2}(e_3;x,y)&=(x+t_1)^2+(y+t_2)^2+\frac{2}{n}\frac{g'(1)}{g(1)}(x+t_1)+\frac{t_1}{n}\\&\quad+\frac{1}{n^2}\frac{g'(1)+g''(1)}{g(1)}+\frac{2}{m}\frac{g'(1)}{g(1)}(y+t_2)+\frac{t_2}{m}+\frac{1}{m^2}\frac{g'(1)+g''(1)}{g(1)}.\end{aligned} \tag{18}$$

Proof. From (5), it is clear that, for all $n,m\in\mathbb{N}$,

$$\begin{aligned}P_{n,m}^{t_1,t_2}(e_0;x,y)&=\frac{e^{-(nt_1+mt_2)}}{g^2(1)}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}P_k(nt_1)P_j(mt_2)\\&=\frac{e^{-(nt_1+mt_2)}}{g^2(1)}g^2(1)e^{nt_1+mt_2}=1.\end{aligned}$$

Also we obtain

$$\begin{aligned}P_{n,m}^{t_1,t_2}(e_1;x,y)&=\frac{e^{-(nt_1+mt_2)}}{g^2(1)}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}P_k(nt_1)P_j(mt_2)(x+\frac{k}{n})\\&=x\frac{e^{-(nt_1+mt_2)}}{g^2(1)}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}P_k(nt_1)P_j(mt_2)\\&\quad+\frac{1}{n}\frac{e^{-(nt_1+mt_2)}}{g^2(1)}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}kP_k(nt_1)P_j(mt_2)\end{aligned}$$

from (6) and (7), we get

$$\begin{aligned}P_{n,m}^{t_1,t_2}(e_1;x,y)&=x\frac{e^{-(nt_1+mt_2)}}{g^2(1)}g^2(1)e^{nt_1+mt_2}\\&\quad+\frac{1}{n}\frac{e^{-(nt_1+mt_2)}}{g^2(1)}\left(g'(1)g(1)+nt_1g^2(1)\right)e^{nt_1+mt_2}\\&=x+t_1+\frac{1}{n}\frac{g'(1)}{g(1)}.\end{aligned}$$

Similarly, we have

$$
\begin{aligned}
P_{n,m}^{t_1,t_2}(e_2;x,y) &= \frac{e^{-(nt_1+mt_2)}}{g^2(1)} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} P_k(nt_1)P_j(mt_2)(y+\frac{j}{m}) \\
&= y\frac{e^{-(nt_1+mt_2)}}{g^2(1)} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} P_k(nt_1)P_j(mt_2) \\
&\quad + \frac{1}{m}\frac{e^{-(nt_1+mt_2)}}{g^2(1)} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} jP_k(nt_1)P_j(mt_2) \\
&= y + t_2 + \frac{1}{m}\frac{g'(1)}{g(1)}.
\end{aligned}
$$

Finally, we can show the formula (18).

$$
\begin{aligned}
P_{n,m}^{t_1,t_2}(e_3;x,y) &= \frac{e^{-(nt_1+mt_2)}}{g^2(1)} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} P_k(nt_1)P_j(mt_2)\left((x+\frac{k}{n})^2+(y+\frac{j}{m})^2\right) \\
&= \frac{e^{-(nt_1+mt_2)}}{g^2(1)} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} P_k(nt_1)P_j(mt_2)(x+\frac{k}{n})^2 \\
&\quad + \frac{e^{-(nt_1+mt_2)}}{g^2(1)} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} P_k(nt_1)P_j(mt_2)(y+\frac{j}{m})^2 \\
&= I_1 + I_2
\end{aligned}
\tag{19}
$$

Now we consider I_1:

$$
\begin{aligned}
I_1 &= \frac{e^{-(nt_1+mt_2)}}{g^2(1)} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} P_k(nt_1)P_j(mt_2)(x+\frac{k}{n})^2 \\
&= x^2\frac{e^{-(nt_1+mt_2)}}{g^2(1)} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} P_k(nt_1)P_j(mt_2) \\
&\quad + \frac{2x}{n}\frac{e^{-(nt_1+mt_2)}}{g^2(1)} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} kP_k(nt_1)P_j(mt_2) \\
&\quad + \frac{1}{n^2}\frac{e^{-(nt_1+mt_2)}}{g^2(1)} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} k^2P_k(nt_1)P_j(mt_2)
\end{aligned}
$$

from (6), (7) and (9), we obtain

$$
I_1 = (x+t_1)^2 + \frac{2}{n}\frac{g'(1)}{g(1)}(x+t_1) + \frac{t_1}{n} + \frac{1}{n^2}\frac{g'(1)+g''(1)}{g(1)} \tag{20}
$$

similarly, we get

$$
I_2 = (y+t_2)^2 + \frac{2}{m}\frac{g'(1)}{g(1)}(y+t_2) + \frac{t_2}{m} + \frac{1}{m^2}\frac{g'(1)+g''(1)}{g(1)}. \tag{21}
$$

Using (20) and (21) in (19), we obtain the desired result. ∎

From Theorem 5, we can immediately give the following Bohman-Korovkin-type theorem:

THEOREM 6. *Let $P_{n,m}^{t_1,t_2}$ be positive on $D=[0,\infty)\times[0,\infty)$ and $x,y\geq 0$ are fixed. If $f\in C_A(D)$, then*

$$\lim_{n,m\to\infty} P_{n,m}^{t_1,t_2}(f;x,y)=f(x+t_1,y+t_2)$$

uniformly on $D_{ab}=[0,a]\times[0,b]$.

Proof. From (15)–(18), we have

$$\lim_{n,m\to\infty} P_{n,m}^{t_1,t_2}(e_i;x,y)=e_i\left(x+t_1,y+t_2\right),\ i\in\{0,1,2,3\}$$

uniformly on D_{ab}. For each fixed $t_1,t_2\geq 0$, if we apply the Korovkin type theorem (see [8]), the proof is completed. ■

THEOREM 7. *Let $P_{n,m}^{t_1,t_2}$ be given by (4) and $f\in C_A(D_{ab})$. Then, for $(x+t_1,y+t_2)\in D_{ab}$, we have*

$$\left\|P_{n,m}^{t_1,t_2}f-f(.+t_1,.+t_2)\right\|_{C_A(D_{ab})}\leq 2K(f;\frac{1}{2}\delta_{n,m})$$

where $K(f;.)$ is Peetre's K-functional defined by (12) *and* $\delta_{n,m}=\max\left\{\frac{1}{n}\frac{g'(1)}{g(1)},\frac{1}{m}\frac{g'(1)}{g(1)},\frac{1}{2}\left(\frac{t_1}{n}+\frac{1}{n^2}\frac{g'(1)+g''(1)}{g(1)}\right),\frac{1}{2}\left(\frac{t_2}{m}+\frac{1}{m^2}\frac{g'(1)+g''(1)}{g(1)}\right)\right\}$.

Proof. Let $\varphi\in C^2(D_{ab})$. Using Taylor's theorem, we can write

$$\begin{aligned}
&\varphi(z_1,z_2)-\varphi(x+t_1,y+t_2)\\
&\qquad=\varphi(z_1,y+t_2)-\varphi(x+t_1,y+t_2)+\varphi(z_1,z_2)-\varphi(z_1,y+t_2)\\
&\qquad=\frac{\partial\varphi(x,y)}{\partial x}(z_1-x-t_1)+\int_{x+t_1}^{z_1}(z_1-\lambda)\frac{\partial^2\varphi(\lambda,y)}{\partial\lambda^2}\,d\lambda\\
&\qquad\quad+\frac{\partial\varphi(x,y)}{\partial y}(z_2-y-t_2)+\int_{y+t_2}^{z_2}(z_2-\mu)\frac{\partial^2\varphi(x,\mu)}{\partial\mu^2}\,d\mu
\end{aligned}$$

Applying the operator $P_{n,m}^{t_1,t_2}$ to both sides, we deduce that

$$\begin{aligned}
&\left|P_{n,m}^{t_1,t_2}(\varphi;x,y)-\varphi(x+t_1,y+t_2)\right|\\
&\quad\leq\left|\frac{\partial\varphi}{\partial x}\right|\left|P_{n,m}(z_1-x-t_1;x,y)\right|+\left|\frac{\partial^2\varphi}{\partial x^2}\right|\left|P_{n,m}^{t_1,t_2}(\frac{1}{2}(z_1-x-t_1)^2;x,y)\right|\\
&\quad\quad+\left|\frac{\partial\varphi}{\partial y}\right|\left|P_{n,m}^{t_1,t_2}(z_2-y-t_2;x,y)\right|+\left|\frac{\partial^2\varphi}{\partial y^2}\right|\left|P_{n,m}^{t_1,t_2}(\frac{1}{2}(z_2-y-t_2)^2;x,y)\right|.
\end{aligned}$$

Since $P_{n,m}^{t_1,t_2}(z_1-x-t_1;x,y)=\frac{1}{n}\frac{g'(1)}{g(1)}$ and $P_{n,m}^{t_1,t_2}(z_2-y-t_2;x,y)=\frac{1}{m}\frac{g'(1)}{g(1)}$, we obtain

$$\begin{aligned}
&\left\|P_{n,m}^{t_1,t_2}(\varphi;x,y)-\varphi(x+t_1,y+t_2)\right\|_{C(D_{ab})}\\
&\qquad\le \frac{1}{n}\frac{g'(1)}{g(1)}\left\|\frac{\partial\varphi}{\partial x}\right\|_{C(D_{ab})}+\frac{1}{m}\frac{g'(1)}{g(1)}\left\|\frac{\partial\varphi}{\partial y}\right\|_{C(D_{ab})}\\
&\qquad+\frac{1}{2}\left\|\frac{\partial^2\varphi}{\partial x^2}\right\|_{C(D_{ab})}\left|P_{n,m}^{t_1,t_2}\left((z_1-x-t_1)^2;x,y\right)\right|\\
&\qquad+\frac{1}{2}\left\|\frac{\partial^2\varphi}{\partial y^2}\right\|_{C(D_{ab})}\left|P_{n,m}^{t_1,t_2}\left((z_2-y-t_2)^2;x,y\right)\right|.
\end{aligned}$$

We estimate $P_{n,m}^{t_1,t_2}\left((z_1-x-t_1)^2;x,y\right)$.

$$\begin{aligned}
P_{n,m}^{t_1,t_2}\left((z_1-x-t_1)^2;x,y\right)&=P_{n,m}^{t_1,t_2}(z_1^2;x,y)-2(x+t_1)P_{n,m}^{t_1,t_2}(z_1;x,y)+(x+t_1)^2\\
&=(x+t_1)^2+\frac{2}{n}\frac{g'(1)}{g(1)}(x+t_1)+\frac{t_1}{n}+\frac{1}{n^2}\frac{g'(1)+g''(1)}{g(1)}\\
&\quad-2(x+t_1)^2-\frac{2}{n}\frac{g'(1)}{g(1)}(x+t_1)+(x+t_1)^2\\
&=\frac{t_1}{n}+\frac{1}{n^2}\frac{g'(1)+g''(1)}{g(1)}
\end{aligned}$$

and we get

$$\left|P_{n,m}^{t_1,t_2}\left((z_1-x-t_1)^2;x,y\right)\right|=\frac{t_1}{n}+\frac{1}{n^2}\frac{g'(1)+g''(1)}{g(1)}. \tag{22}$$

Similarly

$$\left|P_{n,m}^{t_1,t_2}\left((z_2-y-t_2)^2;x,y\right)\right|=\frac{t_2}{m}+\frac{1}{m^2}\frac{g'(1)+g''(1)}{g(1)}. \tag{23}$$

From (22) and (23), we have

$$\begin{aligned}
&\left\|P_{n,m}\varphi-\varphi(.+t_1,.+t_2)\right\|_{C(D_{ab})}\\
&\qquad\le\frac{1}{n}\frac{g'(1)}{g(1)}\left\|\frac{\partial\varphi}{\partial x}\right\|_{C(D_{ab})}+\frac{1}{m}\frac{g'(1)}{g(1)}\left\|\frac{\partial\varphi}{\partial y}\right\|_{C(D_{ab})}\\
&\qquad+\frac{1}{2}\left(\frac{t_1}{n}+\frac{1}{n^2}\frac{g'(1)+g''(1)}{g(1)}\right)\left\|\frac{\partial^2\varphi}{\partial x^2}\right\|_{C(D_{ab})}\\
&\qquad+\frac{1}{2}\left(\frac{t_2}{m}+\frac{1}{m^2}\frac{g'(1)+g''(1)}{g(1)}\right)\left\|\frac{\partial^2\varphi}{\partial y^2}\right\|_{C(D_{ab})}\\
&\qquad\le\delta_{n,m}\left[\left\|\frac{\partial\varphi}{\partial x}\right\|_{C(D_{ab})}+\left\|\frac{\partial\varphi}{\partial y}\right\|_{C(D_{ab})}+\left\|\frac{\partial^2\varphi}{\partial x^2}\right\|_{C(D_{ab})}+\left\|\frac{\partial^2\varphi}{\partial y^2}\right\|_{C(D_{ab})}\right]\\
&\qquad\le\delta_{n,m}\left\|\varphi\right\|_{C^2(D_{ab})}
\end{aligned} \tag{24}$$

where $\delta_{n,m} = \max\left\{\frac{1}{n}\frac{g'(1)}{g(1)}, \frac{1}{m}\frac{g'(1)}{g(1)}, \frac{1}{2}\left(\frac{t_1}{n} + \frac{1}{n^2}\frac{g'(1)+g''(1)}{g(1)}\right), \frac{1}{2}\left(\frac{t_2}{m} + \frac{1}{m^2}\frac{g'(1)+g''(1)}{g(1)}\right)\right\}$.
By the linearity property of $P_{n,m}^{t_1,t_2}$, we get

$$\begin{aligned}
&\left\|P_{n,m}^{t_1,t_2} f - f(.+t_1,.+t_2)\right\|_{C(D_{ab})}\\
&\quad\leq \left\|P_{n,m}^{t_1,t_2} f - P_{n,m}^{t_1,t_2}\varphi\right\|_{C(D_{ab})} + \left\|P_{n,m}^{t_1,t_2}\varphi - \varphi(.+t_1,.+t_2)\right\|_{C(D_{ab})}\\
&\qquad + \left\|f(.+t_1,.+t_2) - \varphi(.+t_1,.+t_2)\right\|_{C(D_{ab})}\\
&\quad\leq \|f-\varphi\|_{C(D_{ab})} + \|f(.+t_1,.+t_2) - \varphi(.+t_1,.+t_2)\|_{C(D_{ab})}\\
&\qquad + \left\|P_{n,m}^{t_1,t_2} g - \varphi(.+t_1,.+t_2)\right\|_{C(D_{ab})} \qquad (25)
\end{aligned}$$

and from (24) and (25), we obtain

$$\left\|P_{n,m}^{t_1,t_2} f - f(.+t_1,.+t_2)\right\|_{C(D_{ab})} \leq 2\left(\|f-\varphi\|_{C(D_{ab})} + \frac{1}{2}\delta_{n,m}\|\varphi\|_{C^2(D_{ab})}\right).$$

We complete the proof by taking the infimum over $\varphi \in C^2(D_{ab})$. ■

Now, we are concerned with the estimate of the order of approximation of a function $f \in C_A(D)$ by means of the positive operator $P_{n,m}^{t_1,t_2}$, using the partial continuity moduli.

Theorem 8. *If $f \in C_A(D)$, then then for all $(x,y) \in D$ and $t_1, t_2 \in [0,\infty)$, we have*

$$\begin{aligned}
\left|P_{n,m}^{t_1,t_2}(f;x,y) - f(x+t_1,y+t_2)\right| &\leq \left(1+\sqrt{t_1 + \frac{1}{n}\frac{g'(1)+g''(1)}{g(1)}}\right)\omega^{(1)}(f;\frac{1}{\sqrt{n}})\\
&\quad + \left(1+\sqrt{t_2 + \frac{1}{m}\frac{g'(1)+g''(1)}{g(1)}}\right)\omega^{(2)}(f;\frac{1}{\sqrt{m}}) \qquad (26)
\end{aligned}$$

where $\omega^{(1)}(f;.)$ and $\omega^{(2)}(f;.)$ are partial continuity modulus of f given by (13) *and* (14).

Proof. Suppose that $f \in C_A(D)$. By (15), we obtain following inequality:

$$\begin{aligned}
&\left|P_{n,m}^{t_1,t_2}(f;x,y) - f(x+t_1,y+t_2)\right|\\
&\quad\leq \frac{e^{-(nt_1+mt_2)}}{g^2(1)}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}P_k(nt_1)P_j(mt_2)\left|f(x+\frac{k}{n},y+\frac{j}{m}) - f(x+t_1,y+\frac{j}{m})\right|\\
&\qquad + \frac{e^{-(nt_1+mt_2)}}{g^2(1)}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}P_k(nt_1)P_j(mt_2)\left|f(x+t_1,y+\frac{j}{m}) - f(x+t_1,y+t_2)\right|\\
&\quad = I_1 + I_2. \qquad (27)
\end{aligned}$$

Now, we consider I_1. By using well-known properties of the modulus of continuity, we obtain the formula

$$\begin{aligned}
I_1 &= \frac{e^{-(nt_1+mt_2)}}{g^2(1)}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}P_k(nt_1)P_j(mt_2)\left|f(x+\frac{k}{n},y+\frac{j}{m}) - f(x+t_1,y+\frac{j}{m})\right|\\
&\leq \omega^{(1)}(f;\delta_n)\left\{1 + \frac{1}{\delta_n}\frac{e^{-(nt_1+mt_2)}}{g^2(1)}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}P_k(nt_1)P_j(mt_2)\left|\tfrac{k}{n} - t_1\right|\right\}.
\end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$I_1 \leq \omega^{(1)}(f;\delta_1)\Big\{1+\frac{1}{\delta_n}\Big(\frac{e^{-(nt_1+mt_2)}}{g^2(1)}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}P_k(nt_1)P_j(mt_2)\left(\tfrac{k^2}{n^2}-2t_1\tfrac{k}{n}+t_1^2\right)\Big)^{1/2}\Big\}$$

and by (6), (7) and (9), we get

$$I_1 \leq \omega^{(1)}(f;\delta_1)\left\{1+\frac{1}{\delta_1}\sqrt{\frac{t_1}{n}+\frac{1}{n^2}\frac{g'(1)+g''(1)}{g(1)}}\right\}$$

By taking $\delta_n = \frac{1}{\sqrt{n}}$, we obtain

$$I_1 \leq \left(1+\sqrt{t_1+\frac{1}{n}\frac{g'(1)+g''(1)}{g(1)}}\right)\omega^{(1)}(f;\frac{1}{\sqrt{n}}). \tag{28}$$

In a similar way, using (6), (8) and (10), we have

$$I_2 \leq \left(1+\sqrt{t_2+\frac{1}{m}\frac{g'(1)+g''(1)}{g(1)}}\right)\omega^{(2)}(f;\frac{1}{\sqrt{m}}). \tag{29}$$

Using (28) and (29) in (27), the proof is completed. ■

Now, we will study error estimation in terms of higher order partial moduli of continuity in simultaneous approximation for the operators (4).

Theorem 9. *If $f \in C_A^r(D)$, then we have*

i) $$\left|\frac{\partial^r}{\partial t_1^r}P_{n,m}^{t_1,t_2}(f;x,y)-\frac{\partial^r}{\partial t_1^r}f(x+t_1,y+t_2)\right|$$
$$\leq \left(1+\sqrt{t_1+\frac{1}{n}\frac{g'(1)+g''(1)}{g(1)}}\right)\omega^{(1)}(\frac{\partial^r}{\partial t_1^r}f;\frac{1}{\sqrt{n}}+\frac{r}{n})$$
$$+\left(1+\sqrt{t_2+\frac{1}{m}\frac{g'(1)+g''(1)}{g(1)}}\right)\omega^{(2)}(\frac{\partial^r}{\partial t_2^r}f;\frac{1}{\sqrt{m}}+\frac{r}{m})$$
$$+\,\omega^{(1)}(\frac{\partial^r}{\partial t_1^r}f;\frac{r}{n}) \tag{30}$$

ii) $$\left|\frac{\partial^r}{\partial t_2^r}P_{n,m}^{t_1,t_2}(f;x,y)-\frac{\partial^r}{\partial t_2^r}f(x+t_1,y+t_2)\right|$$
$$\leq \left(1+\sqrt{t_1+\frac{1}{n}\frac{g'(1)+g''(1)}{g(1)}}\right)\omega^{(1)}(\frac{\partial^r}{\partial t_1^r}f;\frac{1}{\sqrt{n}}+\frac{r}{n})$$
$$+\left(1+\sqrt{t_2+\frac{1}{m}\frac{g'(1)+g''(1)}{g(1)}}\right)\omega^{(2)}(\frac{\partial^r}{\partial t_2^r}f;\frac{1}{\sqrt{m}}+\frac{r}{m})$$
$$+\,\omega^{(2)}(\frac{\partial^r}{\partial t_2^r}f;\frac{r}{m}) \tag{31}$$

Proof. *i*) The partial derivative of Eq. (4) with respect to t_1 may be written as follows:

$$\frac{\partial}{\partial t_1} P_{n,m}^{t_1,t_2}(f;x,y) = \frac{e^{-(nt_1+mt_2)}}{g^2(1)} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} P_k(nt_1)P_j(mt_2)\frac{\Delta^1_{1/n,x} f(x+\frac{k}{n}, y+\frac{j}{m})}{1/n} \tag{32}$$

where $\Delta^1_{1/n,x} f(x+\frac{k}{n}, y+\frac{j}{m}) = f(x+\frac{k+1}{n}, y+\frac{j}{m}) - f(x+\frac{k}{n}, y+\frac{j}{m})$. From (32) one computes the r-th derivative of $P_{n,m}^{t_1,t_2}$ as

$$\begin{aligned}
&\frac{\partial^r}{\partial t_1^r} P_{n,m}^{t_1,t_2}(f;x,y) \\
&\quad= \frac{e^{-(nt_1+mt_2)}}{g^2(1)} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} P_k(nt_1)P_j(mt_2)\frac{\Delta^r_{1/n,x} f(x+\frac{k}{n}, y+\frac{j}{m})}{(1/n)^r} \\
&\quad= r!\frac{e^{-(nt_1+mt_2)}}{g^2(1)} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} P_k(nt_1)P_j(mt_2)\frac{\Delta^r_{1/n,x} f(x+\frac{k}{n}, y+\frac{j}{m})}{r!\,(1/n)^r} \\
&\quad= r!\frac{e^{-(nt_1+mt_2)}}{g^2(1)} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} P_k(nt_1)P_j(mt_2) \\
&\qquad\times \left[x+\frac{k}{n}, x+\frac{k+1}{n}, \ldots, x+\frac{k+r}{n}; f, y+\frac{j}{m}\right] \\
&\quad= r!\frac{e^{-(nt_1+mt_2)}}{g^2(1)} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} P_k(nt_1)P_j(mt_2)h(x+\frac{k}{n}, y+\frac{j}{m})
\end{aligned}$$

where $h(t_1,t_2) = \left[t_1, t_1+\frac{1}{n}, \ldots, t_1+\frac{r}{n}; f, t_2\right]$ and so we obtain

$$\frac{\partial^r}{\partial t_1^r} P_{n,m}^{t_1,t_2}(f;x,y) = r! P_{n,m}^{t_1,t_2}(h;x,y). \tag{33}$$

From (33) the difference $\left|\frac{\partial^r}{\partial t_1^r} P_{n,m}^{t_1,t_2}(f;x,y) - \frac{\partial^r}{\partial t_1^r} f(x+t_1, y+t_2)\right|$ is represented as follows:

$$\begin{aligned}
&\left|\frac{\partial^r}{\partial t_1^r} P_{n,m}^{t_1,t_2}(f;x,y) - \frac{\partial^r}{\partial t_1^r} f(x+t_1, y+t_2)\right| \\
&\qquad\leq r!\left|P_{n,m}^{t_1,t_2}(h;x,y) - h(x+t_1, y+t_2)\right| \\
&\qquad\quad+ \left|r! h(x+t_1, y+t_2) - \frac{\partial^r}{\partial t_1^r} f(x+t_1, y+t_2)\right|
\end{aligned}$$

By using (26),we obtain

$$\left|\frac{\partial^r}{\partial t_1^r} P_{n,m}^{t_1,t_2}(f;x,y) - \frac{\partial^r}{\partial t_1^r} f(x+t_1, y+t_2)\right|$$

$$\leq r!\left(1+\sqrt{t_1+\frac{1}{n}\frac{g'(1)+g''(1)}{g(1)}}\right)\omega^{(1)}(h;\frac{1}{\sqrt{n}})$$
$$+\,r!\left(1+\sqrt{t_2+\frac{1}{m}\frac{g'(1)+g''(1)}{g(1)}}\right)\omega^{(2)}(h;\frac{1}{\sqrt{m}})$$
$$+\left|r!h(x+t_1,y+t_2)-\frac{\partial^r}{\partial t_1^r}f(x+t_1,y+t_2)\right|. \quad (34)$$

On the other hand, we write

$$\left|h(t_1+\delta_1,t_2+\delta_2)-h(t_1,t_2)\right|$$
$$\leq |h(t_1+\delta_1,t_2+\delta_2)-h(t_1+\delta_1,t_2)|+|h(t_1+\delta_1,t_2)-h(t_1,t_2)|$$
$$=I_1+I_2. \quad (35)$$

We estimate I_1:

$$I_1=|h(t_1+\delta_1,t_2+\delta_2)-h(t_1+\delta_1,t_2)|$$
$$=\left|\left[t_2+\delta_2,t_2+\delta_2+\frac{1}{m},\ldots,t_2+\delta_2+\frac{r}{m};f,t_1+\delta_1\right]\right.$$
$$\left.-\left[t_2,t_2+\frac{1}{m},\ldots,t_2+\frac{r}{m};f,t_1+\delta_1\right]\right|$$
$$=\frac{1}{r!}\left|\frac{\partial^r}{\partial t_2^r}f(t_1+\delta_1,t_2+\delta_2+\theta_1\frac{r}{m})-\frac{\partial^r}{\partial t_2^r}f(t_1+\delta_1,t_2+\delta_2+\theta_2\frac{r}{m})\right|$$

where $\theta_1,\theta_2\in(0,1)$. Hence we get

$$I_1\leq\frac{1}{r!}\omega^{(2)}(\frac{\partial^r}{\partial t_2^r}f;\delta_2+|\theta_1-\theta_2|\,\frac{r}{m})\leq\frac{1}{r!}\omega^{(2)}(\frac{\partial^r}{\partial t_2^r}f;\delta_2+\frac{r}{m}) \quad (36)$$

similarly, we have

$$I_2\leq\frac{1}{r!}\omega^{(1)}(\frac{\partial^r}{\partial t_1^r}f;\delta_1+\frac{r}{n}). \quad (37)$$

By inserting (36) and (37) in (35), we get

$$|h(t_1+\delta_1,t_2+\delta_2)-h(t_1,t_2)|\leq\frac{1}{r!}\omega^{(1)}(\frac{\partial^r}{\partial t_1^r}f;\delta_1+\frac{r}{n})+\frac{1}{r!}\omega^{(2)}(\frac{\partial^r}{\partial t_2^r}f;\delta_2+\frac{r}{m}).$$

By taking $\delta_1=\frac{1}{\sqrt{n}}$ and $\delta_2=\frac{1}{\sqrt{m}}$ we obtain

$$\omega^{(1)}(h;\frac{1}{\sqrt{n}})\leq\frac{1}{r!}\omega^{(1)}(\frac{\partial^r}{\partial t_1^r}f;\frac{1}{\sqrt{n}}+\frac{r}{n}) \quad (38)$$
$$\omega^{(2)}(h;\frac{1}{\sqrt{m}})\leq\frac{1}{r!}\omega^{(2)}(\frac{\partial^r}{\partial t_2^r}f;\frac{1}{\sqrt{m}}+\frac{r}{m}). \quad (39)$$

On the other hand, we write

$$\left|r!\,h(x+t_1,y+t_2)-\frac{\partial^r}{\partial t_1^r}f(x+t_1,y+t_2)\right|=$$

$$
\begin{aligned}
&= \left| r! \left[x+t_1, x+t_1+\frac{1}{n}, \ldots, x+t_1+\frac{r}{n}; f, y+t_2 \right] - \frac{\partial^r}{\partial t_1^r} f(x+t_1, y+t_2) \right| \\
&\leq \left| \frac{\partial^r}{\partial t_1^r} f(x+t_1+\theta_3 \frac{r}{n}, y+t_2) - \frac{\partial^r}{\partial t_1^r} f(x+t_1, y+t_2) \right| \\
&\leq \omega^{(1)}\left(\frac{\partial^r}{\partial t_1^r} f; \theta_3 \frac{r}{n}\right) \\
&\leq \omega^{(1)}\left(\frac{\partial^r}{\partial t_1^r} f; \frac{r}{n}\right) \qquad (40)
\end{aligned}
$$

by using (38), (39) and (40) in (34), we have the desired result. The proof of (ii) can be made in a similar way. ■

REFERENCES

[**1**] F. Altomare, M. Campiti, *Korovkin Type Approximation Theory and its Applications*, Walter de Gryter Publ., Berlin, 1994.

[**2**] Ç. Atakut, Í. Büyükyazıcı, *Stancu type generalization of the Favard-Szàsz operators*, Appl. Math. Letters **23** (2010), 1479–1482.

[**3**] G. Bleimann, P.L. Butzer, L. Hahn, *A Bernstein-type operator approximating continuous functions on the semi-axis*, Indag. Math. **42** (1980), 255–262.

[**4**] A. Ciupa, *On a generalized Favard-Szàsz type operator*, Seminar on Numerical and statistical calculus, Babeş-Bolyai Univ., Cluj-Napoca, preprint No. 1 (1994), 33–38.

[**5**] R.A. Devore, G.G. Lorentz, *Constructive Approximation*, Springer, Berlin, 1993.

[**6**] A. Jakimovski, D. Leviatan, *Generalized Szàsz operators for the approximation in the infinite interval*, Mathematica **11** (34) (1969), 97–103.

[**7**] Ovidiu T. Pop, *About some linear operators*, Intern. J. Math. Math. Sci., Volume 2007, Article ID 91781, 13 pages.

[**8**] V.I.Volkov, *On the convergence of sequences of linear positive operators in the space of two variables*, Dokl. Akad. Nauk. SSSR **115** (1957), 17–19.

[**9**] B. Wood, *Generalized Szàsz operators for the approximation in the complex domain*, SIAM J. Appl. Math. **17** (1969), 790–801.

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