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A NEW CHARACTERIZATION OF SPACES WITH LOCALLY COUNTABLE sn-NETWORKS

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Abstract. In this paper we prove that a space X is with a locally countable sn-network (resp., weak base) if and only if it is a compact-covering (resp., compact-covering quotient) compact and ss-image of a metric space, if and only if it is a sequentially-quotient (resp., quotient) π- and ss-image of a metric space, which gives a new characterization of spaces with locally countable sn-networks (or weak bases).

1. Introduction

In 2002, Y. Ikeda, C. Liu and Y. Tanaka introduced the notion of σ-strong networks as a generalization of "development" in developable spaces, and considered certain quotient images of metric spaces in terms of σ-strong networks. By means of σ-strong networks, some characterizations for the quotient and compact images of metric spaces are obtained (see in [4, 18, 19], for example).

In this paper, by means of σ-strong networks, we give a new characterization of spaces with locally countable sn-networks (or weak bases). Throughout this paper, all spaces are assumed to be T_1 and regular, all maps are continuous and onto, $\mathbb{N}$ denotes the set of all natural numbers. Let $\mathcal{P}$ and $\mathcal{Q}$ be two families of subsets of X, and $f : X \longrightarrow Y$ be a map, we denote $\mathcal{P} \bigwedge \mathcal{Q} = \{P \cap Q : P \in \mathcal{P}, Q \in \mathcal{Q}\}$, $\bigcap \mathcal{P} = \bigcap\{P : P \in \mathcal{P}\}$, $\bigcup \mathcal{P} = \bigcup\{P : P \in \mathcal{P}\}$, $\mathrm{st}(x, \mathcal{P}) = \bigcup\{P \in \mathcal{P} : x \in P\}$, and $f(\mathcal{P}) = \{f(P) : P \in \mathcal{P}\}$. For a sequence $\{x_n\}$ converging to x and $P \subset X$, we say that $\{x_n\}$ is *eventually* in P if $\{x\} \bigcup \{x_n : n \geq m\} \subset P$ for some $m \in \mathbb{N}$, and $\{x_n\}$ is *frequently* in P if some subsequence of $\{x_n\}$ is eventually in P.

Definition 1.1. Let X be a space and P be a subset of X.

(1) P is a *sequential neighborhood* of x in X, if each sequence S converging to x is eventually in P.

(2) P is a *sequentially open* subset of X, if P is a sequential neighborhood of x in X for all $x \in P$.

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Definition 1.2. Let $\mathcal{P}$ be a collection of subsets of a space X and let K be a subset of X. Then,

(1) For each $x \in X$, $\mathcal{P}$ is a *network at* x [18], if $x \in P$ for every $P \in \mathcal{P}$, and if $x \in U$ with U is open in X, there exists $P \in \mathcal{P}$ such that $x \in P \subset U$.

(2) $\mathcal{P}$ is a *network* for X [18], if $\{P \in \mathcal{P} : x \in P\}$ is a network at x in X for all $x \in X$.

(3) $\mathcal{P}$ is a *cs*-network* for X [19], if for each sequence S converging to a point $x \in U$ with U is open in X, S is frequently in $P \subset U$ for some $P \in \mathcal{P}$.

(4) $\mathcal{P}$ is a *cs-network* for X [19], if each sequence S converging to a point $x \in U$ with U is open in X, S is eventually in $P \subset U$ for some $P \in \mathcal{P}$.

(5) $\mathcal{P}$ is a *cfp-cover* of K in X [13], if $\mathcal{P}$ is a cover of K in X such that it can be precisely refined by some finite cover of K consisting of compact subsets of K.

(6) $\mathcal{P}$ is a *cfp-cover* for X [13], if whenever K is a compact subset of X, there exists a finite subfamily $\mathcal{G} \subset \mathcal{P}$ such that $\mathcal{G}$ is a cfp-cover of K.

(7) $\mathcal{P}$ is *locally countable*, if for each $x \in X$, there exists a neighborhood V of x such that V meets only countably many members of $\mathcal{P}$.

(8) $\mathcal{P}$ is *point-countable* (resp., *point-finite*), if each point $x \in X$ belongs to only countably (resp., finitely) many members of $\mathcal{P}$.

(9) $\mathcal{P}$ is *star-countable* [15], if each $P \in \mathcal{P}$ meets only countably many members of $\mathcal{P}$.

Definition 1.3. Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a family of subsets of a space X satisfying that, for every $x \in X$, $\mathcal{P}_x$ is a network at x in X, and if $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.

(1) $\mathcal{P}$ is a *weak base* for X [1], if whenever $G \subset X$ satisfying for every $x \in G$, there exists $P \in \mathcal{P}_x$ with $P \subset G$, then G is open in X. Here, $\mathcal{P}_x$ is a *weak neighborhood base* at x in X.

(2) $\mathcal{P}$ is an *sn-network* for X [10], if each member of $\mathcal{P}_x$ is a sequential neighborhood of x for all $x \in X$. Here, $\mathcal{P}_x$ is an *sn-network* at x in X.

Definition 1.4. Let $f : X \longrightarrow Y$ be a map.

(1) f is a *sequence-covering* map [16], if for every convergent sequence S in Y, there exists a convergent sequence L in X such that $f(L) = S$. Note that a sequence-covering map is a *strong sequence-covering* map in the sense of [9].

(2) f is a *compact-covering* map [14], if for each compact subset K of Y, there exists a compact subset L of X such that $f(L) = K$.

(3) f is a *pseudo-sequence-covering* map [8], if for each convergent sequence S in Y, there exists a compact subset K of X such that $f(K) = S$. Note that a pseudo-sequence-covering map is a *sequence-covering* map in the sense of [7].

(4) f is a *subsequence-covering* map [12], if for each convergent sequence S in Y, there exists a compact subset K of X such that $f(K)$ is a subsequence of S.

(5) f is a *sequentially-quotient* map [2], if for each convergent sequence S in Y, there exists a convergent sequence L in X such that $f(L)$ is a subsequence of S.

(6) f is a *quotient* map [3], if whenever $U \subset Y$, U is open in Y if and only if $f^{-1}(U)$ is open in X.

(7) f is an *ss-map* [18], if for each $y \in Y$, there exists a neighborhood U of y such that $f^{-1}(U)$ is separable in X.

(8) f is a *compact* map [19], if $f^{-1}(y)$ is compact in X for all $y \in Y$.

(9) f is a *π-map* [1], if for every $y \in Y$ and for every neighborhood U of y in Y, $d(f^{-1}(y), X - f^{-1}(U)) > 0$, where X is a metric space with a metric d.

Definition 1.5. Let X be a space. Then,

(1) X is a *g-first countable* space [1] (resp., an *sn-first countable* space [3], if there is a countable weak neighborhood base (resp., *sn*-network) at x in X for all $x \in X$.

(2) X is an *$\aleph_0$-space* [14], if it has a countable *cs*-network.

(3) X is a *sequential* space [19], if every sequential open subset of X is open in X.

(4) X is a *Fréchet* space, if for each $x \in \overline{A}$, there exists a sequence in A converging to x in X.

Definition 1.6. [8] Let $\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a sequence of covers of a space X such that $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for every $n \in \mathbb{N}$. $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a *σ-strong network* for X, if $\{\mathrm{st}(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ is a network at x for all $x \in X$.

Notation 1.7. Let $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a σ-strong network for a space X. For each $n \in \mathbb{N}$, put $\mathcal{P}_n = \{P_\alpha : \alpha \in \Lambda_n\}$ and endow Λ_n with the discrete topology. Then,

$$M = \Big\{\alpha = (\alpha_n) \in \prod_{n \in \mathbb{N}} \Lambda_n : \{P_{\alpha_n}\} \text{ forms a network at some point } x_\alpha \in X\Big\}$$

is a metric space and the point x_α is unique in X for every $\alpha \in M$. Define $f : M \longrightarrow X$ by $f(\alpha) = x_\alpha$. Let us call $(f, M, X, \mathcal{P}_n)$ a *Ponomarev's system*, following [13].

For some undefined or related concepts, we refer the reader to [8, 11, 19].

2. Main results

Theorem 2.1. *The following are equivalent for a space X.*

(1) *X is an sn-first countable space with a locally countable cs^*-network;*

(2) *X has a locally countable sn-network;*

(3) *X has a σ-strong network $\mathcal{U} = \bigcup\{\mathcal{U}_n : n \in \mathbb{N}\}$ satisfying the following:*

(a) *Each $\mathcal{U}_n$ is a point-finite cfp-cover;*

(b) $\mathcal{U}$ *is locally countable.*

(4) X *is a compact-covering compact and ss-image of a metric space;*

(5) X *is a pseudo-sequence-covering compact and ss-image of a metric space;*

(6) X *is a subsequence-covering compact and ss-image of a metric space;*

(7) X *is a sequentially-quotient* π *and ss-image of a metric space.*

Proof. (1) $\Longrightarrow$ (2). Similar to the proof of (2) $\Longrightarrow$ (1) in Theorem 2.12 [3].

(2) $\Longrightarrow$ (3). Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\} = \{P_\alpha : \alpha \in \Lambda\}$ be a locally countable sn-network for X, where each $\mathcal{P}_x$ is an sn-network at x. Since X is a regular space, we can assume that each element of $\mathcal{P}$ is closed. Then, for each $x \in X$, there exists an open neighborhood V_x of x such that V_x meets only countably many members of $\mathcal{P}$. Let

$$\mathcal{Q} = \{P \in \mathcal{P} : P \subset V_x \text{ for some } x \in X\}.$$

Then, $\mathcal{Q}$ is a locally countable and star-countable network for X. By Lemma 2.1 in [15], $\mathcal{Q} = \bigcup_{\alpha\in\Lambda} \mathcal{Q}_\alpha$, where each $\mathcal{Q}_\alpha$ is a countable subfamily of $\mathcal{Q}$ and $(\bigcup \mathcal{Q}_\alpha) \cap (\bigcup \mathcal{Q}_\beta) = \emptyset$ for all $\alpha \neq \beta$. For each $\alpha \in \Lambda$, let $\mathcal{Q}_\alpha = \{P_{\alpha,n} : n \in \mathbb{N}\}$, and for each $i \in \mathbb{N}$, denote $\mathcal{H}_i = \{P_{\alpha,i} : \alpha \in \Lambda\}$. Then, $\mathcal{Q} = \bigcup\{\mathcal{Q}_i : i \in \mathbb{N}\}$. Now, for each $i \in \mathbb{N}$, let

$$A_i = \{x \in X : \mathcal{H}_i \cap \mathcal{P}_x = \emptyset\}, \quad \mathcal{G}_i = \mathcal{H}_i \cup \{A_i\}.$$

Then, we have

(a) $\bigcup\{\mathcal{G}_n : n \in \mathbb{N}\}$ is locally countable.

(b) Each $\mathcal{G}_i$ is point-finite.

(c) Each $\mathcal{G}_i$ is a cfp-cover for X. Let K be a non-empty compact subset of X. We shall show that there exists a finite subset of $\mathcal{G}_i$ which forms a cfp-cover of K. In fact, since X has a locally countable sn-network, K is metrizable. Note that each $\bigcup \mathcal{Q}_\alpha$ is sequentially open in X and $(\bigcup \mathcal{Q}_\alpha) \cap (\mathcal{Q}_\beta) = \emptyset$ for all $\alpha \neq \beta$, so the family $\{\alpha \in \Lambda : K \cap (\bigcup \mathcal{Q}_\alpha) \neq \emptyset\}$ is finite. Thus, K meets only finitely many members of $\mathcal{G}_i$. Let $\Gamma_i = \{\alpha : P_\alpha \in \mathcal{H}_i, P_\alpha \cap K \neq \emptyset\}$. For each $\alpha \in \Gamma_i$, put $K_\alpha = P_\alpha \cap K$, then $K_i = \overline{K - \bigcup_{\alpha\in\Gamma_i} K_\alpha}$. It is obvious that all K_α and K_i are closed subset of K, and $K = K_i \cup (\bigcup_{\alpha\in\Gamma_i} K_\alpha)$. Now, we only need to show $K_i \subset A_i$ for all $i \in \mathbb{N}$. Let $x \in K_i$, then there exists a sequence $\{x_n\}$ of $K - \bigcup_{\alpha\in\Gamma_i} K_\alpha$ converging to x. If $P \in \mathcal{P}_x \cap \mathcal{H}_i$, then P is a sequential neighborhood of x and $P = P_\alpha$ for some $\alpha \in \Gamma_i$. Thus, $x_n \in P$ whenever $n \geq m$ for some $m \in \mathbb{N}$. Hence, $x_n \in K_\alpha$ for some $\alpha \in \Gamma_i$, a contradiction. So, $\mathcal{P}_x \cap \mathcal{H}_i = \emptyset$, and $x \in A_i$. This implies that $K_i \subset A_i$ and $\{A_i\} \bigcup\{P_\alpha : \alpha \in \Gamma_i\}$ is a cfp-cover of K.

(d) Each $\{\mathrm{st}(x, \mathcal{G}_n) : n \in \mathbb{N}\}$ is a network at x. Let $x \in U$ with U is open in X. Then, $x \in P \subset U \cap V_x$ for some $P \in \mathcal{P}_x$, so $P \in \mathcal{Q}$. Thus, there exists a unique $\alpha \in \Lambda$ such that $P \in \mathcal{Q}_\alpha$. Hence, $P = P_{\alpha,i} \in \mathcal{H}_i$ for some $i \in \mathbb{N}$. Since $P \in \mathcal{H}_i \cap \mathcal{P}_x$, $x \notin A_i$. Note that $P \cap P_{\alpha,j} = \emptyset$ for all $j \neq i$. Then, $\mathrm{st}(x, \mathcal{G}_i) = P \subset U$. Therefore, $\{\mathrm{st}(x, \mathcal{G}_n) : n \in \mathbb{N}\}$ is a network at x for all $x \in X$.

Next, for each $n \in \mathbb{N}$, put $\mathcal{U}_n = \bigwedge\{\mathcal{G}_i : i \leq n\}$. Then, $\bigcup\{\mathcal{U}_n : n \in \mathbb{N}\}$ is a σ-strong network and each $\mathcal{U}_n$ is a point-finite cfp-cover for X. Now, we shall show that $\bigcup\{\mathcal{U}_n : n \in \mathbb{N}\}$ is locally countable. In fact, since $\mathcal{P}$ is locally countable, $\mathcal{V} = (\{A_i : i \in \mathbb{N}\})\bigcup \mathcal{P}$ is locally countable. Thus, $\{\bigcap \mathcal{F} : \mathcal{F}$ is a finite subfamily of $\mathcal{V}\}$ is locally countable. Furthermore, since $\bigcup\{\mathcal{G}_i : i \in \mathbb{N}\} \subset \mathcal{V}$, we have

$$\bigcup\{\mathcal{U}_n : n \in \mathbb{N}\} \subset \Big\{\bigcap \mathcal{F} : \mathcal{F} \text{ is a finite subfamily of } \mathcal{V}\Big\}.$$

This implies that $\bigcup\{\mathcal{U}_n : n \in \mathbb{N}\}$ is locally countable. Therefore, (3) holds.

(3) $\Longrightarrow$ (4). Let $\mathcal{U} = \bigcup\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a σ-strong network satisfying (3). Consider the Ponomarev's system $(f, M, X, \mathcal{U}_n)$. Because each $\mathcal{U}_n$ is a point-finite and locally countable cfp-cover, it follows from Lemma 2.2 [19] that f is a compact-covering and compact map. We only need to show f is an ss-map. Let $x \in X$, since $\mathcal{U}$ is locally countable, there exists a neighborhood V of x such that V meets only countably many members of $\mathcal{U}$. For each $i \in \mathbb{N}$, let $\Delta_i = \{\alpha \in \Lambda_i : P_\alpha \cap V \neq \emptyset\}$. Then, each Δ_i is countable. On the other hand, since $f^{-1}(V) \subset \prod_{i\in\mathbb{N}} \Delta_i$, $f^{-1}(V)$ is separable in M. Therefore, (4) holds.

(4) $\Longrightarrow$ (5) $\Longrightarrow$ (6). It is obvious.

(6) $\Longrightarrow$ (1). Let $f : M \longrightarrow X$ be a sequentially-quotient π and ss-map. It follows from Corollary 2.6 [4] that X has a σ-strong network $\mathcal{G} = \bigcup\{\mathcal{G}_n : n \in \mathbb{N}\}$, where each $\mathcal{G}_n$ is a cs^*-cover. For each $x \in X$, let $\mathcal{G}_x = \{\mathrm{st}(x, \mathcal{G}_n) : n \in \mathbb{N}\}$. Since each $\mathcal{P}_n$ is a cs^*-cover, it implies that $\bigcup\{\mathcal{G}_x : x \in X\}$ is an sn-network for X. Hence, X is an sn-first countable space. Now, let $\mathcal{B}$ be a point-countable base for M, since f is a sequentially-quotient and ss-map, $f(\mathcal{B})$ is a locally countable cs^*-network for X. Therefore, (1) holds. ∎

Corollary 2.2. *The following are equivalent for a space X.*

(1) *X has a locally countable weak base;*

(2) *X is a sequential space with a σ-strong network $\mathcal{U} = \bigcup\{\mathcal{U}_n : n \in \mathbb{N}\}$ satisfying the following:*

  (a) *Each $\mathcal{U}_n$ is a point-finite cfp-cover;*

  (b) *$\mathcal{U}$ is locally countable.*

(3) *X is a compact-covering quotient compact and ss-image of a metric space;*

(4) *X is a pseudo-sequence-covering quotient compact and ss-image of a metric space;*

(5) *X is a subsequence-covering quotient compact and ss-image of a metric space;*

(6) *X is a quotient π and ss-image of a metric space.*

Example 2.3. Let C_n be a convergent sequence containing its limit point p_n for each $n \in \mathbb{N}$, where $C_m \cap C_n = \emptyset$ if $m \neq n$. Let $\mathbb{Q} = \{q_n : n \in \mathbb{N}\}$ be the set of all rational numbers of the real line $\mathbb{R}$. Put $M = (\bigoplus\{C_n : n \in \mathbb{N}\}) \oplus \mathbb{R}$ and let X be the quotient space obtained from M by identifying each p_n in C_n with q_n in $\mathbb{R}$. Then, by the proof of Example 3.1 [6], X has a countable weak base and X is not a sequence-covering quotient and π-image of a metric space. Hence,

(1) A space with a locally countable sn-network $\not\Rightarrow$ a sequence-covering and π-image of a metric space.

(2) A space with a locally countable weak base $\not\Rightarrow$ a sequence-covering quotient and π-image of a metric space.

EXAMPLE 2.4. Using Example 3.1 [5], it is easy to see that X is Hausdorff, non-regular and X has a countable base, but it is not a sequentially-quotient and π-image of a metric space. This shows that regular properties of X can not be omitted in Theorem 2.1 and Corollary 2.2.

EXAMPLE 2.5. S_ω is a Fréchet and $\aleph_0$-space, but it is not first countable. Thus, S_ω has a locally countable cs-network. Since S_ω is not first countable, it has not locally countable sn-network. Hence, a space with a locally countable cs-network $\not\Rightarrow$ a sequentially-quotient and π-image of a metric space.

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