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FUNCTIONS FROM L_p-SPACES AND TAYLOR MEANS

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Abstract. In this paper, we take up Taylor means to study the degree of approximation of $f \in L_p$ $(p \geqslant 1)$ under the L_p-norm and obtain a general theorem which is used to obtain four more theorems that improve some earlier results obtained by Mohapatra, Holland and Sahney [J. Approx. Theory 45 (1985), 363–374]. One of our theorems provides the Jackson order as the degree of approximation for a subspace of $\mathrm{Lip}(\alpha, p)$ $(0 < \alpha < 1,\ p \geqslant 1)$ and generalizes a result due to Chui and Holland [J. Approx. Theory 39 (1983), 24–38].

1. Definitions and notations

Let $f \in L_p[0, 2\pi]$, $p \geqslant 1$, and let $s_n(f; x)$ denote the partial sum of first $(n+1)$ terms of the Fourier series of f at a point $x \in [0, 2\pi]$. Throughout the paper all norms are taken with respect to x and we write

$$\|f\|_p = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^p \, dx \right\}^{1/p} \quad (1 \leqslant p < \infty), \tag{1.1}$$

$$\|f\|_\infty = \|f\|_c = \sup_{0 \leqslant x \leqslant 2\pi} |f(x)|. \tag{1.2}$$

Suppose that $\omega(\delta; f)$, $\omega_p(\delta; f)$ and $\omega_p^{(2)}(\delta; f)$, respectively, stand for modulus of continuity, integral modulus of continuity and integral modulus of smoothness of f, which are non-negative and non-decreasing (see [8, pp. 42 and 45]). Also see [4, p. 612]. For $0 < \alpha \leqslant 1$, we write: (i) $f \in \mathrm{Lip}\,\alpha$ if $\omega(\delta; f) = O(\delta^\alpha)$ and (ii) $f \in \mathrm{Lip}(\alpha, p)$ if $\omega_p(\delta; f) = O(\delta^\alpha)$. Throughout the paper, $f \in L_p$ $(p \geqslant 1)$ is taken to be non-constant so that (see [8, p. 45])

$$n^{-1} = O(1)\omega_p(n^{-1}; f), \quad \text{as } n \to \infty. \tag{1.3}$$

The space $L_p[0, 2\pi]$, where $p = \infty$, contains the space $C_{2\pi}$. The class $\mathrm{Lip}(\alpha, p)$ with $p = \infty$ will be taken as $\mathrm{Lip}\,\alpha$.

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Let (a_{nk}) be an infinite matrix defined by

$$\frac{(1-r)^{n+1}\theta^n}{(1-r\theta)^{n+1}} = \sum_{k=0}^{\infty} a_{nk}\theta^k \qquad (|r\theta|<1,\ \ n=0,1,\dots,\infty). \tag{1.4}$$

Then the Taylor mean of $(s_n(f,x))$ is given by

$$T_n^r(f;x) = \sum_{k=0}^{\infty} a_{nk} s_n(f,x), \tag{1.5}$$

whenever the series on the right-hand side of (1.5) is convergent for each $n = 0,1,2,\dots$ (see [6]).

In this paper, we shall use the following notations for $0 < r < 1$, $0 < t \leqslant \pi$ and for real x:

$$\phi_x(t) = \frac{1}{2}\{f(x+t)+f(x-t)-2f(x)\}, \tag{1.6}$$

$$B = \frac{r}{2(1-r)^2}, \quad h = (1-r)\sqrt{1+8B\sin^2\tfrac{1}{2}t}, \tag{1.7}$$

$$1 - r\exp(it) = h\exp(i\theta), \quad \theta = \tan^{-1}\Big\{\frac{r\sin t}{1-r\cos t}\Big\}, \tag{1.8}$$

$$L(n,r,t,\theta) = \{(1-r)/h\}^{n+1}\sin\{(n+\tfrac{1}{2})t + (n+1)\theta\}, \tag{1.9}$$

$$a_n = \pi \Big/ \Big\{(n+\tfrac{1}{2}) + (n+1)\frac{r}{1-r}\Big\} \quad \text{and} \quad b_n = a_n^{\delta} \quad (0<\delta<\tfrac{1}{2}), \tag{1.10}$$

$$c_n = (1-r)\pi/n \quad \text{and} \quad d_n = \sqrt{\frac{\log n}{An}} \quad (A>0), \tag{1.11}$$

$$R_n = \int_{c_n}^{d_n} t^{-1}\|\phi_x(t) - \phi_x(t+c_n)\|_p \exp(-Bnt^2)\,dt. \tag{1.12}$$

Define I_n similarly as R_n, with c_n and d_n replaced by a_n and b_n, respectively. We also use the inequality

$$t \leqslant \pi\sin\tfrac{1}{2}t \quad (0 \leqslant t \leqslant \pi). \tag{1.13}$$

2. Introduction

It is known [8, p. 266] that if $f \in L_p$ $(p>1)$ then the Fourier series of f converges in L_p-norm. By using Taylor transform of $s_n(f;x)$, a study has been made to find the rate of its convergence to f in L_p-norm [5, p. 371]. In 1985, Mohapatra, Holland and Sahney [7] obtained a number of results by using Taylor transform; some of them are the following.

Theorem A. *If $f \in L_p$, $p>1$, then for $0<\delta<\frac{1}{2}$,*

$$\|T_n^r(f) - f\|_p = O(1)\omega_p(n^{-1};f) + O(1)\int_{a_n}^{b_n} t^{-1}\omega_p(t;f)\,dt. \tag{2.1}$$

By (1.3), $n^\delta \exp(-Kn^{1-2\delta}) = O(1)\omega_p(n^{-1}; f)$ for $0 < \delta < \frac{1}{2}$, which is used in (2.1) and will be used in (2.7).

It may be observed that

$$\int_{a_n}^{b_n} t^{-1}\omega_p(t; f)\, dt > \frac{1}{2}\omega_p(a_n; f) \log a_n^{-1}. \tag{2.2}$$

In [7], the following result was deduced for the subspace $\mathrm{Lip}(\alpha, p)$ of the L_p space:

Theorem B. *Let $f \in \mathrm{Lip}(\alpha, p)$, where $0 < \alpha \leqslant 1$ and $p > 1$. Then*

$$\|T_n^r(f) - f\|_p = O(n^{-\alpha\delta}), \qquad 0 < \delta < \frac{1}{2}. \tag{2.3}$$

One can see that for $n > 1$, $0 < \alpha \leqslant 1$ and $0 < \delta < \frac{1}{2}$,

$$n^{-\alpha\delta} > n^{-\alpha/2} > n^{-\alpha} \log(n + 1). \tag{2.4}$$

Further, a subclass of functions from L_p was determined in [7], for which the error in approximating a function by the Taylor mean of its Fourier series is of Jackson order. We first state the general result from [7], and then the result for Jackson order will be given.

Theorem C. *Let $f \in L_p$, $p > 1$ and let the following hold:*

$$\omega_p(t; f)/t^r \quad \text{is non-increasing with } t \text{ for } 0 < r < 1, \tag{2.5}$$

$$I_n = O(1)\omega_p(n^{-1}; f), \tag{2.6}$$

where $(1 + r)/(3 + r) \leqslant \delta < \frac{1}{2}$. Then

$$\|T_n^r(f) - f\|_p = O(1)\omega_p(n^{-1}; f). \tag{2.7}$$

Theorem D. *Let $f \in \mathrm{Lip}(\alpha, p)$, $0 < \alpha < 1$, $p > 1$ and let*

$$I_n = O(n^{-\alpha}), \tag{2.8}$$

where $(1 + \alpha)/(3 + \alpha) \leqslant \delta < \frac{1}{2}$. Then

$$\|T_n^r(f) - f\|_p = O(n^{-\alpha}). \tag{2.9}$$

In 2002, one of the authors of the present paper obtained a number of order-estimates including those of “Jackson order” [1]. This has motivated us to proceed to obtain a general result and deduce from it some other order-estimates of Jackson order, as the degree of approximation of f by $B_r(f; x)$ in the L_p-norm. More precisely, we prove five theorems in this paper. Theorems 2 and 3 provide sharper estimates than those which were obtained in Theorems A and B, while, in a different setting, Theorems 4 and 5 determine different subclasses of functions $f \in L_p$, $p \geqslant 1$ to get known estimates (see the remark after the statement of Theorem 5). We first prove the following general theorem, which shall be used in the proofs of the others.

THEOREM 1. *Let $T_n^r(f,x)$ denote the Taylor mean of the Fourier series of $f \in L_p$, $1 \leqslant p \leqslant \infty$. Then*

$$\begin{aligned}\|T_n^r(f) - f\|_p = O(n^{-1}) &\int_{d_n}^{\pi/2} t^{-1}\omega_p^{(2)}(t;f)\,dt \\ &+ O(1)\int_{c_n}^{d_n} \omega_p^{(2)}(t+c_n;f)\exp(-Bnt^2)\,dt \\ &+ O(c_n)\int_{c_n}^{d_n} \frac{\omega_p^{(2)}(t+c_n;f)}{t(t+c_n)}\,dt + O(1)d_n\omega_p^{(2)}(d_n;f) + R_n \\ &+ O(1)\omega_p^{(2)}(n^{-1};f) + O(n^{-1}). \qquad (2.10)\end{aligned}$$

We deduce the following results from Theorem 1.

THEOREM 2. *Let $f \in L_p$, $1 \leqslant p \leqslant \infty$ and let*

$$t^{-1}\omega_p(t;f) \quad \text{be non-increasing with } t. \qquad (2.11)$$

Then

$$\|T_n^r(f) - f\|_p = O(1)\omega_p(n^{-1};f)\log(n+1). \qquad (2.12)$$

For a subclass of $f \in L_p$, $1 \leqslant p \leqslant \infty$, it is clear from (2.2) that Theorem 2 provides sharper estimate than Theorem A. Now for the subspace $\mathrm{Lip}(\alpha, p)$, we give another result which will be deduced from Theorem 1.

THEOREM 3. *Let $f \in \mathrm{Lip}(\alpha,p)$, $0 < \alpha \leqslant 1$, $1 \leqslant p \leqslant \infty$. Then*

$$\|T_n^r(f) - f\|_p = O(n^{-\alpha}\log(n+1)). \qquad (2.13)$$

In view of (2.4), one may observe that the estimate in (2.13) of Theorem 3 is sharper than in (2.3) of Theorem B. For two subclasses of functions $f \in L_p$, $1 \leqslant p \leqslant \infty$, we prove the following two theorems, analogous to Theorems C and D.

THEOREM 4. *Let $f \in L_p$, $1 \leqslant p \leqslant \infty$ and let (2.5) hold. If*

$$R_n = O(1)\omega_p(n^{-1};f), \qquad (2.14)$$

then

$$\|T_n^r(f) - f\|_p = O(1)\omega_p(n^{-1};f). \qquad (2.15)$$

Theorem 5. *Let $f \in \mathrm{Lip}(\alpha, p)$ for $0 < \alpha < 1$ and $1 \leqslant p \leqslant \infty$ and let*

$$R_n = O(n^{-\alpha}). \tag{2.16}$$

Then (2.9) *holds, i.e.*

$$\|T_n^r(f) - f\|_p = O(n^{-\alpha}).$$

Remark. We observe that $a_n < c_n < d_n < b_n$ and

$$R_n = \int_{c_n}^{d_n} \frac{\|\phi_x(t) - \phi_x(t + a_n)\|_p}{t} \exp(-Bnt^2)\, dt + O(n^{-2} \log n). \tag{2.17}$$

Further, the integral on right-hand side of (2.17) is less or equal to I_n. Therefore the conditions (2.14) and (2.16) are not stronger than (2.6) and (2.8), respectively.

3. Lemmas

We require the following lemmas for the proof of the theorems.

Lemma 1 [3].

$$((1-r)/h)^n \leqslant \exp(-Ant^2), \quad A > 0 \text{ and } 0 \leqslant t \leqslant \tfrac{\pi}{2} \tag{3.1}$$

and

$$\big|((1-r)/h)^n - \exp(-Bnt^2)\big| \leqslant Knt^4, \quad t > 0. \tag{3.2}$$

Lemma 2 [6]. *For $0 \leqslant t \leqslant \pi/2$,*

$$|\theta - rt/(1-r)| \leqslant Kt^3. \tag{3.3}$$

Lemma 3. *For $0 \leqslant t \leqslant \pi/2$ and $0 < r < 1$,*

$$\left|\sin\left\{\left(n + \frac{1}{2}\right) t + (n+1)\theta\right\}\right| \leqslant \left(n + \frac{1}{2}\right) t + K(n+1)t^3 + \frac{(n+1)rt}{1-r}. \tag{3.4}$$

This is an easy consequence of Lemma 2.

4. Proofs of the theorems

4.1 *Proof of Theorem 1.* We have (see [7])

$$T_n^r(f, x) - f(x) = \frac{1}{\pi} \int_0^\pi \frac{\phi_x(t)}{\sin \frac{1}{2} t} \left(\sum_{k=0}^\infty a_{nk} \sin(k + \tfrac{1}{2}) t \right) dt,$$

where

$$\sum_{k=0}^\infty a_{nk} \sin(k + \tfrac{1}{2}) t = L(n, r, t, \theta),$$

by using (1.4), (1.8) and (1.9). Now, we write

$$\begin{aligned} T_n^r(f,x) - f(x) &= \frac{1}{\pi}\left(\int_0^{\pi/2} + \int_{\pi/2}^{\pi}\right)\left(\frac{\phi_x(t)}{\sin\frac{1}{2}t} L(n,r,t,\theta)\,dt\right) \\ &= I_1 + I_2, \quad \text{say.} \end{aligned}$$

Then by the generalized Minkowski inequality and (1.7), we have

$$\|T_n^r(f) - f\|_p \leqslant \|I_1\|_p + \|I_2\|_p \tag{4.1.1}$$

and

$$\begin{aligned} \|I_2\|_p &= O(1)\int_{\pi/2}^{\pi} t^{-1}\omega_p^{(2)}(t;f)((1-r)/h)^{n+1}\,dt \\ &= O(1)(1+4B)^{-\frac{1}{2}(n+1)} = O(n^{-1}). \end{aligned} \tag{4.1.2}$$

And for constant $A > 0$ chosen in (3.1) of Lemma 1, we write

$$\begin{aligned} I_1 &= \frac{1}{\pi}\left(\int_0^{c_n} + \int_{c_n}^{d_n} + \int_{d_n}^{\pi/2}\right)\left(\frac{\phi_x(t)}{\sin\frac{1}{2}t} L(n,r,t,\theta)\,dt\right) \\ &= I_{1,1} + I_{1,2} + I_{1,3}, \quad \text{say,} \end{aligned}$$

where c_n and d_n are as in (1.11). Hence by the generalized Minkowski inequality,

$$\|I_1\|_p \leqslant \|I_{1,1}\|_p + \|I_{1,2}\|_p + \|I_{1,3}\|_p, \tag{4.1.3}$$

where by Lemma 1, (1.9), (1.13) and Lemma 3,

$$\begin{aligned} \|I_{1,1}\|_p &\leqslant \int_0^{c_n} \frac{\omega_p^{(2)}(t;f)}{t}\left(\frac{1-r}{h}\right)^{n+1}\left|\left(n+\frac{1}{2}\right)t + K(n+1)t^3 + \frac{r}{1-r}(n+1)t\right|\,dt \\ &\leqslant \int_0^{c_n} \omega_p^{(2)}(t;f)\left\{\left(n+\frac{1}{2}\right) + (n+1)\left(Kt^2 + \frac{r}{1-r}\right)\right\}\,dt \\ &= O(1)\omega_p^{(2)}(n^{-1};f), \end{aligned} \tag{4.1.4}$$

and, once again by the generalized Minkowski inequality, (1.9), (1.13) and (3.1), we get

$$\begin{aligned} \|I_{1,3}\|_p &\leqslant \int_{d_n}^{\pi/2} t^{-1}\omega_p^{(2)}(t;f)\left(\frac{1-r}{h}\right)^{n+1}\,dt \\ &\leqslant \int_{d_n}^{\pi/2} t^{-1}\omega_p^{(2)}(t;f)\exp(-Ant^2)\,dt \\ &\leqslant n^{-1}\int_{d_n}^{\pi/2} t^{-1}\omega_p^{(2)}(t;f)\,dt. \end{aligned} \tag{4.1.5}$$

Now, by (1.9)

$$\begin{aligned} I_{1,2} &= \frac{1}{\pi}\int_{c_n}^{d_n} \frac{\phi_x(t)}{\sin\frac{1}{2}t}\left[\left(\frac{1-r}{h}\right)^{n+1} - \exp(-B(n+1)t^2)\right]\sin\{(n+\tfrac{1}{2})t + (n+1)\theta\}\,dt \\ &\quad + \frac{1}{\pi}\int_{c_n}^{d_n} \frac{\phi_x(t)}{\sin\frac{1}{2}t}\exp(-B(n+1)t^2)\sin\{(n+\tfrac{1}{2})t + (n+1)\theta\}\,dt \\ &= I_{1,2,1} + I_{1,2,2}, \quad \text{say.} \end{aligned}$$

Then, by the generalized Minkowski inequality,

$$\|I_{1,2}\|_p \leqslant \|I_{1,2,1}\|_p + \|I_{1,2,2}\|_p. \tag{4.1.6}$$

Now, proceeding as above and using (3.2) of Lemma 1, we get

$$\|I_{1,2,1}\|_p \leqslant Kn \int_{c_n}^{d_n} \omega_p^{(2)}(t;f)t^3\,dt = O(1)d_n\omega_p^{(2)}(d_n;f) \tag{4.1.7}$$

and

$$\begin{aligned} I_{1,2,2} &= \frac{1}{\pi}\int_{c_n}^{d_n} \frac{\phi_x(t)}{\sin\frac{1}{2}t}\exp(-B(n+1)t^2)\sin\{(n+1)(t+\theta)\}\,dt \\ &\quad + O(1)\int_{c_n}^{d_n} |\phi_x(t)|\exp(-B(n+1)t^2)\,dt \\ &= R_1 + R_2, \quad \text{say.} \end{aligned}$$

Arguing as above,

$$\|R_2\|_p = O(1)\int_{c_n}^{d_n} \omega_p^{(2)}(t;f)\exp\{-B(n+1)t^2\}\,dt$$

and

$$\begin{aligned} R_1 &= \frac{1}{\pi}\int_{c_n}^{d_n} \frac{\phi_x(t)}{\sin\frac{1}{2}t}\exp(-Bnt^2)\sin n(t+\theta)\,dt + O(n^{-1}) \\ &= R_1' + O(n^{-1}), \quad \text{say.} \end{aligned}$$

Therefore, by the generalized Minkowski inequality,

$$\|I_{1,2,2}\|_p = \|R_1'\|_p + O(1)\int_{c_n}^{d_n} \omega_p^{(2)}(t+c_n;f)\exp(-Bnt^2)\,dt + O(n^{-1}). \tag{4.1.8}$$

Now, for $1/(1-r) = q$, we have

$$|\sin n(t+\theta) - \sin nqt| \leqslant n|\theta - rqt| \leqslant Knt^3, \tag{4.1.9}$$

by Lemma 2. Then, arguing as above and using (4.1.9), we have

$$\begin{aligned} \|R_1'\|_p &\leqslant Kn\int_{c_n}^{d_n} t^2\omega_p^{(2)}(t;f)\exp(-Bnt^2)\,dt \\ &\qquad + \left\|\frac{1}{\pi}\int_{c_n}^{d_n} \frac{\phi_x(t)}{\sin\frac{1}{2}t}\exp(-Bnt^2)\sin nqt\,dt\right\|_p \\ &= O(1)d_n\omega_p^{(2)}(d_n;f) + \|J\|_p, \quad \text{say,} \end{aligned} \tag{4.1.10}$$

where

$$\begin{aligned} J &= \frac{1}{\pi}\int_{c_n}^{d_n} \frac{\phi_x(t)}{\sin\frac{1}{2}t}\exp(-Bnt^2)\sin nqt\,dt \\ &= \frac{1}{\pi}\int_{c_n}^{d_n} \phi_x(t)\left\{\operatorname{cosec}\frac{t}{2} - \frac{2}{t}\right\}\exp(-Bnt^2)\sin nqt\,dt \\ &\quad + \frac{2}{\pi}\int_{c_n}^{d_n} t^{-1}\phi_x(t)\exp(-Bnt^2)\sin nqt\,dt \\ &= J_1 + J_2, \quad \text{say.} \end{aligned}$$

Now, proceeding as above and using that cosec $\frac{t}{2} - \frac{2}{t} = O(t)$, we get

$$\begin{aligned}\|J\|_p &= O(1)\int_{c_n}^{d_n} t\omega_p^{(2)}(t;f)\exp(-Bnt^2)\,dt + \|J_2\|_p\\ &= O(1)d_n\omega_p^{(2)}(d_n;f) + \|J_2\|_p. \qquad (4.1.11)\end{aligned}$$

Using the transformation $t \mapsto t + c_n$, we get $\sin nq(t+c_n) = -\sin nqt$ and

$$\begin{aligned}\pi J_2 = &\int_{c_n}^{d_n} \frac{\phi_x(t) - \phi_x(t+c_n)}{t}\exp(-Bnt^2)\sin nqt\,dt\\ &+ \int_{c_n}^{d_n} \frac{\phi_x(t+c_n)}{t}\exp(-Bnt^2)\sin nqt\,dt\\ &- \int_{0}^{d_n-c_n} \frac{\phi_x(t+c_n)}{t+c_n}\exp(-Bn(t+c_n)^2)\sin nqt\,dt\\ = &\ \pi(J_{2,1} + J_{2,2} + J_{2,3}), \quad \text{say.}\end{aligned}$$

Then, by the generalized Minkowski inequality and (1.12), we have

$$\|J_2\|_p \leqslant R_n + \|J_{2,2} + J_{2,3}\|_p \qquad (4.1.12)$$

and

$$\begin{aligned}\pi(J_{2,2} + J_{2,3}) = &\int_{c_n}^{d_n} \frac{\phi_x(t+c_n)}{t}\{\exp(-Bnt^2) - \exp(-Bn(t+c_n)^2)\}\sin nqt\,dt\\ &+ c_n\int_{c_n}^{d_n} \frac{\phi_x(t+c_n)}{t(t+c_n)}\exp(-Bn(t+c_n)^2)\sin nqt\,dt\\ &+ \int_{d_n-c_n}^{d_n} \frac{\phi_x(t+c_n)}{t+c_n}\exp(-Bn(t+c_n)^2)\sin nqt\,dt\\ &- \int_{0}^{c_n} \frac{\phi_x(t+c_n)}{t+c_n}\exp(-Bn(t+c_n)^2)\sin nqt\,dt\\ = &\ \pi(L_1 + L_2 + L_3 + L_4), \quad \text{say.}\end{aligned}$$

Therefore, by the generalized Minkowski inequality,

$$\|J_{2,2} + J_{2,3}\|_p \leqslant \|L_1\|_p + \|L_2\|_p + \|L_3\|_p + \|L_4\|_p. \qquad (4.1.13)$$

Now, we observe that

$$\begin{aligned}\exp(-Bnt^2) - \exp(-Bn(t+c_n)^2) &= 2nB\int_t^{t+c_n} u\exp(-Bnu^2)\,du\\ &= O(t+c_n)\exp(-Bnt^2)\end{aligned}$$

and $t^{-1}(t+c_n)$ is non-increasing. Therefore we get

$$\|L_1\|_p = O(1)\int_{c_n}^{d_n} \omega_p^{(2)}(t+c_n;f)\exp(-Bnt^2)\,dt, \qquad (4.1.14)$$

$$\|L_2\|_p = O(c_n)\int_{c_n}^{d_n} \frac{\omega_p^{(2)}(t+c_n;f)}{t(t+c_n)}\,dt, \tag{4.1.15}$$

$$\|L_3\|_p = O(1)d_n\omega_p^{(2)}(d_n;f), \tag{4.1.16}$$

$$\|L_4\|_p = O(1)\omega_p^{(2)}(n^{-1};f), \tag{4.1.17}$$

Now, collecting (4.1.1) through (4.1.17) except (4.1.2) and (4.1.9), we get (2.11). ■

4.2. *Proof of Theorem 2.* By (1.1) and (2.11), we get

$$n^{-1}\int_{d_n}^{\pi/2} t^{-1}\omega_p^{(2)}(t;f)\,dt \leqslant 2n^{-1}\int_{n^{-1}}^{\pi/2} t^{-1}\omega_p(t;f)\,dt \leqslant \pi\omega_p(n^{-1};f), \tag{4.2.1}$$

$$\begin{aligned}\int_{c_n}^{d_n} &\omega_p^{(2)}(t+c_n;f)\exp(-Bnt^2)\,dt \\ &\leqslant \frac{1}{Bn}\int_{c_n}^{d_n} \frac{\omega_p(t+c_n;f)}{t+c_n}\frac{d}{dt}(-\exp(-Bnt^2))\,dt \\ &= O(1)\omega_p(n^{-1};f),\end{aligned} \tag{4.2.2}$$

$$c_n\int_{c_n}^{d_n} \frac{\omega_p^{(2)}(t+c_n;f)}{t(t+c_n)}\,dt = O(1)\omega_p(n^{-1};f)\log(n+1). \tag{4.2.3}$$

And, by (2.11) and (1.9),

$$d_n\omega_p^{(2)}(d_n;f) = O(1)\omega_p(n^{-1};f)\log(n+1). \tag{4.2.4}$$

Finally we observe that

$$\|\phi_x(t+c_n)-\phi_x(t)\|_p \leqslant \omega_p(c_n;f) \tag{4.2.5}$$

and hence

$$R_n = O(1)\omega_p(n^{-1};f)\log(n+1). \tag{4.2.6}$$

Now, using (4.2.1) through (4.2.6) except (4.2.5) in (2.10), we get (2.12). ■

4.3. *Proof of Theorem 3.* For $f \in \mathrm{Lip}(\alpha,p)$, $0<\alpha\leqslant 1$, $p>1$, we have

$$\omega_p(t;f) = O(t^\alpha) \tag{4.3.1}$$

and hence by (1.1) and (4.3.1) we get

$$n^{-1}\int_{d_n}^{\pi/2} t^{-1}\omega_p^{(2)}(t;f)\,dt = O(n^{-1}), \tag{4.3.2}$$

$$\int_{c_n}^{d_n} \omega_p^{(2)}(t+c_n;f)\exp(-Bnt^2)\,dt = O(n^{-\alpha}), \tag{4.3.3}$$

$$c_n\int_{c_n}^{d_n} \frac{\omega_p^{(2)}(t+c_n;f)}{t(t+c_n)}\,dt = O(1)\begin{cases} n^{-\alpha}, & 0<\alpha<1, \\ n^{-1}\log(n+1), & \alpha=1,\end{cases} \tag{4.3.4}$$

and

$$d_n\omega_p^{(2)}(d_n; f) = O(1)\begin{cases} n^{-\alpha}, & 0 < \alpha < 1, \\ n^{-1}\log(n+1), & \alpha = 1. \end{cases} \tag{4.3.5}$$

Finally, by (4.2.6) and (4.3.1), we get

$$R_n = O(n^{-\alpha})\log(n+1), \qquad 0 < \alpha \leqslant 1. \tag{4.3.6}$$

Now, using (4.3.2) through (4.3.6) in (2.10), we get the required result (2.13). ∎

4.4. *Proof of Theorem 4.* We first observe that (2.5) implies (2.11) and therefore (4.2.1) and (4.2.2) hold. Also, for $0 < \gamma < 1$,

$$c_n \int_{c_n}^{d_n} \frac{\omega_p(t + c_n; f)}{t(t + c_n)}\,dt \leqslant c_n^{1-\gamma}\omega_p(2c_n; f)\int_{c_n}^{\infty} t^{\gamma-2}\,dt = O(1)\omega_p(n^{-1}; f). \tag{4.4.1}$$

and

$$d_n\omega_p^{(2)}(d_n; f) \leqslant 2d_n^{1+\gamma}n^{\gamma}\omega_p(n^{-1}; f) = O(1)\omega_p(n^{-1}; f). \tag{4.4.2}$$

Thus, by using (4.2.1), (4.2.2), (4.4.1), (4.4.2) and (2.14) in (2.10), we get (2.15). ∎

4.5. *Proof of Theorem 5.* Proceeding as in Theorem 3 for $0 < \alpha < 1$ and using (2.16) for (4.3.6), we get (2.9). ∎

This completes the proofs of the theorems.

5. Corollaries

As we have already remarked, for continuous functions f, $L_p[0, 2\pi]$ and $\omega_p(\delta; f)$, respectively, reduce to $C_{2\pi}$ and $\omega(\delta; f)$ for $p = \infty$. Therefore, by letting $p = \infty$ in Theorem 4, we get the following generalization of a theorem due to Chui and Holland [2]:

Corollary 1. *Let $f \in C_{2\pi}$ and let*

$$t^{-\eta}\omega(t; f) \quad \textit{be non-increasing with } t \textit{ for } 0 < \eta < 1$$

and $R_n = O(1)\omega(n^{-1}; f)$. Then

$$\|T_n^r(f) - f\|_c = O(1)\omega(n^{-1}; f).$$

The following result provides Jackson order, which may be deduced from Corollary 1 by letting $\eta = \alpha$ and $\omega(t; f) = t^{\alpha}$:

Corollary 2. *Let $f \in C_{2\pi} \cap \mathrm{Lip}\,\alpha$, where $0 < \alpha < 1$, and let*

$$R_n = O(n^{-\alpha}) \tag{5.1}$$

Then $\|T_n^r(f) - f\|_c = O(n^{-\alpha})$.

It may be observed that Chui and Holland [2] obtained this result by taking $I_n = O(n^{-\alpha})$ for $R_n = O(n^{-\alpha})$ in (5.1) and $R_n < I_n$, since $a_n < c_n < d_n < b_n$.

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