

STRONG CONVERGENCE OF IMPLICIT ITERATES WITH ERRORS FOR NON-LIPSCHITZIAN ASYMPTOTICALLY QUASI-NONEXPANSIVE TYPE MAPPINGS IN BANACH SPACES

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Abstract. In this paper we prove that an implicit iterative process with errors converges strongly to a common fixed point for a finite family of asymptotically quasi-nonexpansive type mappings on unbounded sets in a uniformly convex Banach space. Our results unify, improve and generalize the corresponding results of Ud-din and Khan, Sun, Wittman, Xu and Ori and many others.

1. Introduction

In 1967, Browder [1] studied the iterative construction for fixed points of non-expansive mappings on closed and convex subsets of a Hilbert space (see also [2]). The Ishikawa iteration process in the context of nonexpansive mappings on bounded closed convex subsets of a Banach space has been considered by a number of authors (see, e.g., Tan and Xu [22] and the references therein).

In 1973, Petryshyn and Williamson [17] established a necessary and sufficient condition for a Mann [15] iterative sequence to converge strongly to a fixed point of quasi-nonexpansive mapping. Subsequently, Ghosh and Debnath [5] extended the results of [17] and obtained some necessary and sufficient conditions for an Ishikawa-type iterative sequence to converge to a fixed point of quasi-nonexpansive mapping. In 2001, Liu in [13, 14] extended the results of Ghosh and Debnath [5] to a more general asymptotically quasi-nonexpansive mappings. In 2003, Sahu and Jung [18] studied Ishikawa and Mann iteration process in Banach spaces and they proved some weak and strong convergence theorems for asymptotically quasi-nonexpansive type mapping. In 2006, Shahzad and Udomene [19] gave the necessary and sufficient condition for convergence of common fixed point of two-step modified Ishikawa iterative sequence for two asymptotically quasi-nonexpansive mappings in real Banach space.

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Fixed point results for asymptotically nonexpansive mappings have been obtained by Kirk and Ray [11] on unbounded sets in Banach space. Recently, Hussain and Khan [8] have constructed approximating sequences to fixed points of a class of mappings, containing nonexpansive mappings as a subclass, on closed convex unbounded subsets of a Hilbert space (see [16, 20] as well).

Sun [21] has recently extended an implicit iteration process for a finite family of nonexpansive mapping due to Xu and Ori [26] to the case of asymptotically quasi-nonexpansive mappings. Recently, Ud-din and Khan [4] studied an implicit iteration process with errors for a finite family of asymptotically quasi-nonexpansive mappings on unbounded sets in a uniformly convex Banach space and proved some weak and strong convergence theorems for said mappings.

The main goal of this paper is to prove the strong convergence of an implicit iterative process with errors, in the sense of Xu [25], for a finite family of non-Lipschitzian asymptotically quasi-nonexpansive type mappings on a closed convex unbounded set in a real uniformly convex Banach space. Our results unify, improve and generalize the corresponding results of [4, 21, 23, 26] and many others.

2. Preliminaries and lemmas

Let C be a nonempty subset of a normed space E and $T: C \rightarrow C$ be a given mapping. We assume that the set of fixed points of T , $F(T) = \{x \in C : T(x) = x\}$ is nonempty. The mapping T is said to be

- (1) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.
- (2) quasi-nonexpansive [2] if $\|Tx - p\| \leq \|x - p\|$ for all $x \in C, p \in F(T)$.
- (3) asymptotically nonexpansive [6] if there exists a sequence $\{u_n\}$ in $[0, \infty)$ with $\lim_{n \rightarrow \infty} u_n = 0$ such that $\|T^n x - T^n y\| \leq (1 + u_n)\|x - y\|$ for all $x, y \in C$ and $n \geq 1$.
- (4) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{u_n\}$ in $[0, \infty)$ with $\lim_{n \rightarrow \infty} u_n = 0$ such that $\|T^n x - p\| \leq (1 + u_n)\|x - p\|$ for all $x \in C, p \in F(T)$ and $n \geq 1$.
- (5) uniformly L -Lipschitzian if there exists a positive constant L such that $\|T^n x - T^n y\| \leq L\|x - y\|$ for all $x, y \in C$ and $n \geq 1$.
- (6) asymptotically nonexpansive type [10], if

$$\limsup_{n \rightarrow \infty} \left\{ \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \right\} \leq 0.$$

- (7) asymptotically quasi-nonexpansive type [18], if $F(T) \neq \emptyset$ and

$$\limsup_{n \rightarrow \infty} \left\{ \sup_{x \in C, p \in F(T)} (\|T^n x - p\| - \|x - p\|) \right\} \leq 0.$$

REMARK 2.1. It is easy to see that if $F(T)$ is nonempty, then asymptotically nonexpansive mapping, asymptotically quasi-nonexpansive mapping and asymptotically nonexpansive type mapping are the special cases of asymptotically quasi-nonexpansive type mappings.

The Mann and Ishikawa iteration processes have been used by a number of authors to approximate the fixed points of nonexpansive, asymptotically nonexpansive mappings, and quasi-nonexpansive mappings on Banach spaces (see, e.g., [5, 9, 12–14, 17, 20, 25]).

For a nonempty subset C of a normed space E and $T: C \rightarrow E$, Liu [12] introduced in 1995 the concept of Ishikawa iteration process with errors by the iterative sequence $\{x_n\}$ defined as follows:

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T^n y_n + u_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T^n x_n + v_n, \quad n \geq 1, \end{aligned}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[0, 1]$ satisfying appropriate conditions and $\sum_{n=1}^{\infty} \|u_n\| < \infty$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$. If $\beta_n = 0$, $v_n = 0$ for all $n \geq 1$, then this process becomes the Mann iteration process with errors.

The above definition of Liu depend on the convergence of the error terms u_n and v_n . The occurrence of errors is random and so the conditions imposed on the error terms are unreasonable. Moreover, there is no assurance that the iterates defined by Liu will fall within the domain under consideration.

In 1998, Xu [25] gave the following new definitions in place of these non compatible ones.

For a nonempty convex subset C of a normed space E and $T: C \rightarrow C$, the Ishikawa iteration process with errors is the iterative sequence $\{x_n\}$ defined by

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= \alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n, \\ y_n &= \alpha'_n x_n + \beta'_n T^n x_n + \gamma'_n v_n, \quad n \geq 1, \end{aligned}$$

where $\{u_n\}$, $\{v_n\}$ are bounded sequences in C and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ are sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$ for all $n \geq 1$. It reduces to the Mann iteration process with errors when $\beta'_n = 0 = \gamma'_n$ for all $n \geq 1$.

Clearly, the normal Ishikawa and Mann iteration processes are special cases of the Ishikawa iteration process with errors.

Huang [7] has computed fixed points of asymptotically nonexpansive mappings while Ud-din and Khan [3] have approximated common fixed points of two asymptotically nonexpansive mappings, using iteration process with errors in the sense of Liu [12].

Denote the indexing set $\{1, 2, \dots, N\}$ by I . Let $\{T_i : i \in I\}$ be a finite family of asymptotically quasi-nonexpansive type self-mappings on a convex subset C of a normed space E . The implicit iterative process of Sun [21] with an error term, in the sense of Xu [25], and with an initial value $x_0 \in C$, is defined as follows:

$$x_1 = \alpha_1 x_0 + \beta_1 T_1 x_1 + \gamma_1 u_1,$$

$$\begin{aligned}
x_2 &= \alpha_2 x_1 + \beta_2 T_2 x_2 + \gamma_2 u_2, \\
&\vdots \\
x_N &= \alpha_N x_{N-1} + \beta_N T_N x_N + \gamma_N u_N, \\
x_{N+1} &= \alpha_{N+1} x_N + \beta_{N+1} T_1^2 x_{N+1} + \gamma_{N+1} u_{N+1}, \\
&\vdots \\
x_{2N} &= \alpha_{2N} x_{2N-1} + \beta_{2N} T_N^2 x_{2N} + \gamma_{2N} u_{2N}, \\
x_{2N+1} &= \alpha_{2N+1} x_{2N} + \beta_{2N+1} T_1^3 x_{2N+1} + \gamma_{2N+1} u_{2N+1}, \\
&\vdots
\end{aligned}$$

where $\{u_n\}$ is a bounded sequence in C and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$.

The above sequence can be written in compact form as

$$x_n = \alpha_n x_{n-1} + \beta_n T_i^k x_n + \gamma_n u_n \quad (2.1)$$

with $n \geq 1$, $n = (k-1)N + i$, $i \in I$ and $T_n = T_i \pmod{N} = T_i$.

Denote the indexing set $\{1, 2, \dots, N\}$ by I . Let $\{T_i : i \in I\}$ be N uniformly L -Lipschitzian asymptotically quasi-nonexpansive type self-mappings of C . We show that (2.1) exists. Let $x_0 \in C$ and $x_1 = \alpha_1 x_0 + \beta_1 T_1 x_1 + \gamma_1 u_1$. Define $W : C \rightarrow C$ by: $Wx = \alpha_1 x_0 + \beta_1 T_1 x + \gamma_1 u_1$ for all $x \in C$. The existence of x_1 is guaranteed if W has a fixed point. For any $x, y \in C$, we have

$$\|Wx - Wy\| \leq \beta_1 \|T_1 x - T_1 y\| \leq \beta_1 L \|x - y\|.$$

Now, W is a contraction if $\beta_1 L < 1$ or $L < 1/\beta_1$. As $\beta_1 \in (0, 1)$, therefore W is a contraction even if $L > 1$. By the Banach contraction principle, W has a unique fixed point. Thus, the existence of x_1 is established. Similarly, we can establish the existence of x_2, x_3, x_4, \dots . Thus, the implicit algorithm (2.1) is well defined.

The distance between a point x and a set C and closed ball with center zero and radius r in E are, respectively, defined by

$$d(x, C) = \inf_{y \in C} \|x - y\|, \quad B_r(0) = \{x \in E : \|x\| \leq r\}.$$

DEFINITION 2.1. [21] Let C be a closed subset of a normed space E and let $T : C \rightarrow C$ be a mapping. Then T is said to be semi-compact if for any bounded sequence $\{x_n\}$ in C with $\|x_n - Tx_n\| \rightarrow 0$ as $n \rightarrow \infty$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow x^* \in C$ as $n_k \rightarrow \infty$.

LEMMA 2.1. [22] Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \leq a_n + b_n, \quad n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n \rightarrow \infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to zero, then $\lim_{n \rightarrow \infty} a_n = 0$.

LEMMA 2.2. [24] *Let $p > 1$ and $r > 0$ be two fixed real numbers and E a Banach space. Then E is uniformly convex if and only if there exists a continuous, strictly increasing and convex function $g: [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1 - \lambda)y\|^p \leq \lambda \|x\|^p + (1 - \lambda) \|y\|^p - W_p(\lambda)g(\|x - y\|)$$

for all $x, y \in B_r(0)$ where $0 \leq \lambda \leq 1$ and $W_p(\lambda) = \lambda(1 - \lambda)^p + \lambda^p(1 - \lambda)$.

3. Main results

We begin with a necessary and sufficient condition for convergence of $\{x_n\}$ generated by the implicit iteration process with errors (2.1) to prove the following result.

THEOREM 3.1. *Let C be a nonempty closed convex subset of a Banach space E . Let $T_i: C \rightarrow C$ ($i \in I = \{1, 2, \dots, N\}$) be N asymptotically quasi-nonexpansive type mappings such that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the implicit iteration process with errors defined by (2.1). Put*

$$A_n = \max \left\{ 0, \sup_{p \in F, n \geq 1} (\|T_i^n x_n - p\| - \|x_n - p\|) : i \in I \right\}, \quad (3.1)$$

where $n = (k-1)N + i$, $i \in I$ and $T_n = T_i \pmod{N} = T_i$. Assume that $\sum_{n=1}^{\infty} A_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\{\beta_n\} \subset (s, 1 - s)$ for some $s \in (0, \frac{1}{2})$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point p of the mappings $\{T_i\}_{i=1}^N$ if and only if $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$.

Proof. The necessity is obvious and so it is omitted. Now, we prove the sufficiency. For any $p \in F = \bigcap_{i=1}^N F(T_i)$, from (2.1) and (3.1), we have

$$\begin{aligned} \|x_n - p\| &= \|\alpha_n x_{n-1} + \beta_n T_i^k x_n + \gamma_n u_n - p\| \\ &= \|\alpha_n (x_{n-1} - p) + \beta_n (T_i^k x_n - p) + \gamma_n (u_n - p)\| \\ &\leq \alpha_n \|x_{n-1} - p\| + \beta_n \|T_i^k x_n - p\| + \gamma_n \|u_n - p\| \\ &\leq \alpha_n \|x_{n-1} - p\| + \beta_n (\|x_n - p\| + A_n) + \gamma_n \|u_n - p\| \\ &= \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n - \gamma_n) \|x_n - p\| + \beta_n A_n + \gamma_n \|u_n - p\| \\ &\leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n) \|x_n - p\| + A_n + \gamma_n \|u_n - p\|. \end{aligned} \quad (3.2)$$

Since $\lim_{n \rightarrow \infty} \gamma_n = 0$, there exists a natural number n_1 such that for $n > n_1$, $\gamma_n \leq \frac{s}{2}$. Hence

$$\alpha_n = 1 - \beta_n - \gamma_n \geq 1 - (1 - s) - \frac{s}{2} = \frac{s}{2}, \quad (3.3)$$

for $n > n_1$. Thus, we have from (3.2) and (3.3) that

$$\alpha_n \|x_n - p\| \leq \alpha_n \|x_{n-1} - p\| + A_n + \gamma_n \|u_n - p\|,$$

and

$$\begin{aligned}
\|x_n - p\| &\leq \|x_{n-1} - p\| + \frac{A_n}{\alpha_n} + \frac{\gamma_n}{\alpha_n} \|u_n - p\| \\
&\leq \|x_{n-1} - p\| + \frac{2}{s}A_n + \frac{2\gamma_n}{s} \|u_n - p\| \\
&\leq \|x_{n-1} - p\| + \frac{2}{s}A_n + \frac{2\gamma_n}{s}M, \tag{3.4}
\end{aligned}$$

where $M = \sup_{n \geq 1} \{\|u_n - p\|\}$, since $\{u_n\}$ is a bounded sequence in C . This implies that

$$d(x_n, F) \leq d(x_{n-1}, F) + D_n,$$

where $D_n = \frac{2}{s}A_n + \frac{2\gamma_n}{s}M$. Since by assumptions of the theorem, $\sum_{n=1}^{\infty} A_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$, it follows that $\sum_{n=1}^{\infty} D_n < \infty$. Therefore, from Lemma 2.1, we know that $\lim_{n \rightarrow \infty} d(x_n, F)$ exists. Since by hypothesis $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$, so by Lemma 2.1, we have $\lim_{n \rightarrow \infty} d(x_n, F) = 0$.

Next we prove that $\{x_n\}$ is a Cauchy sequence in C . It follows from (3.4) that for any $m \geq 1$, for all $n \geq n_0$ and for any $p \in F$, we have

$$\begin{aligned}
\|x_{n+m} - p\| &\leq \|x_{n+m-1} - p\| + \frac{2}{s}A_{n+m} + \frac{2M}{s}\gamma_{n+m} \\
&\leq \|x_{n+m-2} - p\| + \frac{2}{s}[A_{n+m} + A_{n+m-1}] + \frac{2M}{s}[\gamma_{n+m} + \gamma_{n+m-1}] \\
&\leq \dots \\
&\leq \|x_n - p\| + \frac{2}{s} \sum_{k=n+1}^{n+m} A_k + \frac{2M}{s} \sum_{k=n+1}^{n+m} \gamma_k.
\end{aligned}$$

So, we have

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq \|x_{n+m} - p\| + \|x_n - p\| \\
&\leq 2\|x_n - p\| + \frac{2}{s} \sum_{k=n+1}^{n+m} A_k + \frac{2M}{s} \sum_{k=n+1}^{n+m} \gamma_k.
\end{aligned}$$

Then, we have

$$\|x_{n+m} - x_n\| \leq 2d(x_n, F) + \frac{2}{s} \sum_{k=n+1}^{n+m} A_k + \frac{2M}{s} \sum_{k=n+1}^{n+m} \gamma_k, \quad \forall n \geq n_0. \tag{3.5}$$

For any given $\varepsilon > 0$, there exists a positive integer $n_1 \geq n_0$ such that for any $n \geq n_1$,

$$d(x_n, F) < \frac{\varepsilon}{6}, \quad \sum_{k=n+1}^{n+m} A_k < \frac{s\varepsilon}{6}, \tag{3.6}$$

and

$$\sum_{k=n+1}^{n+m} \gamma_k < \frac{s\varepsilon}{6M}. \tag{3.7}$$

Thus, from (3.5)–(3.7) and $n \geq n_1$, we have

$$\|x_{n+m} - x_n\| < 2 \cdot \frac{\varepsilon}{6} + \frac{2}{s} \cdot \frac{s\varepsilon}{6} + \frac{2M}{s} \cdot \frac{s\varepsilon}{6M} = \varepsilon.$$

This implies that $\{x_n\}$ is a Cauchy sequence in C . Thus, the completeness of E implies that $\{x_n\}$ must be convergent.

Assume that $\lim_{n \rightarrow \infty} x_n = p$. Now, we have to show that p is a common fixed point of the mappings $\{T_i : i \in I\}$. Indeed, we know that the set $F = \bigcap_{i=1}^N F(T_i)$ is closed. From the continuity of $d(x, F) = 0$ with $\lim_{n \rightarrow \infty} d(x_n, F) = 0$ and $\lim_{n \rightarrow \infty} x_n = p$, we get $d(p, F) = 0$, and so $p \in F$, that is, p is a common fixed point of the mappings $\{T_i\}_{i=1}^N$. This completes the proof. ■

THEOREM 3.2. *Let C be a nonempty closed convex subset of a Banach space E . Let $T_i : C \rightarrow C$ ($i \in I = \{1, 2, \dots, N\}$) be N asymptotically quasi-nonexpansive type mappings such that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the implicit iteration process with errors defined by (2.1). Put*

$$A_n = \max \left\{ 0, \sup_{p \in F, n \geq 1} (\|T_i^n x_n - p\| - \|x_n - p\|) : i \in I \right\},$$

where $n = (k-1)N + i$, $i \in I$ and $T_n = T_i \pmod{N} = T_i$. Assume that $\sum_{n=1}^{\infty} A_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\{\beta_n\} \subset (s, 1-s)$ for some $s \in (0, \frac{1}{2})$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point p of the mappings $\{T_i\}_{i=1}^N$ if and only if there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to p .

Proof. The proof of Theorem 3.2 follows from Lemma 2.1 and Theorem 3.1. ■

We prove a lemma which plays an important role in establishing strong convergence of the implicit iteration process with errors in a uniformly convex Banach space.

LEMMA 3.1. *Let C be a nonempty closed convex subset of a real uniformly convex Banach space E . Let $\{T_i : i \in I\}$ be N uniformly L -Lipschitzian asymptotically quasi-nonexpansive type mappings of C such that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the implicit iteration process with errors defined by (2.1). Put*

$$A_n = \max \left\{ 0, \sup_{p \in F, n \geq 1} (\|T_i^n x_n - p\| - \|x_n - p\|) : i \in I \right\},$$

where $n = (k-1)N + i$, $i \in I$ and $T_n = T_i \pmod{N} = T_i$. Assume that $\sum_{n=1}^{\infty} A_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\{\beta_n\} \subset (s, 1-s)$ for some $s \in (0, \frac{1}{2})$. Then $\lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = 0$ for all $i \in I$.

Proof. Set $\sigma_n = \|T_n^k x_n - x_{n-1}\|$, $n = (k-1)N + i$, $i \in I$. As in the proof of Theorem 3.1, $\lim_{n \rightarrow \infty} \|x_n - q\|$ exists for all $q \in F$, so $\{x_n - q, T_i^k x_n - q\}$ is a bounded set. Hence, we can obtain a closed ball $B_r(0) \supset \{x_n - q, T_i^k x_n - q\}$ for some $r > 0$. By Lemma 2.2, the scheme (2.1) and equation (3.1), we get

$$\|x_n - q\|^2 = \|\alpha_n(x_{n-1} - q) + (1 - \alpha_n)(T_i^k x_n - q) + \gamma_n(u_n - T_i^k x_n)\|^2$$

$$\begin{aligned}
&\leq \|\alpha_n(x_{n-1} - q) + (1 - \alpha_n)(T_i^k x_n - q)\|^2 + \gamma_n K \quad \text{for some } K > 0 \\
&\leq \alpha_n \|x_{n-1} - q\|^2 + (1 - \alpha_n) \|T_i^k x_n - q\|^2 - W_2(\alpha_n)g(\sigma_n) + \gamma_n K \\
&\leq \alpha_n \|x_{n-1} - q\|^2 + (1 - \alpha_n)[\|x_n - q\| + A_n]^2 - W_2(\alpha_n)g(\sigma_n) + \gamma_n K \\
&\leq \alpha_n \|x_{n-1} - q\|^2 + (1 - \alpha_n)[\|x_n - q\|^2 + \rho_n] - W_2(\alpha_n)g(\sigma_n) + \gamma_n K \\
&= \alpha_n \|x_{n-1} - q\|^2 + (1 - \alpha_n) \|x_n - q\|^2 + (1 - \alpha_n)\rho_n \\
&\quad - W_2(\alpha_n)g(\sigma_n) + \gamma_n K,
\end{aligned}$$

where $\rho_n = A_n^2 + 2\|x_n - q\|A_n$, since $\sum_{n=1}^{\infty} A_n < \infty$, it follows that $\sum_{n=1}^{\infty} \rho_n < \infty$. Thus from the above inequality and (3.3), we have that

$$\|x_n - q\|^2 \leq \|x_{n-1} - q\|^2 + \left(\frac{2}{s} - 1\right)\rho_n - (1 - \alpha_n)g(\sigma_n) + \frac{2\gamma_n}{s}K. \quad (3.8)$$

Therefore, as in Theorem 3.1, it can be shown that $\lim_{n \rightarrow \infty} \|x_n - q\|^2 = d$ exists. From (3.8), it follows that

$$(1 - \alpha_n)g(\sigma_n) \leq \|x_{n-1} - q\|^2 - \|x_n - q\|^2 + \left(\frac{2}{s} - 1\right)\rho_n + \frac{2\gamma_n}{s}K.$$

From $(1 - \alpha_n) \geq (1 - s/2)$, we have

$$\left(\frac{2-s}{2}\right)g(\sigma_n) \leq \|x_{n-1} - q\|^2 - \|x_n - q\|^2 + \left(\frac{2}{s} - 1\right)\rho_n + \frac{2\gamma_n}{s}K.$$

Let m be a positive integer such that $m \geq n$. Then

$$\begin{aligned}
\sum_{n=1}^m g(\sigma_n) &\leq \left(\frac{2}{2-s}\right) [\|x_0 - q\|^2 - \|x_m - q\|^2] + \frac{2}{s} \sum_{n=1}^m \rho_n + \frac{4K}{s(2-s)} \sum_{n=1}^m \gamma_n \\
&\leq \left(\frac{2}{2-s}\right) \|x_0 - q\|^2 + \frac{2}{s} \sum_{n=1}^m \rho_n + \frac{4K}{s(2-s)} \sum_{n=1}^m \gamma_n. \quad (3.9)
\end{aligned}$$

When $m \rightarrow \infty$ in (3.9), we have that $\lim_{n \rightarrow \infty} g(\sigma_n) = 0$. Since g is strictly increasing and continuous with $g(0) = 0$, it follows that $\lim_{n \rightarrow \infty} \sigma_n = 0$. Hence

$$\begin{aligned}
\|x_n - x_{n-1}\| &\leq \beta_n \|T_n^k x_n - x_{n-1}\| + \gamma_n \|u_n - x_{n-1}\| \\
&\leq (1 - \alpha_n) \|T_n^k x_n - x_{n-1}\| + \gamma_n Q, \quad \text{for some } Q > 0 \\
&\leq (1 - s/2) \|T_n^k x_n - x_{n-1}\| + \gamma_n Q,
\end{aligned}$$

which implies that $\lim_{n \rightarrow \infty} \|x_n - x_{n-1}\| = 0$. That is, $\lim_{n \rightarrow \infty} \|x_n - x_{n+l}\| = 0$ for all $l > 2N$. For $n > N$, we have

$$\begin{aligned}
\|x_{n-1} - T_n x_n\| &\leq \|x_{n-1} - T_n^k x_n\| + \|T_n^k x_n - T_n x_n\| \\
&\leq \sigma_n + L \|T_n^{k-1} x_n - x_n\| \\
&\leq \sigma_n + L [\|T_n^{k-1} x_n - T_{n-N}^{k-1} x_{n-N}\| + \|T_{n-N}^{k-1} x_{n-N} - x_{(n-N)-1}\|] \\
&\quad + L \|x_{(n-N)-1} - x_n\|.
\end{aligned}$$

By $n \equiv (n - N) \pmod{N}$, we get $T_n = T_{n-N}$. Now the above inequality becomes

$$\|x_{n-1} - T_n x_n\| \leq \sigma_n + L^2 \|x_n - x_{n-N}\| + L\sigma_{n-N} + L \|x_{(n-N)-1} - x_n\|,$$

which yields that $\lim_{n \rightarrow \infty} \|x_{n-1} - T_n x_n\| = 0$. Since

$$\|x_n - T_n x_n\| \leq \|x_n - x_{n-1}\| + \|x_{n-1} - T_n x_n\|,$$

so we have that $\lim_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0$.

Hence, for all $l \in I$, we have

$$\begin{aligned} \|x_n - T_{n+l} x_n\| &\leq \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l} x_{n+l}\| + \|T_{n+l} x_{n+l} - T_{n+l} x_n\| \\ &\leq (1 + L) \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l} x_n\|, \end{aligned}$$

which implies that

$$\lim_{n \rightarrow \infty} \|x_n - T_{n+l} x_n\| = 0 \quad \forall l \in I.$$

Thus $\lim_{n \rightarrow \infty} \|x_n - T_l x_n\| = 0 \quad \forall l \in I$. ■

Now, we are in a position to prove our strong convergence theorems.

THEOREM 3.3. *Let C be a nonempty closed convex subset of a real uniformly convex Banach space E . Let $T_i: C \rightarrow C$ ($i \in I = \{1, 2, \dots, N\}$) be N uniformly L -Lipschitzian asymptotically quasi-nonexpansive type mappings such that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the implicit iteration process with errors defined by (2.1). Put*

$$A_n = \max \left\{ 0, \sup_{p \in F, n \geq 1} (\|T_i^n x_n - p\| - \|x_n - p\|) : i \in I \right\},$$

where $n = (k-1)N + i$, $i \in I$ and $T_n = T_i \pmod{N} = T_i$. Assume that $\sum_{n=1}^{\infty} A_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\{\beta_n\} \subset (s, 1-s)$ for some $s \in (0, \frac{1}{2})$. If at least one member T in $\{T_i : i \in I\}$ is semi-compact, then the implicitly defined sequence $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i\}_{i=1}^N$.

Proof. By Lemma 3.1, it follows that

$$\lim_{n \rightarrow \infty} \|x_n - T_l x_n\| = 0 \quad \forall l \in I. \quad (3.10)$$

Without any loss of generality, assume that T_1 is semi-compact. Therefore, by (3.10), it follows that $\lim_{n \rightarrow \infty} \|x_n - T_1 x_n\| = 0$. Since T_1 is semi-compact, therefore there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightarrow x^* \in C$. Now consider

$$\|x^* - T_l x^*\| = \lim_{n_j \rightarrow \infty} \|x_{n_j} - T_l x_{n_j}\| = 0 \quad \forall l \in I.$$

This proves that $x^* \in F$. As $\lim_{n \rightarrow \infty} \|x_n - q\|$ exists for all $q \in F$, therefore $\{x_n\}$ converges to $x^* \in F$, and hence the result. ■

REMARK 3.1. Theorem 3.3 extends and improves Theorem 3.3 due to Sun [21] to the case of more general class of asymptotically quasi-nonexpansive mapping and

implicit iteration process with errors and without the boundedness of C which in turn generalizes Theorem 2 by Wittmann [23] from Hilbert spaces to uniformly convex Banach spaces.

DEFINITION 3.1. (condition $(*)$ [4]) The family $\{T_i : i \in I\}$ of N -self mappings on a subset C of a normed space E satisfies condition $(*)$ if there exists a nondecreasing function $f: [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that $\frac{1}{N} \sum_{i=1}^N \|x - T_i x\| \geq f(d(x, F))$ for all $x \in C$ where $d(x, F) = \inf\{\|x - p\| : p \in F\}$.

Note that condition $(*)$ defined above reduces to the condition (A) [22] if we choose $T_i = T$ (say) for all $i \in I$.

Finally, an application of the convergence criteria established in Theorem 3.1 is given below to obtain yet another strong convergence result in our setting.

THEOREM 3.4. *Let C be a nonempty closed convex subset of a real uniformly convex Banach space E . Let $T_i: C \rightarrow C$ ($i \in I = \{1, 2, \dots, N\}$) be N uniformly L -Lipschitzian asymptotically quasi-nonexpansive type mappings such that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ and satisfy the condition $(*)$. Let $\{x_n\}$ be the implicit iteration process with errors defined by (2.1). Put*

$$A_n = \max \left\{ 0, \sup_{p \in F, n \geq 1} (\|T_i^n x_n - p\| - \|x_n - p\|) : i \in I \right\},$$

where $n = (k-1)N + i$, $i \in I$ and $T_n = T_i \pmod{N} = T_i$. Assume that $\sum_{n=1}^{\infty} A_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\{\beta_n\} \subset (s, 1-s)$ for some $s \in (0, \frac{1}{2})$. Then the iterative sequence $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i\}_{i=1}^N$.

Proof. As in the proof of Theorem 3.3, (3.10) holds. Taking \liminf on both sides of condition $(*)$ and using (3.10), we have that $\liminf_{n \rightarrow \infty} f(d(x_n, F)) = 0$. Since f is a nondecreasing function with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$, it follows that $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$. Now by Theorem 3.1, $x_n \rightarrow p \in F$, that is, $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i\}_{i=1}^N$. This completes the proof. ■

REMARK 3.2. (i) Our results extend the corresponding results of Ud-din and Khan [4] to the case of more general class of asymptotically quasi-nonexpansive mappings considered in this paper.

(ii) Our results also generalize and improve the corresponding results of Sun [21], Wittmann [23] and Xu and Ori [26] to the case of more general class of nonexpansive, asymptotically quasi-nonexpansive mappings and implicit iteration process with errors considered in this paper.

EXAMPLE 3.1. Let E be the real line with the usual norm $|\cdot|$ and $K = [0, 1]$. Define $T: K \rightarrow K$ by

$$T(x) = \sin x, \quad x \in [0, 1],$$

for $x \in K$. Obviously $T(0) = 0$, that is, 0 is a fixed point of T , that is, $F(T) = \{0\}$. Now we check that T asymptotically quasi-nonexpansive type mapping. In fact, if

$x \in [0, 1]$ and $p = 0 \in [0, 1]$, then

$$|T(x) - p| = |T(x) - 0| = |\sin x - 0| = |\sin x| \leq |x| = |x - 0| = |x - p|,$$

that is, $|T(x) - p| \leq |x - p|$. Thus, T is quasi-nonexpansive. It follows that T is asymptotically quasi-nonexpansive with the constant sequence $\{k_n\} = \{1\}$ for each $n \geq 1$ and hence it is asymptotically quasi-nonexpansive type mapping (by Remark 2.1). But the converse does not hold in general.

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