

## FUZZY IDEALS IN LASKERIAN RINGS

Tariq Shah and Muhammad Saeed

**Abstract.** We introduce strongly primary fuzzy ideals and strongly irreducible fuzzy ideals in a unitary commutative ring and fixed their role in a Laskerian ring. We established that: A finite intersection of prime fuzzy ideals (resp. primary fuzzy ideals, irreducible fuzzy ideals and strongly irreducible fuzzy ideals) is a prime fuzzy ideal (resp. primary fuzzy ideal, irreducible fuzzy ideal and strongly irreducible fuzzy ideal). We also find that, a fuzzy ideal of a ring is prime if and only if it is semiprime and strongly irreducible. Furthermore we characterize that: (1) Every nonzero fuzzy ideal of a one dimensional Laskerian domain can be uniquely expressed as a product of primary fuzzy ideals with distinct radicals, (2) A unitary commutative ring is (strongly) Laskerian if and only if its localization is (strongly) Laskerian with respect to every fuzzy ideal.

### 1. Introduction

L. A. Zadeh [22] proposed theory of fuzzy sets, which provides a useful mathematical tool for describing the behavior of the multifaceted or distracted systems to admit accurate mathematical analysis by classical technique. The study of fuzzy algebraic structure has started by Rosenfeld [18] and since then this concept has been applied to various algebraic structures. Liu introduced the concept of a fuzzy ideal of a ring in [10]. The notions of prime fuzzy ideals, maximal fuzzy ideals, primary fuzzy deals were introduced in [12–14]. Also, Malik [11], Mukererjee and Sen [16] studied rings with chain conditions with the help of fuzzy ideals. The notion of fuzzy quotient ring was introduced by Kumar [7], Kuroaka and Kuroki [8]. In [9] K. H. Lee examined some properties of fuzzy quotient rings and used them to characterize Artinian and Noetherian rings.

Following [22], a fuzzy subset of a non-empty set  $X$  is a function  $\mu : X \rightarrow [0, 1]$ . By [21], the set of all fuzzy subsets of a set  $X$  with the relation  $\mu \subseteq \nu$  if  $\mu(x) \leq \nu(x)$ ,  $\forall x \in X$  is a complete lattice. Whereas for a non empty family  $\{\mu_i : i \in I\}$  of fuzzy subsets of  $X$ ,

$$\inf\{\mu_i : i \in I\} : X \rightarrow [0, 1], \quad x \mapsto \inf\{\mu_i(x) : i \in I\} \quad \text{and}$$
$$\sup\{\mu_i : i \in I\} : X \rightarrow [0, 1], \quad x \mapsto \sup\{\mu_i(x) : i \in I\}, \quad \forall x \in X.$$

---

2010 AMS Subject Classification: 13A15, 03E72, 13C12

Keywords and phrases: Fuzzy ideal; prime fuzzy ideal; primary fuzzy ideal; (strongly) irreducible fuzzy ideal; (strongly) Laskerian ring.

A fuzzy subset  $\mu$  of a ring  $R$  is a fuzzy left (respectively right) ideal of  $R$  if for every  $x, y \in R$ ;  $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$  and  $\mu(xy) \geq \mu(y)$  (respectively  $\mu(xy) \geq \mu(x)$ ). If  $\mu$  is a fuzzy subset of  $R$ , then for  $t \in [0, 1]$  the set  $\mu_t = \{x \in R : \mu(x) \geq t\}$  is called a level subset of  $R$  with respect to  $\mu$ . A fuzzy subset is a fuzzy left ideal if and only if  $\mu(0) \geq \mu(x) \forall x \in R$  and  $\mu_t$  is a left ideal of  $R, \forall t \in [0, \mu(0)]$ . We denote  $\mu_* = \{x \in R : \mu(x) = \mu(0)\}$ . A fuzzy subset  $\mu$  of  $R$  is a fuzzy ideal of  $R$  if it is a left and right fuzzy ideal. Following [20] a fuzzy ideal  $\mu$  of a ring  $R$  is called a prime fuzzy ideal if for any two fuzzy ideals  $\sigma$  and  $\theta$  of  $R$  the condition  $\sigma\theta \subseteq \mu$  implies that  $\sigma \subseteq \mu$  or  $\theta \subseteq \mu$ . Following [13,14], for a fuzzy ideal  $\mu$  of a ring  $R$ , the fuzzy radical of  $\mu$ , denoted by  $\sqrt{\mu}$ , is defined by  $\sqrt{\mu} = \bigcap \{\sigma : \sigma \text{ is a fuzzy prime ideal of } R, \sigma \subseteq \mu, \sigma_* \subseteq \mu_*\}$ . In [17], V. Murali and B. Makaba has discussed concepts of primary decomposition of fuzzy ideals and the radicals of such ideals over a Noetherian ring.

The operations of sums and products on fuzzy subsets  $\mu$  and  $\nu$  of  $R$  are defined as follows:

$$\begin{aligned}
 (\mu + \nu)(x) &= \sup\{\mu(r)\wedge\nu(s) : x = r + s; r, s \in R\} \\
 (\mu\nu)(x) &= \sup\{\mu(r)\wedge\nu(s) : x = rs; r, s \in R\}, \\
 (\mu\nu)(x) &= \sup\{\wedge_{i=1}^n(\mu(r_i)\wedge\nu(s_i)) : x = \sum_{i=1}^n r_i s_i; r_i, s_i \in R, n \in \mathbb{N}\}.
 \end{aligned}$$

If  $x$  has no such decomposition, the sum and product take the value 0 at  $x$ .

The following text presupposes all rings to be commutative rings possessing an identity element.

Recall that a ring  $R$  is Laskerian if each ideal of  $R$  admits a shortest primary representation;  $R$  is strongly Laskerian, if  $R$  is Laskerian and each primary ideal of  $R$  contains a power of its radical [5, p. 455]. It is equivalent to say that a ring  $R$  is Laskerian (resp. strongly Laskerian) if every ideal of  $R$  can be represented as a finite intersection of primary ideals (resp. strongly primary ideals) [4, pp. 295,298]. There does not exist chain condition on principal ideals of a Laskerian ring (see [19, Remark 2.2]), but a strongly Laskerian ring has ACCP (cf. [19, Proposition 2.1]). By [1], ring  $R$  is said to be a  $Q$ -ring, if every ideal in  $R$  is a finite product of primary ideals. A ring  $R$  is a  $Q$ -ring if and only if  $R$  is a Laskerian ring in which every non-maximal prime ideal is quasi-principal (cf. [1, Theorem 13]). The followings are non reversible implications.

$$\begin{array}{ccccc}
 & & & & Q - \text{ring} \\
 & & & & \downarrow \\
 \text{Noetherian} & \implies & \text{Strongly Laskerian} & \implies & \text{Laskerian}
 \end{array}$$

In this paper we introduce strongly primary fuzzy ideals and strongly irreducible fuzzy ideals in rings and fixed their role in a Laskerian ring. We determine that: A finite intersection of prime fuzzy ideals (resp. primary fuzzy ideals, irreducible fuzzy ideals and strongly irreducible fuzzy ideals) is a prime fuzzy ideal (resp. primary fuzzy ideal, irreducible fuzzy ideal and strongly irreducible fuzzy

ideal). Furthermore we prove that: (1) Every nonzero fuzzy ideal of a one dimensional Laskerian domain can be uniquely expressed as a product of primary fuzzy ideals with distinct radicals, (2) A ring  $R$  is Laskerian if and only if  $R_\mu$  is Laskerian for every fuzzy ideal  $\mu$  of  $R$ .

## 2. Finite intersection of fuzzy ideals

We initiate with the following proposition.

**PROPOSITION 1.** *Finite intersection of prime fuzzy ideals of a ring is a prime fuzzy ideal.*

*Proof.* Let  $\{\mu_i : 1 \leq i \leq n\}$  be a family of prime fuzzy ideals of a ring  $R$  and let  $\theta\sigma \subseteq \bigwedge_{i=1}^n \mu_i$  for any two fuzzy ideals  $\theta$  and  $\sigma$  of  $R$ . This means  $\theta\sigma \subseteq \mu_i$  for all  $1 \leq i \leq n$ . Since  $\mu_i$  are prime fuzzy ideals, therefore if  $\theta \not\subseteq \mu_i$  for all  $1 \leq i \leq n$ , then  $\sigma \subseteq \mu_i$ , for all  $1 \leq i \leq n$ . Similarly, if  $\sigma \not\subseteq \mu_i$  for all  $1 \leq i \leq n$ , then  $\theta \subseteq \mu_i$ , for all  $1 \leq i \leq n$ . This implies that if  $\theta \not\subseteq \bigwedge_{i=1}^n \mu_i$ , then  $\sigma \subseteq \bigwedge_{i=1}^n \mu_i$ . Hence  $\bigwedge_{i=1}^n \mu_i$  is a prime fuzzy ideal of  $R$ . ■

Following [20], a fuzzy ideal  $\mu$  of a ring  $R$  is known as a primary fuzzy ideal if for any two fuzzy ideals  $\sigma$  and  $\theta$  of  $R$  the conditions  $\sigma\theta \subseteq \mu$  and  $\sigma \not\subseteq \mu$  together imply that  $\theta \subseteq \sqrt{\mu}$ .

**PROPOSITION 2.** *Finite intersection of primary fuzzy ideals of a ring is a primary fuzzy ideal.*

*Proof.* Let  $\{\mu_i : 1 \leq i \leq n\}$  be a family of primary fuzzy ideals of a ring  $R$  and let  $\theta\sigma \subseteq \bigwedge_{i=1}^n \mu_i$  for any two fuzzy ideals  $\theta$  and  $\sigma$ . This implies that  $\theta\sigma \subseteq \mu_i$  for all  $1 \leq i \leq n$ . Since  $\mu_i$  are primary fuzzy ideals therefore if  $\theta \not\subseteq \mu_i$  for all  $1 \leq i \leq n$  then  $\sigma \subseteq \sqrt{\mu_i}$  for all  $1 \leq i \leq n$ . So, if  $\theta \not\subseteq \bigwedge_{i=1}^n \mu_i$  for some  $i$  then  $\sigma \subseteq \sqrt{\bigwedge_{i=1}^n \mu_i}$ . Consequently  $\bigwedge_{i=1}^n \mu_i$  is a primary fuzzy ideal of  $R$ . ■

**REMARK 1.** Every prime fuzzy ideal of a ring is a primary fuzzy ideal.

**DEFINITION 1.** A fuzzy ideal  $\mu$  of a ring  $R$  is called strongly primary fuzzy ideal in  $R$  if  $\mu$  is a primary fuzzy ideal and  $(\sqrt{\mu})^n \subseteq \mu$  for some  $n \in \mathbb{N}$ .

Following [17, Definition 4.1], a fuzzy ideal  $\mu$  of a Noetherian ring  $R$  is said to be irreducible if  $\mu \neq R$  and whenever  $\mu = \mu_1 \wedge \mu_2$ , where  $\mu_1, \mu_2$  are fuzzy ideals of  $R$ , then  $\mu = \mu_1$  or  $\mu = \mu_2$ . A prime fuzzy ideal is necessarily irreducible; however, the converse is not true (see [17, Example 4.6]).

In the rest of text instead of a Noetherian ring we consider an arbitrary ring.

**DEFINITION 2.** A proper fuzzy ideal  $\mu$  of a ring  $R$  is said to be strongly irreducible if for each pair of fuzzy ideals  $\theta$  and  $\sigma$  of  $R$ , if  $\theta\sigma \subseteq \mu$ , then either  $\theta \subseteq \mu$  or  $\sigma \subseteq \mu$ .

**REMARK 2.** A strongly irreducible fuzzy ideal is irreducible.

PROPOSITION 3. *Finite intersection of strongly irreducible fuzzy ideals of a ring is a strongly irreducible fuzzy ideal.*

*Proof.* Let  $\{\mu_i : 1 \leq i \leq n\}$  be a family of strongly irreducible fuzzy ideals of a ring  $R$  and let  $\theta \wedge \sigma \subseteq \bigwedge_{i=1}^n \mu_i$  for any two fuzzy ideals  $\theta$  and  $\sigma$ . This means  $\theta \wedge \sigma \subseteq \mu_i$  for all  $1 \leq i \leq n$ . This implies  $\theta \subseteq \mu_i$  or  $\sigma \subseteq \mu_i$  for all  $1 \leq i \leq n$ . Since  $\mu_i$  are strongly irreducible fuzzy ideals, therefore if  $\theta \subseteq \mu_i$  or  $\sigma \subseteq \mu_i$  for all  $1 \leq i \leq n$ , then  $\theta \subseteq \bigwedge_{i=1}^n \mu_i$  or  $\sigma \subseteq \bigwedge_{i=1}^n \mu_i$  for all  $i$ . Hence  $\bigwedge_{i=1}^n \mu_i$  is a strongly irreducible fuzzy ideal of  $R$ . ■

Recall that a fuzzy ideal  $\mu$  of a ring is known as radical ideal if  $\mu = \sqrt{\mu}$ .

PROPOSITION 4. *A strongly irreducible fuzzy ideal in a ring is a prime fuzzy ideal if and only if it is a radical ideal.*

*Proof.* If  $\mu$  is a prime fuzzy ideal of a ring  $R$ , then by [3, Theorem 5.10]  $\mu = \sqrt{\mu}$ . Conversely assume that  $\mu = \sqrt{\mu}$ . Let  $\mu_1 \wedge \mu_2 \subseteq \mu$ , where  $\mu_1$  and  $\mu_2$  are fuzzy ideals of  $R$ . Then  $\mu_1 \mu_2 \subseteq \mu_1 \wedge \mu_2 \subseteq \sqrt{\mu_1 \wedge \mu_2} = \sqrt{\mu_1 \mu_2} \subseteq \sqrt{\mu} = \mu$ , and since  $\mu$  is a strongly irreducible fuzzy ideal of  $R$ . Hence  $\mu_1 \subseteq \mu$  or  $\mu_2 \subseteq \mu$ . ■

Following [6], a fuzzy ideal  $\mu$  of a ring  $R$  is called semiprime if  $\mu^2(x) = \mu(x)$  for all  $x \in R$ .

PROPOSITION 5. *A fuzzy ideal  $\mu$  of a ring  $R$  is prime if and only if it is semiprime and strongly irreducible.*

*Proof.* Suppose  $\mu$  is a fuzzy prime ideal of  $R$ . Obviously  $\mu$  is fuzzy semiprime ideal. Moreover, if  $\sigma$  and  $\theta$  are fuzzy ideals of  $R$ , satisfying  $\sigma \wedge \theta \subseteq \mu$ , then  $\sigma \theta \subseteq \mu$ , since  $\sigma \theta \subseteq \sigma \wedge \theta$ . This implies  $\sigma \subseteq \mu$  or  $\theta \subseteq \mu$ . Hence  $\mu$  is strongly irreducible. Conversely, assume that  $\mu$  is a strongly irreducible semiprime fuzzy ideal of  $R$ . Suppose  $\sigma$  and  $\theta$  be two fuzzy ideals of  $R$  with  $\sigma \theta \subseteq \mu$ . Consider

$$\begin{aligned}
(\sigma \wedge \theta)^2(x) &= ((\sigma \wedge \theta)(\sigma \wedge \theta))(x) \\
&= \sup\{\bigwedge_{i=1}^n ((\sigma \wedge \theta)(r_i) \wedge (\sigma \wedge \theta)(s_i)) : x = \sum_{i=1}^n r_i s_i; r_i, s_i \in R, n \in \mathbb{N}\} \\
&= \sup\{\bigwedge_{i=1}^n \min\{(\sigma \wedge \theta)(r_i), (\sigma \wedge \theta)(s_i)\} : x = \sum_{i=1}^n r_i s_i; r_i, s_i \in R, n \in \mathbb{N}\} \\
&= \sup\{\bigwedge_{i=1}^n \min\{\min\{\sigma(r_i), \theta(r_i)\}, \min\{\sigma(s_i), \theta(s_i)\}\} : \\
&\quad x = \sum_{i=1}^n r_i s_i; r_i, s_i \in R, n \in \mathbb{N}\} \\
&\leq \sup\{\bigwedge_{i=1}^n \min\{\sigma(r_i), \theta(s_i)\} : x = \sum_{i=1}^n r_i s_i; r_i, s_i \in R, n \in \mathbb{N}\} \\
&= (\sigma \theta)(x).
\end{aligned}$$

Therefore  $(\sigma \wedge \theta)^2 \subseteq \sigma \theta \subseteq \mu$ . But, since  $\mu$  is semiprime, therefore  $\sigma \wedge \theta \subseteq \mu$ . Hence  $\sigma \subseteq \mu$  or  $\theta \subseteq \mu$ , as  $\mu$  is strongly irreducible. ■

### 3. Fuzzy ideals in Laskerian rings

Following [17, Definitions 3.1 and 3.2], if for a collection  $\{v_i : i = 1, 2, \dots, n\}$  of primary fuzzy ideals of  $R$  and  $\{\mu_i : i = 1, 2, \dots, n\}$  a finite collection of  $v_i$ -primary fuzzy ideals of  $R$ , then,  $\mu = \Lambda_{i=1}^n \mu_i$  is called a primary decomposition of  $\mu$ . This decomposition is said to be reduced or irredundant if

- (a) the  $v_1, v_2, \dots, v_n$  are all distinct and
- (b)  $\mu_j \not\subseteq \Lambda_{i=1, i \neq j}^n \mu_i$ , for all  $j = 1, 2, \dots, n$ .

Every irreducible fuzzy ideal of a Noetherian ring is a primary fuzzy ideal (cf. [17, Proposition 4.7]). In the following proposition we generalize [17, Proposition 4.7], in case of Laskerian rings.

**PROPOSITION 6.** *In a Laskerian ring an irreducible fuzzy ideal is a primary fuzzy ideal.*

*Proof.* By passage to the quotient rings, it is enough to assume that the (0) ideal is an irreducible fuzzy ideal and show that it is primary. So, suppose that  $\mu v = 0$  and  $\mu \neq 0$ , where  $\mu$  and  $v$  are fuzzy ideals of  $R$ . Consider the chain of ideals  $\text{ann}(v) \subseteq \text{ann}(v^2) \subseteq \dots \subseteq \text{ann}(v^n) \dots$ . Since  $R$  is Laskerian (strongly Hopfian), this chain stabilizes: there exists a positive integer  $n$  such that  $\text{ann}(v^n) = \text{ann}(v^{n+k})$  for all  $k$ . It follows that  $\mu \Lambda v^n = 0$ . Indeed, if  $a \in \mu$ , then  $a\mu = 0$ , and if  $a \in v^n$ , then  $a = bv^n$  for some  $b \in R$ . Hence  $bv^{n+1} = 0$ , so  $b \in \text{ann}(v^{n+1}) = \text{ann}(v^n)$ . Hence  $bv^n = 0$ ; that is,  $a = 0$ . Since the (0) ideal is irreducible and  $\mu \neq 0$ , we must then have  $v^n = 0$ , and this shows that (0) is primary. ■

**PROPOSITION 7.** *If every primary fuzzy ideal of a ring  $R$  is a strongly irreducible fuzzy ideal, then every fuzzy minimal primary decomposition for each fuzzy ideal of  $R$  is unique.*

*Proof.* Let a fuzzy ideal  $\mu$  of  $R$  has two fuzzy minimal primary decompositions  $\Lambda_{i=1}^n \mu_i$  and  $\Lambda_{i=1}^m \sigma_i$ . For  $n \leq m$ , we have  $\Lambda_{i=1}^n \mu_i \subseteq \Lambda_{i=1}^m \sigma_i$  and since  $\sigma_1$  is strongly irreducible fuzzy ideal for some  $j$ ,  $1 \leq j \leq n$ , therefore  $\mu_j \subseteq \sigma_1$ . On the other hand,  $\Lambda_{i=1}^m \sigma_i \subseteq \mu_j$ . Since  $\mu_j$  is strongly irreducible fuzzy ideal, for some  $k$ ,  $1 \leq k \leq m$ , we have  $\sigma_k \subseteq \mu_j \subseteq \sigma_1$ . Since  $\Lambda_{i=1}^m \sigma_i$  is a fuzzy minimal primary decomposition,  $\sigma_k = \sigma_1$  and so  $k = 1$ . Hence  $\sigma_1 = \mu_j$ . Without loss of generality, let  $\sigma_1 = \mu_1$ . Similarly we can show that  $\sigma_2 = \mu_t$  for some  $t$ ,  $1 \leq t \leq n$ , and since  $\sigma_2 \neq \sigma_1$ ,  $\mu_t \neq \mu_1$ . That is,  $t \neq 1$ . Therefore, without loss of generality, we can assume that  $\sigma_2 = \mu_2$ . The same argument will show that for each  $t$ ,  $1 \leq t \leq m$ ,  $\sigma_t = \mu_t$  and  $n = m$ . ■

Two ideals  $I$  and  $J$  in a ring  $R$  are said to be co-prime (or co-maximal) if  $I + J = R$  [2, p. 7].

**PROPOSITION 8.** *Let  $D$  be a Laskerian domain of dimension 1. Then every nonzero fuzzy ideal  $\mu$  of  $D$  can be uniquely expressed as a product of primary fuzzy ideals with distinct radicals.*

*Proof.* Let  $\mu$  be a nonzero fuzzy ideal of a Laskerian domain  $D$ . Then  $\mu = \bigwedge_{i=1}^n \mu_i$ , where each  $\mu_i$  is  $P$ -primary fuzzy ideal of  $D$ . The ideals  $P_1, \dots, P_n$  are maximal since they stem from reduced primary decomposition, they are pair-wise different and therefore pair-wise coprime. Therefore  $\mu_1, \dots, \mu_n$  are pair-wise coprime and by [2, Proposition 1.10.]  $\prod \mu_i = \bigwedge \mu_i$ . Hence  $\mu = \prod \mu_i$ .

Conversely, if  $\mu = \prod \mu_i$ , the same argument shows that  $\mu = \bigwedge \mu_i$ ; this is a minimal primary decomposition of  $\mu$ , in which each  $\mu_i$  is isolated primary component, and by [2, Theorem 4.10], is therefore unique. ■

The following proposition will help us to find the rings for which every primary fuzzy ideal is a strongly irreducible fuzzy ideal.

PROPOSITION 9. *In a ring  $R$ , the following are equivalent.*

- (1) *Every fuzzy ideal of  $R$  is a strongly irreducible fuzzy ideal.*
- (2) *Every two fuzzy ideals of  $R$  are comparable.*

*Proof.* (1) $\Rightarrow$ (2). Let  $\mu$  and  $\sigma$  be two fuzzy ideals of  $R$ . Note that  $\mu \wedge \sigma$  is a strongly irreducible fuzzy ideal, and  $\mu \wedge \sigma \subseteq \mu \wedge \sigma$ . So  $\mu \subseteq \mu \wedge \sigma \subseteq \sigma$  or  $\sigma \subseteq \mu \wedge \sigma \subseteq \mu$ .

(2) $\Rightarrow$ (1) The proof is obvious. ■

Let  $\mu$  be a fuzzy ideal of a ring  $R$  and  $x \in R$ . By [9], the fuzzy subset of  $R$   $\mu_x^*(r) = \{\mu(r-x) \text{ for all } r \in R\}$  is termed as the fuzzy coset determined by  $x$  and  $\mu$ . The set of all cosets of  $\mu$  in  $R$  is a ring under the binary operations  $\mu_x^* + \mu_y^* = \mu_{x+y}^*$  and  $\mu_x^* \mu_y^* = \mu_{xy}^*$  for all  $x, y \in R$  and it is denoted by  $R_\mu$ , and known as fuzzy quotient ring of  $R$  induced by the fuzzy ideal  $\mu$ .

In [9], K.H. Lee has characterized Artinian and Noetherian rings, respectively, using fuzzy quotient rings (see [9, Propositions 3.9 and 3.10]). In the following proposition we characterize Laskerian rings by fuzzy quotient ring using the same technique of [9].

PROPOSITION 10. *A ring  $R$  is strongly Laskerian if and only if  $R_\mu$  is strongly Laskerian for every fuzzy ideal  $\mu$  of  $R$ .*

*Proof.* Let  $\bar{\nu} : R_\mu \rightarrow [0, 1]$  be a fuzzy ideal of  $R_\mu$ . To show that  $\bar{\nu}$  has a finite reduced strongly primary decomposition, define a map  $\theta : R \rightarrow [0, 1]$  by  $\theta(x) = \bar{\nu}(\mu_x^*)$  for every  $x \in R$ . Then  $\theta$  is a fuzzy ideal of  $R$  and it has a finite reduced strongly primary decomposition. Since the set of values of  $\theta$  is same to the set of values of  $\bar{\nu}$  therefore  $\bar{\nu}$  also has a finite reduced strongly primary decomposition. Hence  $R_\mu$  is strongly Laskerian.

Conversely, let  $\mu$  be a fuzzy ideal of a ring  $R$ . Then the fuzzy ideal  $\bar{\nu}$  of  $R_\mu$  defined by  $\bar{\nu}(\mu_x^*) = \mu(x)$  for every  $x \in R$  has a finite reduced primary decomposition, so that  $\mu$  also has finite reduced strongly primary decomposition. Hence  $R$  is strongly Laskerian. ■

PROPOSITION 11. *A ring  $R$  is Laskerian if and only if  $R_\mu$  is Laskerian for every fuzzy ideal  $\mu$  of  $R$ .*

*Proof.* Let  $\bar{\nu}$  be a fuzzy ideal of  $R_\mu$ . To show that  $\bar{\nu}$  has a finite reduced primary decomposition, define a map  $\theta : R \rightarrow [0, 1]$  by  $\theta(x) = \bar{\nu}(\mu_x^*)$  for every  $x \in R$ . Then  $\theta$  is a fuzzy ideal of  $R$  and it has finite primary decomposition. Since the set of values of  $\theta$  is same to the set of values of  $\bar{\nu}$  therefore  $\bar{\nu}$  also has a finite reduced primary decomposition. This implies that  $R_\mu$  is Laskerian.

Conversely, let  $\mu$  be a fuzzy ideal of a ring  $R$ . Then the fuzzy ideal  $\bar{\nu}$  of  $R_\mu$  defined by  $\bar{\nu}(\mu_x^*) = \mu(x)$  for every  $x \in R$  has a finite reduced primary decomposition, so that  $\mu$  also has finite fuzzy primary decomposition. Hence  $R$  is Laskerian. ■

The following table summarizes findings of [9, Propositions 3.9 and 3.10] and Proposition 3.11.

$R$		$R_\mu$
Artinian	$\iff$	Artinian
Noetherian	$\iff$	Noetherian
Strongly Laskerian	$\iff$	Strongly Laskerian
Laskerian	$\iff$	Laskerian
$Q$ -ring	$\implies$	Laskerian

#### REFERENCES

- [1] D.D. Anderson, L.A. Mahaney, *Commutative rings in which every ideal is a product of primary ideals*, J. Algebra **106** (1987), 528–535.
- [2] M.F. Atiyah, I.G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley Publishing company, Inc., 1969.
- [3] S.K. Bhakat, P. Das, *Fuzzy subrings and ideals redefined*, Fuzzy Sets and Systems **81** (1996), 383–393.
- [4] N. Bourbaki, *Commutative Algebra*, Addison-Wesley, Reading, Mass., 1972.
- [5] R. Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, New York, 1972.
- [6] H.V. Kambhojkar, M.S. Bapat, *On prime and primary fuzzy ideals and their radicals*, Fuzzy Sets and Systems **53** (1993), 203–216.
- [7] R. Kumar, *Fuzzy subgroups, fuzzy ideals, and fuzzy cosets: Some properties*, Fuzzy Sets and Systems **48** (1992), 267–274.
- [8] T. Kuraoka, N. Kuroki, *On fuzzy quotient rings induced by fuzzy ideals*, Fuzzy Sets and Systems **47** (1992), 381–386.
- [9] K.H. Lee, *On fuzzy quotient rings and chain conditions*, J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. **7** (2000), 33–40.
- [10] W. Liu, *Operations on fuzzy ideals*, Fuzzy Sets and Systems **8** (1983), 31–41.
- [11] D.S. Malik, *Fuzzy ideals of Artinian rings*, Fuzzy Sets and Systems **37** (1990), 111–115.
- [12] D.S. Malik, J.N. Mordeson, *Fuzzy prime ideals of a ring*, Fuzzy Sets and Systems **37** (1990), 93–98.
- [13] D.S. Malik, J.N. Mordeson, *Fuzzy maximal, radical, and primary ideals of a ring*, Information Sci. **53** (1991), 237–250.
- [14] D.S. Malik, J.N. Mordeson, *Radical of fuzzy ideals*, Information Sci. **65** (1992), 239–252.
- [15] T.K. Mukherjee, M.K. Sen, *On fuzzy ideals of a ring*, Fuzzy Sets and Systems **21** (1987), 99–104.
- [16] T.K. Mukherjee, M.K. Sen, *Rings with chain conditions*, Fuzzy Sets and Systems **39** (1991), 117–123.
- [17] V. Muraly, B.B. Makaba, *On Krull intersection theorem of fuzzy ideals*, Intern. J. Math. Math. Sci. **4** (2003), 251–262.

- [18] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl. **35** (1971), 512–517.
- [19] T. Shah, M. Saeed, *A note on strongly Laskerian domains*, Advances in Algebra 3 (2010), 79–86.
- [20] A. Sheikabdullah, K. Jeyaraman, *Anti-homomorphisms in fuzzy ideals of rings*, Intern. J. Contemp. Math. Sciences **5** (2010), 2717–2721.
- [21] Xiang-Yun Xie, *Fuzzy ideal extensions of semigroups*, Soochow J. Math. **27** (2001), 125–138.
- [22] L.A. Zadeh, *Fuzzy sets*, Inform. Control **8** (1965), 338–353.

(received 11.04.2011; in revised form 22.01.2012; available online 15.03.2012)

Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan

*E-mail*: [stariqshah@gmail.com](mailto:stariqshah@gmail.com), [saeedharris@hotmail.com](mailto:saeedharris@hotmail.com)