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TWO-SIDED BOUNDS FOR THE COMPLETE BUTZER-FLOCKE-HAUSS OMEGA FUNCTION

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Dedicated to the memory of Milorad Bertolino (1929–1981)

Abstract. The main aim of this short note is to obtain two sided bounding inequalities for the real argument Butzer-Flocke-Hauss complete Omega function improving and developing a recent result by Pogány and Srivastava [Some two-sided bounding inequalities for the Butzer-Flocke-Hauss Omega function, Math. Inequal. Appl. 10 (2007), 587–595]. The main tools are the ODE whose particular solution is the Omega function and the related Čaplygin type differential inequality.

1. Introduction and preliminaries

In the course of their investigation of the *complex-index* Euler function $E_\alpha(z)$, Butzer, Flocke and Hauss (BHF) [8] introduced the following special function:

$$\Omega(w) = 2\int_{0+}^{\frac{1}{2}} \sinh(uw)\cot(\pi u)\,\mathrm{d}u, \qquad w \in \mathbb{C}, \tag{1}$$

which they called the *complete Omega function* (see also [6, Definition 7.1]). On the other hand, in view of the definition of the Hilbert transform, the complete Omega function $\Omega(w)$ is the Hilbert transform $\mathcal{H}(\mathrm{e}^{-xw})_1(0)$ of the 1-periodic continuation of e^{-xw}, $x \in [-1/2, 1/2]$; $w \in \mathbb{C}$ at 0, that is,

$$\Omega(w) = \mathcal{H}(\mathrm{e}^{-xw})_1(0) = \mathrm{P.V.}\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{e}^{wu}\cot(\pi u)\,\mathrm{d}u$$

where the integral is taken in the sense of Cauchy's P.V. at zero [6, p. 67].

We also recall the following partial-fraction expansion of the Omega function (see [6, Theorem 1.3] and [8]):

$$\frac{\pi\Omega(2\pi w)}{2\sinh(\pi w)} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\,n}{n^2+w^2}, \qquad w \in \mathbb{C}. \tag{2}$$

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Additional links to the various applications of the Omega function $\Omega(w)$, $w \in \mathbb{C}$ in generating-function descriptions and allied considerations of the complex-index Euler $E_\alpha(z)$ and the complex-index Bernoulli function $B_\alpha(z)$ include (for example) [6, 7, 8].

Butzer *et al.* [9, Theorem 1] showed that the real-argument complete BHF Omega function $\Omega(x)$ is a particular solution of the linear ODE

$$y' = \frac{1}{2}\coth\left(\frac{x}{2}\right) y - \frac{x}{2\pi^3}\sinh\left(\frac{x}{2}\right)\widetilde{S}\left(\frac{x}{2\pi}\right), \qquad x \in \mathbb{R}, \tag{3}$$

where

$$\widetilde{S}(w) = \begin{cases} \dfrac{1}{w}\displaystyle\int_0^\infty \frac{t\sin(wt)}{e^t+1}\,\mathrm{d}t, & w \neq 0, \\ 2\eta(3), & w = 0, \end{cases} \tag{4}$$

and

$$\eta(s) := \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^s} =: (1-2^{1-s})\,\zeta(s), \qquad \Re(s) > 0;\ s \neq 1$$

denotes the Dirichlet Eta function, $\zeta(s)$ being the Riemann Zeta function.

To make precise the structure of (3), we point out that the celebrated Mathieu series

$$S(x) = \sum_{n=1}^\infty \frac{2n}{(n^2+x^2)^2}, \qquad x \in \mathbb{R},$$

has been considered for the first time by É. L. Mathieu in his book [23] devoted to mathematical physics investigations on the elasticity of rigid bodies. (For the sake of completeness, various generalizations of Mathieu series can be found in the exhaustive research paper [31] and the references therein). According to proposal by Tomovski, the alternating Mathieu series $\widetilde{S}(x)$ was introduced by Pogány *et al.* in [31, p. 72, Eq. (2.7)]. Thus

$$\widetilde{S}(x) = \sum_{n=1}^\infty (-1)^{n-1}\frac{2n}{(n^2+x^2)^2}, \qquad x \in \mathbb{R}.$$

In the same article [31, Eq. (2.8)] the authors reported on the integral representation

$$\widetilde{S}(x) = \frac{1}{x}\int_0^\infty \frac{t\sin(xt)}{e^t+1}\,\mathrm{d}t, \quad x > 0.$$

Now obvious steps lead to (4).

Our aim in this section is first to derive a two-sided bounding inequality for $\Omega(x)$ with the help of the linear first-order ODE (3) and the Čaplygin Comparison Theorem associated with the Omega function (see, for details, [10, 11, 12], [5, Section 15] and [25, Section I.1].

Consider the Cauchy problem given by

$$y' = f(x,y) \quad \text{and} \quad y(x_0) = y_0\,. \tag{5}$$

For a given interval $\mathcal{I} \subseteq \mathbb{R}$, let $x_0 \in \mathcal{I}$ and let the functions $\varphi, \psi \in \mathrm{C}^1(\mathcal{I})$. We say that φ and ψ are the *lower* and the *upper functions*, respectively, if

$$\varphi'(x) \le f\big(x, \varphi(x)\big) \quad \text{and} \quad \psi'(x) \ge f\big(x, \psi(x)\big), \qquad x \in \mathcal{I};$$
$$\varphi(x_0) = \psi(x_0) = y_0\,.$$

Suppose also that the function $f(x,y)$ is continuous on some domain $\mathbb{D}$ in the (x,y)-plane containing the interval $\mathcal{I}$ with the lower and upper functions φ and ψ, respectively. Then the solution $y(x)$ of the Cauchy problem (5) satisfies the following two-sided inequality:

$$\varphi(x) \le y(x) \le \psi(x), \qquad x \in \mathcal{I}\,. \tag{6}$$

This is actually the so-called *Čaplygin type Differential Inequality* or the *Čaplygin type Comparison Theorem* (see [5, p. 202] and [25, pp. 3–4]).

Finally, it is not hard to see that

$$\widetilde{S}(x) = S(x) - \frac{1}{4} S\Big(\frac{x}{2}\Big), \qquad x \in \mathbb{R}\,. \tag{7}$$

So, having certain two-sided bounding inequality $L(x) < S(x) < R(x)$, say, we conclude

$$L(x) - \frac{1}{4} R\Big(\frac{x}{2}\Big) < \widetilde{S}(x) < R(x) - \frac{1}{4} L\Big(\frac{x}{2}\Big), \qquad x \in \mathbb{R}\,. \tag{8}$$

2. Two-sided inequalities associated with the class $\boldsymbol{R}$

The bilateral bounds for Mathieu series $S(x)$ attracted many mathematicians like Schröder [35], Emersleben [16], Berg [3], Makai [22], Diananda [13] and more recently we have the works by Alzer, Guo, Lampret, Mortici, Pogány, Qi, Srivastava, Tomovski and coworkers (see [1, 2, 14, 16–20, 24, 26, 27, 29–35, 40–42] among others), while Mathieu himself conjectured [23, Ch. X, pp. 256–258] only the upper bound $S(x) < x^{-2}, x > 0$, proved first by Berg [3] (see also the paper by van der Corput and Heflinger [12]). Then, the bilateral bounding inequality of the same type like Berg's:

$$\frac{1}{x^2 + \frac{1}{2}} < S(x) < \frac{1}{x^2 + \frac{1}{6}}\,,$$

has been given Makai [22] who proved it in a highly elegant manner (compare to (9)).

There are three different kind of bounds L, R upon $S(x)$: (i) the class $\boldsymbol{R}$ of rational bounds [1, 2, 6, 12, 13, 15–17, 19, 20, 22, 23, 27, 29, 31, 34]; (ii) a class $\boldsymbol{A}$ of bounds consisting of combination of rational, algebraic, exponential, hyperbolic and logarithmic functions [18, 31–33, 37] and (iii) the class $\boldsymbol{O}$ of bounds containing definite integrals of certain kind differential operators [14] and higher transcendental functions [29, 30, 39].

Let us mention that the recent paper by Mortici [27] contains exhaustive efficiency discussion upon the whole class $\boldsymbol{R}$ of rational bounds (except Lampret's results), giving good account to our further considerations.

Naturally, we are looking for the at tightest possible couple (L_0, R_0) of lower and upper bound of bilateral approximation (8), such that attains minimal deviation

$$\delta(x) := R(x) - L(x) + \frac{1}{4}\left(R\left(\frac{x}{2}\right) - L\left(\frac{x}{2}\right)\right) .$$

Obviously, tighter L, R result in tighter bounds for $\widetilde{S}(x)$.

The famous result by Alzer *et al.* [2] states that

$$\frac{1}{x^2 + \frac{1}{2\zeta(3)}} < S(x) < \frac{1}{x^2 + \frac{1}{6}}, \qquad x > 0\,, \tag{9}$$

where the constants $1/(2\zeta(3))$ (first conjectured by Elbert [15]) and $1/6$ are sharp in the sense that cannot be replaced with another smaller lower and bigger upper ones. Here $\zeta(3) \approx 1.2020569$ stands for the celebrated Apèry's constant. The main advantage of Alzer's bound is its simple structure. However, Mortici [27] states the following result such that turns out to be superior to the bounds by Alzer [2], Hoorfar and Qi [19], Qi [32] and Qi *et al.* [33]. According to [27, p. 910, Theorem 1], we have

$$a(x) < S(x) < b(x), \qquad x > 0\,, \tag{10}$$

where

$$\begin{aligned} a(x) &= \frac{5(42x^6 + 341x^4 + 885x^2 + 814)}{6(x^2+1)(x^2+4)(35x^4 + 115x^2 + 72)} \\ b(x) &= \frac{1\,680x^{10} + 22\,460x^8 + 130\,092x^6 + 403\,017x^4 + 665570x^2 + 499\,305}{6(x^2+1)^2(280x^8 + 3\,230x^6 + 15\,583x^4 + 36\,627x^2 + 34\,614)} \,. \end{aligned}$$

Mortici give two another simpler bounds [27, p. 910, Corollary 1]; the first one reads as follows:

$$\frac{1}{x^2 + \frac{1}{6} + \frac{13}{210x^2}} < S(x) < \frac{1}{x^2 + \frac{1}{6} + \frac{11}{180x^2}} \,. \tag{11}$$

Here the left-hand side inequality holds true for all $x \ge x_L$, where $x_L \approx 9.59595556$ is the greatest real root of the polynomial

$$P_6(x) \equiv 7x^6 - 292x^4 - 32\,581x^2 + 10\,582\,, \tag{12}$$

and the right-hand side inequality holds even for every $x \ge x_R$, being $x_R \approx 0.603078$ the greatest real positive root of the polynomial

$$P_8(x) \equiv 11\,960x^8 + 2\,956x^6 + 48\,213x^4 + 15\,082\,700x^2 - 5\,492\,355\,.$$

The second estimate is

$$S(x) < \frac{1}{x^2 + \frac{1}{2\mu} - \frac{32}{25}\,x}, \qquad x > 0,\ \mu = \frac{166\,435}{138\,456}\,. \tag{13}$$

Let us mention that these bounds improve the earlier mentioned ones in their ranges of validity.

Now, applying (8), from (11) we deduce for all $x \geq x_L$:

$$\frac{x^2}{x^4+\frac{1}{6}\,x^2+\frac{13}{210}}-\frac{x^2}{x^4+\frac{2}{3}\,x^2+\frac{44}{45}}<\widetilde{S}(x)<\frac{x^2}{x^4+\frac{1}{6}\,x^2+\frac{11}{180}}-\frac{x^2}{x^4+\frac{2}{3}\,x^2+\frac{104}{105}}\,. \tag{14}$$

To estimate $S(x)$ with some tight two-sided bounding inequality, we point out that

$$\delta(x) \approx \frac{5}{4}\big\{R(x)-L(x)\big\}\,,$$

in both cases when $x \to 0$ or $x \to \infty$, therefore, in the case $x \geq q > 0$, reasonable candidate for the couple (L_0, R_0) is Mortici's bound (11), since (10) gives hardly handleable upper and lower Čaplygin's ODEs.

Theorem 1. *For all $x \geq x_L \approx 9.59595556$, where x_L denotes the greatest real root of the polynomial $P_6(x)$ (12), the following two-sided inequality holds true for the complete real parameter Butzet-Flocke-Hauss Omega-function:*

$$\varphi_1(x) < \Omega(x) < \psi_1(x)\,, \tag{15}$$

where

$$\begin{aligned}
\varphi_1(x) &= \sinh\Big(\frac{x}{2}\Big)\left(\frac{1}{2\pi}\ln\frac{77\left(x^4+\frac{8}{3}\pi^2x^2+\frac{1\,664}{105}\pi^4\right)}{78\left(x^4+\frac{2}{3}\pi^2x^2+\frac{44}{45}\pi^4\right)}\right.\\
&\qquad\left.+\,\frac{1}{\pi}\sqrt{\frac{5}{39}}\,\arctan\frac{3\sqrt{195}\,x^2}{15x^2+44\pi^2}-\frac{1}{\pi}\sqrt{\frac{35}{277}}\,\arctan\frac{3\sqrt{9\,695}\,x^2}{105x^2+1\,248\pi^2}\right)\\
\psi_1(x) &= \sinh\Big(\frac{x}{2}\Big)\left(\frac{1}{2\pi}\ln\frac{78\left(x^4+\frac{8}{3}\pi^2x^2+\frac{704}{45}\pi^4\right)}{77\left(x^4+\frac{2}{3}\pi^2x^2+\frac{104}{105}\pi^4\right)}\right.\\
&\qquad\left.-\,\frac{1}{\pi}\sqrt{\frac{5}{39}}\,\arctan\frac{3\sqrt{195}\,x^2}{15x^2+176\pi^2}+\frac{1}{\pi}\sqrt{\frac{35}{277}}\,\arctan\frac{3\sqrt{9\,695}\,x^2}{105x^2+312\pi^2}\right)\,.
\end{aligned}$$

Moreover, for $x < 0$ opposite inequalities hold true.

Proof. Consider the Cauchy problem

$$\Omega\,' - \frac{1}{2}\coth\Big(\frac{x}{2}\Big)\cdot\Omega = -\frac{x}{2\pi^3}\,\sinh\Big(\frac{x}{2}\Big)\cdot\widetilde{S}\Big(\frac{x}{2\pi}\Big)\,, \qquad \Omega(0)=0\,. \tag{16}$$

Evaluating $\widetilde{S}(x)$ with the bilateral estimate (14), we deduce the Čaplygin lower and upper ODEs respectively:

$$\begin{aligned}
&\varphi_1' - \frac{1}{2}\coth\Big(\frac{x}{2}\Big)\,\varphi_1\\
&\quad=\frac{2x^3}{\pi}\,\sinh\Big(\frac{x}{2}\Big)\left(\frac{1}{x^4+\frac{8}{3}\pi^2x^2+\frac{1\,664}{105}\pi^4}-\frac{1}{x^4+\frac{2}{3}\,\pi^2x^2+\frac{44}{45}\pi^4}\right)
\end{aligned} \tag{17}$$

$$\begin{aligned}&\psi_1' - \frac{1}{2}\coth\left(\frac{x}{2}\right)\,\psi_1\\&\quad= \frac{2x^3}{\pi}\,\sinh\left(\frac{x}{2}\right)\left(\frac{1}{x^4+\frac{8}{3}\pi^2\,x^2+\frac{704}{45}\pi^4} - \frac{1}{x^4+\frac{2}{3}\,\pi^2x^2+\frac{104}{105}\pi^4}\right),\end{aligned} \tag{18}$$

on the interval $\mathcal{I} = \mathbb{R}_+$. The solutions of these linear ODEs are:

$$\begin{aligned}\varphi_1(x) = \sinh\left(\frac{x}{2}\right)\Bigg(&C_1 + \frac{1}{2\pi}\ln\frac{x^4+\frac{8}{3}\pi^2\,x^2+\frac{1\,664}{105}\pi^4}{x^4+\frac{2}{3}\pi^2\,x^2+\frac{44}{45}\pi^4}\\&+\frac{1}{\pi}\sqrt{\frac{5}{39}}\,\arctan\frac{\sqrt{5}\,(3x^2+\pi^2)}{\pi^2\sqrt{39}} - \frac{1}{\pi}\sqrt{\frac{35}{277}}\,\arctan\frac{\sqrt{35}\,(3x^2+4\pi^2)}{4\pi^2\sqrt{277}}\Bigg),\end{aligned} \tag{19}$$

$$\begin{aligned}\psi_1(x) = \sinh\left(\frac{x}{2}\right)\Bigg(&C_2 + \frac{1}{2\pi}\ln\frac{x^4+\frac{8}{3}\pi^2\,x^2+\frac{704}{45}\pi^4}{x^4+\frac{2}{3}\pi^2\,x^2+\frac{104}{105}\pi^4}\\&+\frac{1}{\pi}\sqrt{\frac{35}{277}}\,\arctan\frac{\sqrt{35}\,(3x^2+\pi^2)}{\pi^2\sqrt{277}} - \frac{1}{\pi}\sqrt{\frac{5}{39}}\,\arctan\frac{\sqrt{5}\,(3x^2+4\pi^2)}{4\pi^2\sqrt{39}}\Bigg).\end{aligned} \tag{20}$$

We point out that the initial condition $\Omega(0) = 0$ is chosen in accordance with the behaviour of the Omega function $\Omega(x)$ given by

$$\Omega(x) = 8\pi\,\sinh\left(\frac{x}{2}\right)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,n}{x^2+4\pi^2n^2} \sim \frac{2}{\pi}\,\eta(1)\sinh\left(\frac{x}{2}\right) = \frac{2\ln 2}{\pi}\,\sinh\left(\frac{x}{2}\right) = o(x)\,, \tag{21}$$

as $x \to 0$, provided by the partial-fraction expansions (2). Thus, by (19) and (20), we get

$$\varphi_1(x) \sim \sinh\left(\frac{x}{2}\right)\big(C_1 + C_\varphi\big) \quad \text{and} \quad \psi_1(x) \sim \sinh\left(\frac{x}{2}\right)\big(C_2 + C_\psi\big), \qquad x \to 0\,,$$

where

$$\begin{aligned}C_\varphi &= \frac{1}{2\pi}\ln\frac{1\,248}{77} + \frac{1}{\pi}\sqrt{\frac{5}{39}}\,\arctan\sqrt{\frac{5}{39}} - \frac{1}{\pi}\sqrt{\frac{35}{277}}\,\arctan\sqrt{\frac{35}{277}},\\C_\psi &= \frac{1}{2\pi}\ln\frac{616}{39} + \frac{1}{\pi}\sqrt{\frac{35}{277}}\,\arctan\sqrt{\frac{35}{277}} - \frac{1}{\pi}\sqrt{\frac{5}{39}}\,\arctan\sqrt{\frac{5}{39}}\,.\end{aligned}$$

So, by the constraint (6), near to the origin

$$\varphi_1(x) \sim \sinh\left(\frac{x}{2}\right)\big(C_1 + C_\varphi\big) \le \frac{2\ln 2}{\pi}\sinh\left(\frac{x}{2}\right) \le \sinh\left(\frac{x}{2}\right)\big(C_2 + C_\psi\big) \sim \psi_1(x)\,,$$

that is

$$\begin{aligned}C_1 &= \frac{1}{2\pi}\ln\frac{77}{78} - \frac{1}{\pi}\sqrt{\frac{5}{39}}\,\arctan\sqrt{\frac{5}{39}} + \frac{1}{\pi}\sqrt{\frac{35}{277}}\,\arctan\sqrt{\frac{35}{277}},\\C_2 &= \frac{1}{2\pi}\ln\frac{78}{77} + \frac{1}{\pi}\sqrt{\frac{5}{39}}\,\arctan\sqrt{\frac{5}{39}} - \frac{1}{\pi}\sqrt{\frac{35}{277}}\,\arctan\sqrt{\frac{35}{277}}\,.\end{aligned}$$

Now, obvious steps lead to the assertion of the Theorem. ∎

Remark 1. Let us mention that the numerical values of the integration constants read as follows:

$$C_1 \approx -0.00259805\,, \qquad C_2 \approx 0.00259805\,.$$

The calculations throughout have been performed by `Mathematica 8`.

Since Mortici's bound comparison analysis does not include Lampret's bounds, we list this result as well. Using Euler-Maclaurin summation formula for the $(m-1)$-th partial sum for Mathieu series, Lampret derived [20, p. 2274, Eq. (17)] a set of two-sided inequalities:

$$a_m(x) < S(x) < b_m(x) \qquad x > 0\,, \tag{22}$$

where for all $m \ge 1$ it is

$$\begin{aligned} a_m(x) &= \Big(1 - \frac{1}{2(m^2+x^2)^2 + 2m^3 + m^2}\Big)\sigma_m(x) \\ b_m(x) &= \Big(1 + \frac{1}{2(m^2+x^2)^2 + 2m^3 + m^2}\Big)\sigma_m(x) \\ \sigma_m(x) &= \sum_{j=1}^{m-1} \frac{2j}{(j^2+x^2)^2} + \frac{1}{m^2+x^2} + \frac{m}{(m^2+x^2)^2} + \frac{3m^2-x^2}{6(m^2+x^2)^3}\,. \end{aligned}$$

Obviously, for all $x > 0$ we have

$$\lim_{m\to\infty} a_m(x) = \lim_{m\to\infty} b_m(x) = \lim_{m\to\infty} \sigma_m(x) = S(x)\,,$$

where the relative convergence rate has the magnitude $\mathcal{O}(m^{-4})$, see [20, Eq. (15)].

In what follows we consider Lampret's bounds $a_2(x) < S(x) < b_2(x)$, $x > 0$ as an illustrative example of his set of results. First, setting $m = 2$ in (22) we arrive at

$$\Big(1 - \frac{1}{2(x^2+4)^2 + 20}\Big)\sigma_2(x) < S(x) < \Big(1 + \frac{1}{2(x^2+4)^2 + 20}\Big)\sigma_2(x)\,,$$

where

$$\sigma_2(x) = \frac{2}{(x^2+1)^2} + \frac{1}{x^2+4} + \frac{2}{(x^2+4)^2} + \frac{12-x^2}{6(x^2+4)^3}\,.$$

By virtue of (8) we get

$$\begin{aligned} &\Big(1 - \frac{1}{2(x^2+4)^2 + 20}\Big)\sigma_2(x) - \frac{1}{4}\left(1 + \frac{1}{2((x/2)^2+4)^2 + 20}\right)\sigma_2\Big(\frac{x}{2}\Big) \\ &\quad < \widetilde{S}(x) < \Big(1 + \frac{1}{2(x^2+4)^2 + 20}\Big)\sigma_2(x) - \frac{1}{4}\left(1 - \frac{1}{2((x/2)^2+4)^2 + 20}\right)\sigma_2\Big(\frac{x}{2}\Big)\,. \end{aligned}$$

As $x > 0$ obviously

$$1 + \frac{1}{2(x^2+4)^2 + 20} < \frac{21}{20} \quad \text{and} \quad 1 - \frac{1}{2(x^2+4)^2 + 20} > \frac{19}{20},$$

so the results

$$\frac{19}{20}\,\sigma_2(x) - \frac{21}{80}\,\sigma_2\left(\frac{x}{2}\right) < \widetilde{S}(x) < \frac{21}{20}\,\sigma_2(x) - \frac{19}{80}\,\sigma_2\left(\frac{x}{2}\right). \tag{23}$$

Now, we are ready to expose our next main result.

THEOREM 2. *For all $x > 0$ the complete, real argument* BHF *Omega function has the following two-sided bounding inequality*

$$\varphi_2(x) < \Omega(x) < \psi_2(x)\,, \tag{24}$$

where

$$\begin{aligned}\varphi_2(x) = \sinh\left(\frac{x}{2}\right)\Bigg(&\frac{37}{20\pi} + \frac{1}{\pi}\ln\left(64\pi^{23/5}\right) + \frac{42\pi}{5}\,\frac{1}{x^2+4\pi^2} - \frac{213\pi}{5}\,\frac{1}{x^2+16\pi^2}\\ &- \frac{112\pi^3}{5}\,\frac{1}{(x^2+16\pi^2)^2} - \frac{461\pi}{15}\,\frac{1}{x^2+64\pi^2} - \frac{1\,216\pi^3}{15}\,\frac{1}{(x^2+64\pi^2)^2}\\ &- \frac{21}{10\pi}\,\ln(x^2+16\pi^2) + \frac{19}{20\pi}\,\ln(x^2+64\pi^2)\Bigg)\\ \psi_2(x) = \sinh\left(\frac{x}{2}\right)\Bigg(&\frac{379}{240\pi} + \frac{1}{\pi}\ln\frac{16}{\pi^{2/5}} + \frac{38\pi}{5}\,\frac{1}{x^2+4\pi^2} - \frac{799\pi}{30}\,\frac{1}{x^2+16\pi^2}\\ &- \frac{304}{15\pi}\,\frac{1}{(x^2+16\pi^2)^2} - \frac{329\pi}{3}\,\frac{1}{x^2+64\pi^2} - \frac{448\pi^2}{5}\,\frac{1}{(x^2+64\pi^2)^2}\\ &- \frac{19}{10\pi}\,\ln(x^2+16\pi^2) + \frac{21}{20\pi}\,\ln(x^2+64\pi^2)\Bigg).\end{aligned}$$

Proof. The Cauchy problem (16) in conjunction with the previous estimates (23) upon $\widetilde{S}(x)$ enables us to formulate the Čaplygin lower and upper ODEs respectively:

$$\begin{aligned}\varphi_2' - \frac{1}{2}\coth\left(\frac{x}{2}\right)\,\varphi_2 &= -\frac{x}{2\pi^3}\,\sinh\left(\frac{x}{2}\right)\,\left(\frac{21}{20}\,\sigma_2\left(\frac{x}{2\pi}\right) - \frac{19}{80}\,\sigma_2\left(\frac{x}{4\pi}\right)\right)\\ \psi_2' - \frac{1}{2}\coth\left(\frac{x}{2}\right)\,\psi_2 &= -\frac{x}{2\pi^3}\,\sinh\left(\frac{x}{2}\right)\,\left(\frac{19}{20}\,\sigma_2\left(\frac{x}{2\pi}\right) - \frac{21}{80}\,\sigma_2\left(\frac{x}{4\pi}\right)\right),\end{aligned}$$

on the interval $\mathcal{I} = \mathbb{R}_+$. The solutions of these linear ODEs are:

$$\begin{aligned}\varphi_2(x) = \sinh\left(\frac{x}{2}\right)\Bigg(&C_3 - \frac{23}{10\pi}\ln 2\pi + \frac{42\pi}{5}\,\frac{1}{x^2+4\pi^2} - \frac{213\pi}{5}\,\frac{1}{x^2+16\pi^2}\\ &- \frac{112\pi^3}{5}\,\frac{1}{(x^2+16\pi^2)^2} - \frac{461\pi}{15}\,\frac{1}{x^2+64\pi^2} - \frac{1\,216\pi^3}{15}\,\frac{1}{(x^2+64\pi^2)^2}\\ &-\frac{21}{10\pi}\,\ln(x^2+16\pi^2) + \frac{19}{20\pi}\,\ln(x^2+64\pi^2)\Bigg)\end{aligned}$$

$$\psi_2(x) = \sinh\left(\frac{x}{2}\right)\Bigg(C_4 - \frac{17}{10\pi}\ln 2\pi + \frac{38\pi}{5}\,\frac{1}{x^2+4\pi^2} - \frac{799\pi}{30}\,\frac{1}{x^2+16\pi^2}$$
$$- \frac{304}{15\pi}\,\frac{1}{(x^2+16\pi^2)^2} - \frac{329\pi}{3}\,\frac{1}{x^2+64\pi^2} - \frac{448\pi^2}{5}\,\frac{1}{(x^2+64\pi^2)^2}$$
$$-\frac{19}{10\pi}\,\ln(x^2+16\pi^2) + \frac{21}{20\pi}\,\ln(x^2+64\pi^2)\Bigg)\,.$$

Letting $x \to 0$, we conclude:

$$\varphi_2(x) \sim \sinh\left(\frac{x}{2}\right)\big(C_3 + A\big) \le \frac{2\ln 2}{\pi}\sinh\left(\frac{x}{2}\right) \le \sinh\left(\frac{x}{2}\right)\big(C_4 + B\big) \sim \psi_1(x)\,,$$

where

$$A = -\frac{37}{20\pi} - \frac{1}{\pi}\,\ln\left(32\,\pi^{23/5}\right), \qquad B = -\frac{379}{240\pi} - \frac{1}{\pi}\,\ln\left(8\pi^{13/10}\right)\,.$$

Obviously

$$C_3 = \frac{37}{20\pi} + \frac{1}{\pi}\,\ln\left(128\pi^{23/5}\right), \qquad C_4 = \frac{379}{240\pi} + \frac{1}{\pi}\,\ln\left(32\pi^{13/10}\right)\,,$$

such that finishes the proof of the Theorem. ■

Remark 2. Routine calculations give us the approximate values of the constants

$$C_3 \approx 3.8094651 \quad \text{and} \quad C_4 \approx 2.0795349\,.$$

Remark 3. Pogány *et al.* [31, Proposition 2] reported on the estimate (related to (9)):

$$\frac{4\zeta(3)-3}{(3x^2+2)(2\zeta(3)x^2+1)} < \widetilde{S}(x) < \frac{12-\zeta(3)}{(6x^2+1)(\zeta(3)x^2+2)} \qquad x > 0\,.$$

By virtue of this result, using Čaplygin's Comparison Theorem, Pogány and Srivastava [29, Theorem 3] proved that for all $x \ge 0$, there holds true

$$\frac{1}{\pi}\sinh\left(\frac{x}{2}\right)\,\ln\left(\frac{\zeta(3)x^2+8\pi^2}{3x^2+2\pi^2}\right) < \Omega(x) < \frac{1}{\pi}\sinh\left(\frac{x}{2}\right)\,\ln\left(\frac{3x^2+8\pi^2}{\zeta(3)x^2+2\pi^2}\right). \quad (25)$$

For $x < 0$, we have the reversed inequalities.

3. Two-sided inequalities for the class $\boldsymbol{A}$

In this section we present some bilateral bounding inequalities associated with bounding functions $L, R \in \boldsymbol{A}$. This kind of bounds have been considered by Guo, Qi and coworkers (see [18, 19, 31–33]). Here we report on a new result.

Theorem 3. *For all $x \ge 0$, the following two-sided bounding inequalities hold true*

$$\varphi_3(x) < \Omega(x) < \psi_3(x)\,, \quad (26)$$

where

$$\varphi_3(x) = \sinh\left(\frac{x}{2}\right)\left(\frac{8\pi}{x^2+4\pi^2} - \frac{1}{\pi^2}\ln\frac{e^{2\pi^2/x}}{e^{2\pi^2/x}-1} - \frac{1}{\pi(e^{2\pi^2/x}-1)} + \frac{2(\ln 2-1)}{\pi}\right)$$

$$\psi_3(x) = \sinh\left(\frac{x}{2}\right)\left(\frac{4\pi}{x^2+4\pi^2} + \frac{1}{\pi^2}\ln\frac{e^{2\pi^2/x}}{e^{2\pi^2/x}-1} + \frac{1}{\pi(e^{2\pi^2/x}-1)} + \frac{2\ln 2-1}{\pi}\right).$$

Moreover, for $x<0$, the inequality (26) *reverses.*

Proof. Consider the two-sided bounding inequality for the alternating Mathieu series $\widetilde{S}(x)$ by Tomovski and Leškovski [37, Theorem 2.3]:

$$\widetilde{L}(x) < \widetilde{S}(x) < \widetilde{R}(x), \qquad x>0 \tag{27}$$

where

$$\widetilde{L}(x) = \frac{2}{(1+x^2)^2} - \frac{1}{(1+x^2)^2(1-e^{-\pi/x})} - \frac{\pi e^{-\pi/x}}{2x(1+x^2)(1-e^{-\pi/x})^2}$$

$$\widetilde{R}(x) = \frac{1}{(1+x^2)^2} + \frac{1}{(1+x^2)^2(1-e^{-\pi/x})} + \frac{\pi e^{-\pi/x}}{2x(1+x^2)(1-e^{-\pi/x})^2},$$

and take $\mathcal{I}=\mathbb{R}_+$. Applying bounds (27) to the ODE (3), that is, for the Cauchy problem

$$\Omega' - \frac{1}{2}\coth\left(\frac{x}{2}\right)\Omega = -\frac{x}{2\pi^3}\sinh\left(\frac{x}{2}\right)\widetilde{S}\left(\frac{x}{2\pi}\right), \qquad \Omega(0)=0, \tag{28}$$

and using some elementary inequalities such as $x^2+4\pi^2 \ge 4\pi x$, $x^2+4\pi^2\ge x^2$, we deduce the related lower and upper ODEs:

$$\begin{aligned}
&\varphi_3' - \frac{1}{2}\coth\left(\frac{x}{2}\right)\varphi_3\\
&\quad = -\sinh\left(\frac{x}{2}\right)\cdot\left(\frac{16\pi x}{(x^2+4\pi^2)^2} + \frac{2}{x^2(e^{2\pi^2/x}-1)} + \frac{2\pi e^{2\pi^2/x}}{x^2(e^{2\pi^2/x}-1)^2}\right)\\
&\psi_3' - \frac{1}{2}\coth\left(\frac{x}{2}\right)\psi_3\\
&\quad = \sinh\left(\frac{x}{2}\right)\cdot\left(-\frac{8\pi x}{(x^2+4\pi^2)^2} + \frac{2}{x^2(e^{2\pi^2/x}-1)} + \frac{2\pi e^{2\pi^2/x}}{x^2(e^{2\pi^2/x}-1)^2}\right),
\end{aligned}$$

respectively; the initial condition $\Omega(0)=0$ has been used in accordance with the definition (1). The solutions of above lower and upper ODEs become

$$\varphi_3(x) = \sinh\left(\frac{x}{2}\right)\cdot\left(C_5 + \frac{8\pi}{x^2+4\pi^2} - \frac{1}{\pi^2}\ln\frac{e^{2\pi^2/x}}{e^{2\pi^2/x}-1} - \frac{1}{\pi(e^{2\pi^2/x}-1)}\right) \tag{29}$$

$$\psi_3(x) = \sinh\left(\frac{x}{2}\right)\cdot\left(C_6 + \frac{4\pi}{x^2+4\pi^2} + \frac{1}{\pi^2}\ln\frac{e^{2\pi^2/x}}{e^{2\pi^2/x}-1} + \frac{1}{\pi(e^{2\pi^2/x}-1)}\right), \tag{30}$$

with the integration constants C_5, C_6. Bearing in mind (21), we clearly conclude that

$$\varphi_3(x) \sim \sinh\Big(\frac{x}{2}\Big)\Big(C_5 + \frac{2}{\pi}\Big) \quad \text{and} \quad \psi_3(x) \sim \sinh\Big(\frac{x}{2}\Big)\Big(C_6 + \frac{1}{\pi}\Big) \qquad x \to 0\,.$$

Hence

$$\varphi_3(x) \sim \sinh\Big(\frac{x}{2}\Big)\Big(C_5 + \frac{2}{\pi}\Big) \le \frac{2\ln 2}{\pi}\sinh\Big(\frac{x}{2}\Big) \le \sinh\Big(\frac{x}{2}\Big)\Big(C_6 + \frac{1}{\pi}\Big) \sim \psi_3(x)\,,$$

that is

$$C_5 = \frac{2(\ln 2 - 1)}{\pi} \approx -0.1953486, \qquad C_6 = \frac{2\ln 2 - 1}{\pi} \approx 0.1229613\,.$$

The second assertion of the theorem follows from the fact that $\Omega(x)$ is odd. ∎

4. Two-sided inequalities associated with the class *O*

In this section, at the beginning, we recall a refinement of Alzer's bounds (9) by Draščić and Pogány [14]. In that paper considering a special case of a more general integral representation by Pogány [28], the authors derive the following result. Denote

$$U(x) = 2\int_1^\infty \frac{[\sqrt{t}]^2}{(x^2+t)^3}\,\mathrm{d}t, \qquad V(x) = 4\int_1^\infty \frac{[\sqrt{t}\,]}{(x^2+t)^3}\,\mathrm{d}t\,,$$

where $[\alpha]$ stands for the integer part of the argument α. The inequality

$$\frac{1}{x^2 + \frac{1}{2\zeta(3)}} \le U(x) \le S(x) \tag{31}$$

holds for all $x \in I_1 = [x_1, x_2]$, where x_1, x_2 are the real positive roots of the equation

$$\frac{x^2+3}{(x^2+1)^2} - \frac{8}{3(x+1)^3} + \frac{4x}{(x+1)^4} - \frac{8x^2}{5(x+1)^5} = \frac{2\zeta(3)}{2\zeta(3)x^2+1}\,.$$

We remark that both equalities in (31) cannot happen simultaneously. Moreover, the inequality

$$S(x) < U(x) + V(x) \le \frac{1}{x^2 + \frac{1}{6}} \tag{32}$$

holds for all $x \in I_2 = (0, x_3] \cup [x_4, \infty)$, where x_3, x_4 are the positive real roots of the equation

$$\frac{\pi x^4 + 2(4+\pi)x^2 + \pi - 4}{4x^2(1+x^2)^2} - \frac{6}{6x^2+1} = 0.$$

In (31), (32) for $x_j, j = \overline{1,4}$ we have equalities. Calculations with `Mathematica 8` give

$$x_1 \approx 0.394443, \quad x_2 \approx 5.04572; \quad x_3 \approx 0.660463, \quad x_4 \approx 2.74663.$$

Consequently, with the aid of (7) we conclude

$$\begin{aligned} \widetilde{L}_4(x) &:= 2\int_1^\infty \left(\frac{[\sqrt{t}\,]^2}{(x^2+t)^3} - \frac{[\sqrt{t}\,]^2+2[\sqrt{t}\,]}{(x^2+4t)^3}\right)\,\mathrm{d}t \\ &< \widetilde{S}(x) < 2\int_1^\infty \left(\frac{[\sqrt{t}\,]^2+2[\sqrt{t}\,]}{(x^2+t)^3} - \frac{[\sqrt{t}\,]^2}{(x^2+4t)^3}\right)\,\mathrm{d}t =: \widetilde{R}_4(x)\,. \end{aligned} \tag{33}$$

Now, following the lines of here exposed results in previous sections, we can easily conclude the similar fashion results by using Čaplygin's Comparison Theorem. Although the same procedure, we cannot apply directly the results by Draščić and Pogány [14], since solutions φ, ψ of Čaplygin's upper and lower linear ODEs contain terms

$$\int_{0+}^x \xi\widetilde{R}_4\left(\frac{\xi}{2\pi}\right)\,\mathrm{d}\xi, \qquad \int_{0+}^x \xi\widetilde{R}_4\left(\frac{\xi}{2\pi}\right)\,\mathrm{d}\xi\,,$$

respectively. Both integrands do not allow integration order exchange, because the resulting integrals diverge for all $x \in \mathbb{R}_+$. On the other hand, the Čaplygin's upper and lower functions expressed *via* above functions are not applicable for Cauchy problem solving directly. To skip these problems, let us evaluate $U(x), V(x)$, by the obvious estimate $a-1 < [a] < a$, $a \in \mathbb{R}$. Hence

$$\frac{x-\arctan x}{x^3} = 2\int_1^\infty \frac{(\sqrt{t}-1)^2\,\mathrm{d}t}{(x^2+t)^3} < U(x) < 2\int_1^\infty \frac{t\,\mathrm{d}t}{(x^2+t)^3} = \frac{x^2+2}{(x^2+1)^2}\,.$$

Similar estimates can be achieved for $V(x)$:

$$\begin{aligned} \frac{\arctan x}{x^3} - \frac{1}{x^2(x^2+1)^2} &= 4\int_1^\infty \frac{(\sqrt{t}-1)\,\mathrm{d}t}{(x^2+t)^3} < V(x) \\ &< 4\int_1^\infty \frac{\sqrt{t}\,\mathrm{d}t}{(x^2+t)^3} = \frac{\arctan x}{x^3} + \frac{x^2-1}{x^2(x^2+1)^2}\,. \end{aligned}$$

Therefore, so do *a fortiori* for $\widetilde{L}_4, \widetilde{R}_4$. As the left-hand-side bounds in both estimates are positive on $\mathbb{R}_+$, we have

$$\ln\sqrt{x^2+1} + \frac{\arctan x}{x^2+1} - 1 < \int_{0+}^x \xi U(\xi)\,\mathrm{d}\xi < \ln\sqrt{x^2+1} + \frac{x^2}{2(x^2+1)} \tag{34}$$

$$1 - \frac{\arctan x}{x^2+1} < \int_{0+}^x \xi V(\xi)\,\mathrm{d}\xi < \ln\sqrt{x^2+1} + 1 - \frac{\arctan x}{x}\,. \tag{35}$$

We need these bounds in the proving procedure of our next main result.

Theorem 4. *For all $x > 0$ we have*

$$-\int_{0+}^x \xi\widetilde{R}_4\left(\frac{\xi}{2\pi}\right)\,\mathrm{d}\xi < \frac{2\pi^3\Omega(x)}{\sinh\left(\frac{x}{2}\right)} - \pi^2\ln 16 < -\int_{0+}^x \xi\widetilde{L}_4\left(\frac{\xi}{2\pi}\right)\,\mathrm{d}\xi\,, \tag{36}$$

where the lower and upper guard-bound functions $\widetilde{L}_4(x), \widetilde{R}_4(x)$ of $\widetilde{S}(x)$ are defined by (33).

Proof. The Čaplygin lower and upper ODEs are built with the help of the upper and lower bounds $\widetilde{R}_4, \widetilde{L}_4$ respectively:

$$\varphi_4' - \frac{1}{2}\coth\left(\frac{x}{2}\right)\,\varphi_4 = 64\pi^3\,x\sinh\left(\frac{x}{2}\right)\int_1^\infty\left(\frac{[\sqrt{t}\,]^2}{(x^2+16\pi^2t)^3} - \frac{[\sqrt{t}\,]^2+2[\sqrt{t}\,]}{(x^2+4\pi^2t)^3}\right)\mathrm{d}t \tag{37}$$

$$\psi_4' - \frac{1}{2}\coth\left(\frac{x}{2}\right)\,\psi_4 = 64\pi^3\,x\sinh\left(\frac{x}{2}\right)\int_1^\infty\left(\frac{[\sqrt{t}\,]^2+2[\sqrt{t}\,]}{(x^2+16\pi^2t)^3} - \frac{[\sqrt{t}\,]^2}{(x^2+4\pi^2t)^3}\right)\mathrm{d}t\,. \tag{38}$$

The solutions are

$$\varphi_4(x) = \sinh\left(\frac{x}{2}\right)\left(C_7 - \frac{1}{2\pi^3}\int_{0+}^x \xi\widetilde{R}_4\left(\frac{\xi}{2\pi}\right)\mathrm{d}\xi\right),$$
$$\psi_4(x) = \sinh\left(\frac{x}{2}\right)\left(C_8 - \frac{1}{2\pi^3}\int_{0+}^x \xi\widetilde{L}_4\left(\frac{\xi}{2\pi}\right)\mathrm{d}\xi\right),$$

where we describe $\widetilde{L}_4, \widetilde{R}_4$ in (33).

Now, it is not hard to build by (7) the associated bounding guard-functions $\widetilde{L}_4(x), \widetilde{R}_4(x)$ and to conclude that the both Cauchy problems have boundary conditions

$$\varphi(0+) = \psi(0+) = 0\,.$$

Bearing in mind (34) and (35), we get

$$\varphi_4(x) \sim C_7\,\sinh\left(\frac{x}{2}\right) \quad \text{and} \quad \psi_4(x) \sim C_8\,\sinh\left(\frac{x}{2}\right), \qquad x \to 0\,,$$

yielding

$$\varphi_4(x) \sim C_7\,\sinh\left(\frac{x}{2}\right) \le \frac{2\ln 2}{\pi}\sinh\left(\frac{x}{2}\right) \le C_8\,\sinh\left(\frac{x}{2}\right) \sim \psi_4(x)\,.$$

Finally, we have

$$C_7 = C_8 = \frac{2\ln 2}{\pi}\,.$$

This finishes the proof of the Theorem. ∎

5. Two-sided inequalities associated with the explicit bounds on $\widetilde{S}(x)$

Finally, bounds on alternating Mathieu series $\widetilde{S}(x)$ have been given exclusively by Pogány, Srivastava, Tomovski and Leškovski alone and in collaboration in [29–31, 36–39]. In that cases, faced with explicit lower and upper bounds $\widetilde{L}, \widetilde{R}$ we derive in this section certain representative examples too.

First, we recall a very simple upper bound [39, p. 11, Theorem 3.1]TP by Tomovski and Pogány:

$$\big|\widetilde{S}(x)\big| \le 16\sqrt{\frac{2\pi}{x}}, \qquad x > 0\,.$$

It is worth to mention another similar type upper bound of magnitude $\boldsymbol{O}(x^{-1/(2p)})$, $p > 1$ such that readily follows by [39, p. 12, Theorem 3.2]. However, we present the [30, p. 319, Theorem 2] for the Mathieu-series $S(x), x \in \mathbb{R}$, such that links to the modulus of the alternating Mathieu series:

$$\big|\widetilde{S}(x)\big| \le \sqrt{\frac{3\zeta(3)}{1+4x^2}}\,. \tag{39}$$

THEOREM 5. *For all $x \ge 0$, we have*

$$\Big|\,\Omega(x) - \frac{\ln 4}{\pi}\,\sinh\Big(\frac{x}{2}\Big)\Big| \le \frac{2\sqrt{3\zeta(3)}\,x^2\,\sinh\big(\frac{x}{2}\big)}{\pi^3(1+\sqrt{1+4x^2})}\,. \tag{40}$$

Proof. Both Čaplygin's ODEs are

$$y' - \frac{1}{2}\,\coth\Big(\frac{x}{2}\Big)\,y = \pm\frac{\sqrt{3\zeta(3)}}{2\pi^3}\,\frac{x\sinh\big(\frac{x}{2}\big)}{\sqrt{1+4x^2}} \qquad y \in \{\varphi_5, \psi_5\}\,,$$

and the related solutions are

$$y(x) = \sinh\Big(\frac{x}{2}\Big)\,\bigg(C_{9,10} \pm \frac{\sqrt{3\zeta(3)}}{8\pi^3}\,\sqrt{1+4x^2}\bigg)\,.$$

Now, easy steps show that

$$C_{9,10} = \frac{\ln 4}{\pi} \pm \frac{\sqrt{3\zeta(3)}}{2\pi^3}\,,$$

such that lead to the asserted two-sided bound (40). ∎

Finally, we mention the upper bound [30, p. 320, Theorem 3], whose special case relates to alternating Mathieu series. Namely:

$$\widetilde{S}(x) \le \frac{4\pi^2}{3}\,\left(\frac{\pi\,x}{5(1+2x^2)}\,{}_2F_1\left[\begin{matrix}1,\,1/2\\ 3/2\end{matrix}\,\Big|\,\frac{4x^4}{(1+2x^2)^2}\right]\right)^{1/2}, \qquad x \ge 0\,.$$

Here ${}_2F_1[\cdot]$ stands for the familiar Gauss hypergeometric function. However, the use of this bound will result only in upper Čaplygin's function.

6. Efficiency discussion. Further remarks

In this section we analyze the efficiency of bounds presented by Theorems 1–3 and Theorem 5. Of course, we are looking for at most tighter couple (φ, ψ) of lower and upper bound of bilateral approximations, such that permits the least deviation

$$\min_{1\le j\le 5}\big\{\delta_j(x) := \psi_j(x) - \varphi_j(x)\big\}\,.$$

Obviously, tighter L, R result in tighter bounds for $\widetilde{S}(x)$, that is for the complete real argument BHF Omega function $\Omega(x)$. In other words as closer is $\delta_j(x)$ to the real axis, as tighter the lower and upper Čaplygin's functions are.

Building the deviation $\delta_j(x)$ associated with bilateral bounds by Theorem j, $j = 1, 2, 3, 5$, we get

$$\begin{aligned}\delta_1(x) = \sinh\left(\frac{x}{2}\right) &\left(\frac{1}{2\pi}\ln\frac{6\,084\left(x^4+\frac{2}{3}\pi^2x^2+\frac{44}{45}\pi^4\right)\left(x^4+\frac{8}{3}\pi^2x^2+\frac{704}{45}\pi^4\right)}{5\,929\left(x^4+\frac{2}{3}\pi^2x^2+\frac{104}{105}\pi^4\right)\left(x^4+\frac{8}{3}\pi^2x^2+\frac{1\,664}{105}\pi^4\right)}\right.\\ &- \frac{1}{\pi}\sqrt{\frac{5}{39}}\left(\arctan\frac{3\sqrt{195}\,x^2}{15x^2+176\pi^2}+\arctan\frac{3\sqrt{195}\,x^2}{15x^2+44\pi^2}\right)\\ &\left.+\frac{1}{\pi}\sqrt{\frac{35}{277}}\left(\arctan\frac{3\sqrt{9\,695}\,x^2}{105x^2+1\,248\pi^2}+\arctan\frac{3\sqrt{9\,695}\,x^2}{105x^2+312\pi^2}\right)\right),\end{aligned} \tag{41}$$

$$\begin{aligned}\delta_2(x) = \sinh\left(\frac{x}{2}\right) &\left(-\frac{13}{48\pi}+\frac{1}{\pi}\ln\frac{4\pi^5(x^2+64\pi^2)^{1/10}}{(x^2+16\pi^2)^{1/5}}-\frac{4\pi}{5}\,\frac{1}{x^2+4\pi^2}\right.\\ &+\frac{479\pi}{30}\,\frac{1}{x^2+16\pi^2}+\frac{32}{15\pi}\,\frac{1}{(x^2+16\pi^2)^2}\\ &\left.-\frac{1184\pi}{15}\,\frac{1}{x^2+64\pi^2}-\frac{128\pi^3}{15}\,\frac{1}{(x^2+64\pi^2)^2}\right),\end{aligned} \tag{42}$$

$$\delta_3(x) = \sinh\left(\frac{x}{2}\right)\left(\frac{3}{4\pi^3}-\frac{4\pi}{x^2+4\pi^3}+\frac{2}{\pi^2}\ln\frac{\mathrm{e}^{2\pi^2/x}}{\mathrm{e}^{2\pi^2/x}-1}+\frac{2}{\pi(\mathrm{e}^{2\pi^2/x}-1)}\right), \tag{43}$$

$$\delta_4(x) = \frac{1}{\pi}\sinh\left(\frac{x}{2}\right)\ln\frac{(3x^2+8\pi^2)(3x^2+2\pi^2)}{(\zeta(3)x^2+2\pi^2)(\zeta(3)x^2+8\pi^2)}, \tag{44}$$

$$\delta_5(x) = \sinh\left(\frac{x}{2}\right)\frac{4\sqrt{3\zeta(3)}\,x^2}{\pi^3(1+\sqrt{1+4x^2})}. \tag{45}$$

Here, since Theorem 4 improves, by the Draščić-Pogány definite integral bounds (31) and (32), the result (9) by Alzer *et al.* we consider their simple rational bound instead of the result obtained in Theorem 4. Being the bilateral integral bound (36) tighter then the one by Alzer *et al.* for all $x \in [0.394443, 5.04572]$ to the left and for all $x \in \mathbb{R}_+ \setminus [0.660463, 2.74663]$ on the right, we can easily deduce the tightness of the bounds exposed in Theorem 4. Hence, the deviation $\delta_4(x)$ associated with Alzer's bounds we form by (25) in Remark 3, compare also [29, Theorem 3].

It is straightforward that Mortici's bounds give the tighter bounding region for the complete BHF Omega function. However, near to the origin another bounds give precise approximations, while for larger values of the argument x the width of the bounding regions, measured pointwise by the defiation functions $\delta_j(x), 1 \le j \le 5$ are of exponential growth (thanks to sinh factors).

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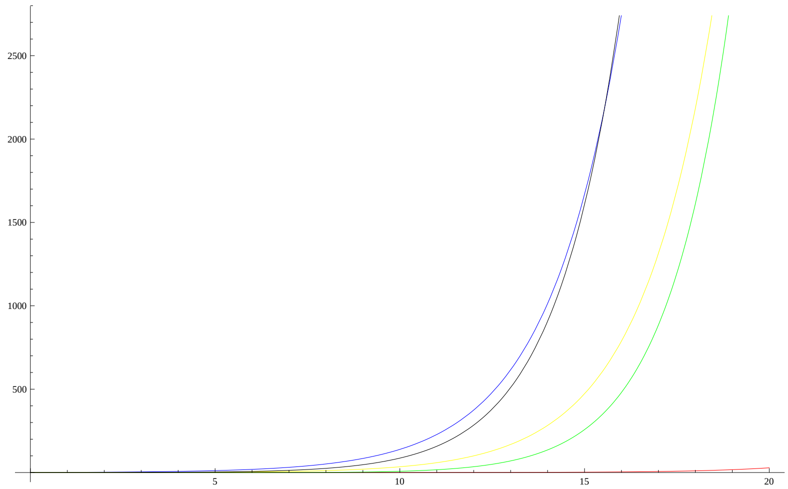

Figure 1. Deviation functions $\delta_j(x), 1 \le j \le 5$ presented for $x \in [0, 20]$.
From above at $x = 10$ the deviation functions are $\delta_2 > \delta_5 > \delta_4 > \delta_3 > \delta_1$.

had the pleasure to enjoy Professor Bertolino's excellent lectures, and it is the 30th anniversary of his passing away; we decided to submit this article to Matematički Vesnik because Professor Bertolino had carried out a number of different editorial duties in Matematički Vesnik over an extended period; and last, but not least, because Professor Bertolino had introduced the first author to one of his favorite subjects - Chaplygin's Differential Inequality, see [4].

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