

## REMARKS ON COUPLED FIXED POINT THEOREMS IN CONE METRIC SPACES

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**Abstract.** In this paper, we first show that some coupled fixed point theorems in cone metric spaces are proper consequences of relevant fixed point theorems. Then we give and prove some corresponding coupled fixed point theorems in partially ordered cone metric spaces. Some examples are also given to illustrate our work.

### 1. Introduction and preliminaries

The well-known Banach contraction principle is one of the pivotal results of analysis and has applications in a number of branches of mathematics. This principle has been extended and generalized in various directions for recent years by putting conditions on the mappings or on the spaces. Huang and Zhang in [16] introduced the notion of cone metric spaces, investigated the convergence in these spaces, introduced the notion of their completeness, and proved some fixed point theorems for contractive mappings on cone metric spaces. After that, many authors have focused on cone metric spaces and its topological properties, given and proved fixed point theorems in cone metric spaces (see [1–6, 12–14, 16–18, 20–26, 33–40, 42–43] and references therein).

Now we first recall some definitions and properties of cone metric spaces.

**DEFINITION 1.** [15] Let  $E$  be a real Banach space. A subset  $P$  of  $E$  is called a cone if and only if:

- (a)  $P$  is closed, non-empty and  $P \neq \{\theta\}$ ,
- (b)  $a, b \in \mathbb{R}, a, b \geq 0, x, y \in P$  imply that  $ax + by \in P$ ,
- (c)  $P \cap (-P) = \{\theta\}$ .

Given a cone, define a partial ordering  $\leq$  with respect to  $P$  by  $x \leq y$  if and only if  $y - x \in P$ . We shall write  $\ll$  for  $y - x \in \text{Int } P$ , where  $\text{Int } P$  is the interior

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of  $P$ . Also we shall use  $\prec$  to indicate that  $x \leq y$  and  $x \neq y$ . The cone  $P$  in normed space  $E$  is called normal whenever there is a number  $k > 0$  such that for all  $x, y \in E$ ,  $\theta \leq x \leq y$  implies  $\|x\| \leq k\|y\|$ . The least positive number  $k$  satisfying this norm inequality is called the normal constant of  $P$ . It is clear that  $k \geq 1$ . It is known that there exists ordered Banach space  $E$  with cone  $P$  which is not normal but with  $\text{Int } P \neq \emptyset$ .

DEFINITION 2. [16] Let  $X$  be a non-empty set. Suppose that the mapping  $d : X \times X \rightarrow E$  satisfies:

- (d1)  $\theta \leq d(x, y)$  for all  $x, y \in X$  and  $d(x, y) = \theta$  if and only if  $x = y$ ,
- (d2)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ,
- (d3)  $d(x, y) \leq d(x, z) + d(z, y)$  for all  $x, y, z \in X$ .

Then  $d$  is called a cone metric on  $X$  and  $(X, d)$  is called a cone metric space.

The concept of a cone metric space is more general than that of a metric space.

DEFINITION 3. [16] Let  $(X, d)$  be a cone metric space. We say that a sequence  $\{x_n\}$  in  $X$  is:

(a) a Cauchy sequence if for every  $c \in E$  with  $0 \ll c$ , there exists an  $N$  such that for all  $n, m > N$ ,  $d(x_n, x_m) \ll c$ .

(b) a convergent sequence if for every  $c \in E$  with  $0 \ll c$ , there exists an  $N$  such that for all  $n > N$ ,  $d(x_n, x) \ll c$  for some fixed  $x \in X$ .

A cone metric space  $X$  is said to be complete if every Cauchy sequence in  $X$  is convergent in  $X$ .

Let  $(X, d)$  be a cone metric space; then we have the following properties

(p1) If  $E$  is a real Banach space with a cone  $P$  and  $a \leq ha$  where  $a \in P$  and  $h \in (0, 1)$  then  $a = \theta$ .

(p2) if  $\theta \leq u \leq c$  for each  $\theta \leq c$  then  $u = \theta$ .

(p3) if  $u \leq v$  and  $v \ll w$  then  $u \ll w$ .

(p4) if  $a \leq b + c$  for each  $\theta \leq c$  then  $a = b$ .

(p5) if  $c \in \text{Int } P$ ,  $0 \leq a_n$  and  $a_n \rightarrow \theta$  then there exists a  $K$  such that for all  $n > K$ , we have  $a_n \ll c$ .

For the details about these properties see [21, 24].

It is known that the sequence  $\{x_n\}$  converges to  $x \in X$  if  $d(x_n, x) \rightarrow \theta$  as  $n \rightarrow \infty$  and  $\{x_n\}$  is a Cauchy sequence if  $d(x_n, x_m) \rightarrow \theta$  as  $n, m \rightarrow \infty$ . In the case when the cone is not necessarily normal, the fact that  $d(x_n, y_n) \rightarrow d(x, y)$  if  $x_n \rightarrow x$  and  $y_n \rightarrow y$  is not applicable.

DEFINITION 4. [3] Let  $f, g : X \rightarrow X$  be two self-mappings on  $X$ . An element  $x \in X$  is called a coincidence point of  $f$  and  $g$  if  $fx = gx$ .  $f$  and  $g$  are said to be weakly compatible if they commute at their coincidence points, that is  $gfx = fgx$  if  $fx = gx$ .

Using the concept of weakly compatible mappings, many authors have studied the existence and uniqueness of common fixed points of self-mappings in cone metric spaces (see, for example, [3, 22, 23] and references therein). For our purpose, we now state the result of Jungck et.al. [22].

**Theorem 5.** [22] *Let  $(X, d)$  be a cone metric space,  $P$  a cone with non-empty interior and mappings  $f, g : X \rightarrow X$ . Suppose that there exist non-negative constants  $a_i, i = 1, 2, \dots, 5$  satisfying  $\sum_{i=1}^5 a_i < 1$  such that, for all  $x, y \in X$ ,*

$$d(fx, fy) \leq a_1 d(gx, gy) + a_2 d(gx, fx) + a_3 d(gy, fy) + a_4 d(gx, fy) + a_5 d(gy, fx) \quad (1)$$

*If  $f(X) \subseteq g(X)$  and  $f(X)$  or  $g(X)$  is a complete subspace of  $X$  then  $f$  and  $g$  have a unique coincidence point in  $X$ . Moreover, if  $f$  and  $g$  are weakly compatible, then  $f$  and  $g$  have a unique common fixed point.*

Recently, existence of fixed points for contraction type mappings in partially ordered metric spaces has been considered in [7–11, 19, 27–32, 41] and references therein, where some applications to matrix equations, ordinary differential equations, and integral equations has been presented. Bhashkar and Lakshmikantham [10] introduced the concept of a coupled fixed point of a mapping  $F : X \times X \rightarrow X$  (a non-empty set) and established some coupled fixed point theorems in partially ordered complete metric spaces which can be used to discuss the existence and uniqueness of solution for periodic boundary value problems. Later, Lakshmikantham and Ćirić [27] proved coupled coincidence and coupled common fixed point results for nonlinear mappings  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$  satisfying certain contractive conditions in partially ordered complete metric spaces. Using the concepts of coupled fixed point and coupled coincidence point, some authors have proved coupled (coincidence, fixed) point theorems in cone metric spaces (see [1, 14, 25, 40, 42]). Some of them are in non-ordered cone metric spaces.

**DEFINITION 6.** [10] Let  $(X, \preceq)$  be a partially ordered set and  $F : X \times X \rightarrow X$ . The mapping  $F$  is said to have the mixed monotone property if  $F$  is monotone non-decreasing in  $x$  and  $F$  is monotone non-increasing in  $y$ , that is, for any  $x, y \in X$ ,

$$\begin{aligned} x_1, x_2 \in X, x_1 \preceq x_2 &\Rightarrow F(x_1, y) \preceq F(x_2, y) \\ y_1, y_2 \in X, y_1 \preceq y_2 &\Rightarrow F(x, y_1) \succeq F(x, y_2). \end{aligned}$$

**DEFINITION 7.** [10] An element  $(x, y) \in X \times X$  is called a coupled fixed point of the mapping  $F : X \times X \rightarrow X$  if  $x = f(x, y)$  and  $y = f(y, x)$ .

**DEFINITION 8.** [27] Let  $(X, \preceq)$  be a partially ordered set and  $F : X \times X \rightarrow X, g : X \rightarrow X$  be two mappings. The mapping  $F$  is said to have the mixed g-monotone property if  $F$  is monotone g-non-decreasing in its first argument and  $F$  is monotone g-non-increasing in its second argument, that is, for any  $x, y \in X$ ,

$$\begin{aligned} x_1, x_2 \in X, gx_1 \preceq gx_2 &\Rightarrow F(x_1, y) \preceq F(x_2, y) \\ y_1, y_2 \in X, gy_1 \preceq gy_2 &\Rightarrow F(x, y_1) \succeq F(x, y_2). \end{aligned}$$

DEFINITION 9. [27] An element  $(x, y) \in X \times X$  is called

- (1) a coupled coincidence point of the mapping  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$  if  $gx = F(x, y)$  and  $gy = F(y, x)$ .
- (2) a coupled common fixed point of the mapping  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$  if  $x = gx = F(x, y)$  and  $y = gy = F(y, x)$ .

DEFINITION 10. [27] The mappings  $F$  and  $g$  where  $F : X \times X \rightarrow X$ ,  $g : X \rightarrow X$  are said to commute if  $F(gx, gy) = g(Fx, Fy)$  for all  $x, y \in X$ .

In [40], Sabetghadam et al. proved the following coupled fixed point theorems.

THEOREM 11. [40] Let  $(X, d)$  be a cone metric space,  $P$  a cone with non-empty interior. Suppose that the mapping  $F : X \times X \rightarrow X$  satisfies the following contractive condition for all  $x, y, u, v \in X$ ,

$$d(F(x, y), F(u, v)) \leq kd(x, u) + ld(y, v), \quad (2)$$

where  $k, l$  are non-negative constants with  $k + l < 1$ . Then  $F$  has a unique coupled fixed point.

THEOREM 12. [40] Let  $(X, d)$  be a cone metric space,  $P$  a cone with non-empty interior. Suppose that the mapping  $F : X \times X \rightarrow X$  satisfies the following contractive condition for all  $x, y, u, v \in X$ ,

$$d(F(x, y), F(u, v)) \leq kd(F(x, y), x) + ld(F(u, v), u), \quad (3)$$

where  $k, l$  are non-negative constants with  $k + l < 1$ . Then  $F$  has a unique coupled fixed point.

THEOREM 13. [40] Let  $(X, d)$  be a cone metric space,  $P$  a cone with non-empty interior. Suppose that the mapping  $F : X \times X \rightarrow X$  satisfies the following contractive condition for all  $x, y, u, v \in X$ ,

$$d(F(x, y), F(u, v)) \leq kd(F(x, y), u) + ld(F(u, v), x), \quad (4)$$

where  $k, l$  are non-negative constants with  $k + l < 1$ . Then  $F$  has a unique coupled fixed point.

Abbas et al. [1] introduced the concept of w-compatible mappings and proved some coupled coincidence point theorems which generalized the results of Sabetghadam et al. [40].

DEFINITION 14. [1] The mappings  $F$  and  $g$  where  $F : X \times X \rightarrow X$ ,  $g : X \rightarrow X$  are said to be w-compatible if  $gF(x, y) = F(gy, gx)$  whenever  $gx = F(x, y)$  and  $gy = F(y, x)$ .

THEOREM 15. [1] Let  $(X, d)$  be a cone metric space with a cone  $P$  having non-empty interior,  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$  be mappings satisfying.

$$\begin{aligned} d(F(x, y), F(u, v)) \leq & a_1d(gx, gu) + a_2d(F(x, y), gx) + a_3d(gy, gv) \\ & + a_4d(F(u, v), gu) + a_5d(F(x, y), gu) + a_6d(F(u, v), gx) \end{aligned} \quad (5)$$

for all  $x, y, u, v \in X$ , where  $a_i, i = 1, 2, \dots, 6$  are non-negative real numbers such that  $\sum_{i=1}^6 a_i < 1$ . If  $F(X \times X) \subseteq g(X)$  and  $g(X)$  is a complete subspace of  $X$  then  $F$  and  $g$  have a coupled coincidence point in  $X$ . Moreover, if  $F$  and  $g$  are  $w$ -compatible, then  $F$  and  $g$  have a unique common coupled fixed point and common coupled fixed point of  $F$  and  $g$  is of the form  $(u, u)$  for some  $u \in X$ .

In this paper, we first show that Theorem 15 is a real consequence of Theorem 5 and so are Theorems 11, 12 and 13. Then we give and prove some coupled fixed point results in partially ordered cone metric spaces that are relevant to Theorem 15. The results unify and extend some recent results.

## 2. Main results

LEMMA 16. Let  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$  be  $w$ -compatible mappings. If the mapping  $f : X \rightarrow X$  is defined by  $fx = F(x, x)$  for all  $x \in X$ , then  $f$  and  $g$  are weakly compatible mappings.

*Proof.* Suppose that  $x$  is a coincidence point of  $f$  and  $g$ , that is,  $fx = gx$ . By the definition of  $f$ , we have  $F(x, x) = gx$ . Since  $F$  and  $g$  are weakly compatible, we have  $F(gx, gx) = gF(x, x)$ . Therefore  $fgx = gfx$ , that is,  $f$  commute  $g$  at their coincidence point. ■

THEOREM 17. Theorem 15 is a consequence of Theorem 5.

*Proof.* Let  $f : X \rightarrow X$  be the mapping defined by  $fx = F(x, x)$  for all  $x \in X$ . In (5), take  $x = y, u = v$ , we have

$$\begin{aligned} d(fx, fu) &= d(F(x, y), F(u, v)) \\ &\leq a_1 d(gx, gu) + a_2 d(F(x, x), gx) + a_3 d(gx, gu) \\ &\quad + a_4 d(F(u, u), gu) + a_5 d(F(x, x), gu) + a_6 d(F(u, u), gx) \\ &= a_1 d(gx, gu) + a_2 d(fx, gx) + a_3 d(gx, gu) \\ &\quad + a_4 d(fu, gu) + a_5 d(fx, gu) + a_6 d(fu, gx) \\ &= (a_1 + a_3) d(gx, gu) + a_2 d(fx, gx) \\ &\quad + a_4 d(fu, gu) + a_5 d(fx, gu) + a_6 d(fu, gx). \end{aligned}$$

Moreover, we have  $f(X) \subseteq F(X \times X) \subseteq g(X)$ ,  $g(X)$  is a complete subspace of  $X$ . Applying Theorem 5,  $f$  and  $g$  have a coincidence point  $x \in X$ , that is,  $fx = gx$ . This implies that  $F(x, x) = gx$ , that is,  $(x, x)$  is coupled coincidence point of  $F$  and  $g$ . Since  $f$  and  $g$  are weakly compatible,  $x$  is unique and  $x = fx = gx$ , that is  $x = F(x, x) = gx$ . Therefore  $F$  and  $g$  have unique common coupled fixed point of the form  $(x, x)$ . ■

The following example shows that Theorem 15 is a proper consequence of Theorem 5.

EXAMPLE 18. Let  $X = \mathbb{R}$  with the cone metric  $d(x, y) = |x - y|$ , for all  $x, y \in X$ . Let  $F : X \times X \rightarrow X$  be given by

$$F(x, y) = \begin{cases} x/4, & \text{if } x = y \\ x + y, & \text{if } x \neq y, \end{cases}$$

and  $g : X \rightarrow X$  be given by  $gx = x, \forall x \in X$ . Then  $F$  and  $g$  do not satisfy the condition (5) for all  $x, y, u, v \in X$ . Indeed, suppose (5) holds for all  $x, y, u, v \in X$ , take  $x = 2u \neq 0, y = v = 0$ , we have

$$\begin{aligned} |u| &= |x - u| = d(F(x, y), F(u, v)) \\ &\leq a_1 d(gx, gu) + a_2 d(F(x, y), gx) + a_3 d(gy, gv) \\ &\quad + a_4 d(F(u, v), gu) + a_5 d(F(x, y), gu) + a_6 d(F(u, v), gx) \\ &= a_1 |x - u| + a_3 |u| + a_5 |x - u| + a_6 |u - x| \\ &= (a_1 + a_3 + a_5 + a_6) |u|, \end{aligned}$$

which is a contradiction.

However, if we define  $f : X \rightarrow X$  by  $fx = F(x, x)$  for all  $x \in X$  then  $f$  and  $g$  satisfy all the conditions of Theorem 5. Applying Theorem 5, we conclude that  $f$  and  $g$  have the unique common fixed point 0. Therefore,  $F$  and  $g$  have the common coupled fixed point  $(0, 0)$ .

We next give and prove some coupled fixed point results in partially ordered cone metric space for compatible mappings.

**DEFINITION 19.** Let  $(X, d)$  be a cone metric space. The mappings  $F$  and  $g$  where  $F : X \times X \rightarrow X, g : X \rightarrow X$  are said to be compatible if

$$\lim_{n \rightarrow \infty} d(gF(x_n, y_n), F(gx_n, gy_n)) = \theta \text{ and } \lim_{n \rightarrow \infty} d(gF(y_n, x_n), F(gy_n, gx_n)) = \theta,$$

where  $\{x_n\}$  and  $\{y_n\}$  are sequences in  $X$  such that

$$\lim_{n \rightarrow \infty} F(x_n, y_n) = \lim_{n \rightarrow \infty} gx_n = x \text{ and } \lim_{n \rightarrow \infty} F(y_n, x_n) = \lim_{n \rightarrow \infty} gy_n = y$$

for all  $x, y \in X$  are satisfied.

It is easy to see that if  $F$  and  $g$  commute then they are compatible.

**THEOREM 20.** Let  $(X, \preceq)$  be a partially ordered set and suppose there is a metric  $d$  such that  $(X, d)$  is a complete cone metric space. Let  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$  be such  $F$  has the mixed  $g$ -monotone property and there exist non-negative constants  $\alpha, \beta, \gamma$  and  $\lambda$  satisfying  $\alpha + \beta + 2\gamma + 2\lambda < 1$  such that

$$\begin{aligned} d(F(x, y), F(u, v)) &\leq \alpha d(gx, gu) + \beta d(gy, gv) + \gamma [d(F(x, y), gx) + d(F(u, v), gu)] \\ &\quad + \lambda [d(F(x, y), gu) + d(F(u, v), gx)] \end{aligned} \quad (6)$$

for all  $x, y, u, v \in X$  with  $gx \preceq gu$  and  $gy \succeq gv$ . Further suppose that  $F(X \times X) \subseteq g(X)$ ,  $g$  is continuous and  $g$  and  $F$  are compatible. Suppose either

- (a)  $F$  is continuous or
- (b)  $X$  has the following property
  - (i) If  $\{x_n\}$  is a non-decreasing sequence and  $\lim_{n \rightarrow \infty} x_n = x$  then  $gx_n \preceq gx$  for all  $n$ ,

(ii) If  $\{y_n\}$  is a non-increasing sequence and  $\lim_{n \rightarrow \infty} y_n = y$  then  $gy \preceq gy_n$  for all  $n$ .

If there exist  $x_0, y_0 \in X$  such that  $gx_0 \preceq F(x_0, y_0)$  and  $gy_0 \succeq F(y_0, x_0)$  then  $F$  and  $g$  have a coupled coincidence point.

*Proof.* Let  $x_0, y_0 \in X$  be such that  $gx_0 \preceq F(x_0, y_0)$  and  $gy_0 \succeq F(y_0, x_0)$ . Since  $F(X \times X) \subseteq g(X)$ , we construct sequences  $\{x_n\}$  and  $\{y_n\}$  in  $X$  as follows

$$gx_{n+1} = F(x_n, y_n) \text{ and } gy_{n+1} = F(y_n, x_n), \text{ for all } n \geq 0 \quad (7)$$

We shall show that

$$gx_n \preceq gx_{n+1}, \text{ for all } n \geq 0 \quad (8)$$

and

$$gy_n \succeq gy_{n+1}, \text{ for all } n \geq 0 \quad (9)$$

Since  $gx_0 \preceq F(x_0, y_0)$  and  $gy_0 \succeq F(y_0, x_0)$  and as  $gx_1 = F(x_0, y_0)$  and  $gy_1 = F(y_0, x_0)$ , we have  $gx_0 \preceq gx_1$  and  $gy_0 \succeq gy_1$ . Thus (8) and (9) hold for  $n = 0$ .

Suppose that (8) and (9) hold for some  $n \geq 0$ . Then, since  $gx_n \preceq gx_{n+1}$  and  $gy_n \succeq gy_{n+1}$ , and by the  $g$ -mixed monotone property of  $F$ , we have

$$gx_{n+2} = F(x_{n+1}, y_{n+1}) \succeq F(x_n, y_{n+1}) \succeq F(x_n, y_n) = gx_{n+1} \quad (10)$$

and

$$gy_{n+2} = F(y_{n+1}, x_{n+1}) \preceq F(y_n, x_{n+1}) \preceq F(y_n, x_n) = gy_{n+1}. \quad (11)$$

Now from (10) and (11), we obtain

$$gx_{n+1} \preceq gx_{n+2} \text{ and } gy_{n+1} \succeq gy_{n+2}$$

Thus by the mathematical induction we conclude that (8) and (9) hold for all  $n \geq 0$ .

Since  $gx_{n-1} \preceq gx_n$  and  $gy_{n-1} \succeq gy_n$ , from (6) and (7), we have

$$\begin{aligned} d(gx_n, gx_{n+1}) &= d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\leq \alpha d(gx_{n-1}, gx_n) + \beta d(gy_{n-1}, gy_n) \\ &\quad + \gamma [d(F(x_{n-1}, y_{n-1}), gx_{n-1}) + d(F(x_n, y_n), gx_n)] \\ &\quad + \lambda [d(F(x_{n-1}, y_{n-1}), gx_n) + d(F(x_n, y_n), gx_{n-1})] \\ &\leq \alpha d(gx_{n-1}, gx_n) + \beta d(gy_{n-1}, gy_n) + \gamma [d(gx_n, gx_{n-1}) + d(gx_{n+1}, gx_n)] \\ &\quad + \lambda d(gx_{n+1}, gx_{n-1}) \\ &\leq \alpha d(gx_{n-1}, gx_n) + \beta d(gy_{n-1}, gy_n) + \gamma [d(gx_n, gx_{n-1}) + d(gx_{n+1}, gx_n)] \\ &\quad + \lambda [d(gx_{n+1}, gx_n) + d(gx_n, gx_{n-1})] \end{aligned} \quad (12)$$

Therefore,

$$d(gx_n, gx_{n+1}) \leq \frac{\alpha + \gamma + \lambda}{1 - \gamma - \lambda} d(gx_{n-1}, gx_n) + \frac{\beta}{1 - \gamma - \lambda} d(gy_{n-1}, gy_n). \quad (13)$$

Similarly,  $gy_n \preceq gy_{n-1}$  and  $gx_n \succeq gx_{n-1}$ , from (6) and (7), and we have

$$\begin{aligned}
d(gy_{n+1}, gy_n) &= d(F(y_n, x_n), F(y_{n-1}, x_{n-1})) \\
&\leq \alpha d(gy_n, gy_{n-1}) + \beta d(gx_n, gx_{n-1}) \\
&\quad + \gamma [d(F(y_n, x_n), gy_n) + d(F(y_{n-1}, x_{n-1}), gy_{n-1})] \\
&\quad + \lambda [d(F(y_n, x_n), gy_{n-1}) + d(F(y_{n-1}, x_{n-1}), gy_n)] \\
&\leq \alpha d(gy_n, gy_{n-1}) + \beta d(gx_n, gx_{n-1}) + \gamma [d(gy_{n+1}, gy_n) + d(gy_n, gy_{n-1})] \\
&\quad + \lambda d(gy_{n+1}, gy_{n-1}) \\
&\leq \alpha d(gy_n, gy_{n-1}) + \beta d(gx_n, gx_{n-1}) + \gamma [d(gy_{n+1}, gy_n) + d(gy_n, gy_{n-1})] \\
&\quad + \lambda [d(gy_{n+1}, gy_n) + d(gy_n, gy_{n-1})] \tag{14}
\end{aligned}$$

Therefore,

$$d(gy_n, gy_{n+1}) \leq \frac{\alpha + \gamma + \lambda}{1 - \gamma - \lambda} d(gy_{n-1}, gy_n) + \frac{\beta}{1 - \gamma - \lambda} d(gx_{n-1}, gx_n). \tag{15}$$

From (13) and (15), we have

$$d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) \leq \frac{\alpha + \beta + \gamma + \lambda}{1 - \gamma - \lambda} [d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)]. \tag{16}$$

for all  $n$ . Set  $k = \frac{\alpha + \beta + \gamma + \lambda}{1 - \gamma - \lambda} < 1$ ; from (16), we have

$$\begin{aligned}
d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) &\leq k [d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)] \\
&\leq k^2 [d(gx_{n-2}, gx_{n-1}) + d(gy_{n-2}, gy_{n-1})] \\
&\quad \vdots \\
&\leq k^n [d(gx_0, gx_1) + d(gy_0, gy_1)]
\end{aligned}$$

This implies

$$d(gx_n, gx_{n+1}) \leq k^n [d(gx_0, gx_1) + d(gy_0, gy_1)],$$

and

$$d(gy_n, gy_{n+1}) \leq k^n [d(gx_0, gx_1) + d(gy_0, gy_1)].$$

We shall show that  $\{gx_n\}$  and  $\{gy_n\}$  are Cauchy sequences.

For  $m > n$ , we have

$$\begin{aligned}
d(gx_n, gx_m) &\leq d(gx_n, gx_{n+1}) + \cdots + d(gx_{m-1}, gx_m) \\
&\leq k^n [d(gx_0, gx_1) + d(gy_0, gy_1)] + \cdots + k^{m-1} [d(gx_0, gx_1) + d(gy_0, gy_1)] \\
&\leq \frac{k^n}{1 - k} [d(gx_0, gx_1) + d(gy_0, gy_1)].
\end{aligned}$$

Let  $\theta \ll c$  be given. Then there is a neighborhood of  $\theta$

$$N_\delta(\theta) = \{y \in E : \|y\| \leq \delta\},$$

where  $\delta > 0$ , such that  $c + N_\delta(\theta) \subseteq \text{Int}P$ . Since  $k < 1$ , choose  $N \in \mathbb{N}$  such that

$$\left\| -\frac{k^n}{1-k} [d(gx_0, gx_1) + d(gy_0, gy_1)] \right\| < \delta.$$

Then

$$-\frac{k^n}{1-k} [d(gx_0, gx_1) + d(gy_0, gy_1)] \in N_\delta(\theta)$$

for all  $n > N$ . Hence

$$c - \frac{k^n}{1-k} [d(gx_0, gx_1) + d(gy_0, gy_1)] \in c + N_\delta(\theta) \subseteq \text{Int}P.$$

Therefore,

$$\frac{k^n}{1-k} [d(gx_0, gx_1) + d(gy_0, gy_1)] \ll c,$$

for all  $n > N$ . This means,

$$d(gx_n, gx_m) \ll c, \text{ for all } m > n > N.$$

Hence we conclude that  $\{gx_n\}$  is a Cauchy sequence. Similarly, one can show that  $\{gy_n\}$  is also a Cauchy sequence. Since  $X$  is a complete cone metric space, there exist  $x, y \in X$  such that

$$\lim_{n \rightarrow \infty} gx_n = x \text{ and } \lim_{n \rightarrow \infty} gy_n = y. \quad (17)$$

Thus

$$\lim_{n \rightarrow \infty} F(x_n, y_n) = \lim_{n \rightarrow \infty} gx_n = x \text{ and } \lim_{n \rightarrow \infty} F(y_n, x_n) = \lim_{n \rightarrow \infty} gy_n = y. \quad (18)$$

Since  $F$  and  $g$  are compatible, from (18) we have

$$\lim_{n \rightarrow \infty} d(gF(x_n, y_n), F(gx_n, gy_n)) = \theta \quad (19)$$

and

$$\lim_{n \rightarrow \infty} d(gF(y_n, x_n), F(gy_n, gx_n)) = \theta. \quad (20)$$

Now, suppose that assumption (a) holds. Since  $F, g$  is continuous, by (18),  $gF(x_n, y_n) \rightarrow gx$  and  $F(gx_n, gy_n) \rightarrow F(x, y)$  as  $n \rightarrow \infty$ . Let  $\theta \ll c$  be given; there exists  $k \in \mathbb{N}$ , such that, for all  $n > k$ ,

$$\begin{aligned} d(gx, gF(x_n, y_n)) &\ll \frac{c}{3}, \quad d(F(gx_n, gy_n), F(x, y)) \ll \frac{c}{3} \\ \text{and } d(gF(x_n, y_n), F(gx_n, gy_n)) &\ll \frac{c}{3} \end{aligned}$$

Therefore,

$$\begin{aligned} d(gx, F(x, y)) &\leq d(gx, gF(x_n, y_n)) + d(gF(x_n, y_n), F(gx_n, gy_n)) \\ &\quad + d(F(gx_n, gy_n), F(x, y)) \ll c \end{aligned}$$

for all  $n > k$ . Since  $c$  is arbitrary, we get

$$d(gx, F(x, y)) \ll \frac{c}{m}, \forall m \in \mathbb{N}$$

Notice that  $\frac{c}{m} \rightarrow \theta$  as  $m \rightarrow \infty$ , and we conclude that  $\frac{c}{m} - d(gx, F(x, y)) \rightarrow -d(gx, F(x, y))$  as  $m \rightarrow \infty$ . Since  $P$  is closed, we get  $-d(gx, F(x, y)) \in P$ . Thus  $d(gx, F(x, y)) \in P \cap (-P)$ . Hence  $d(gx, F(x, y)) = \theta$ . Therefore,  $gx = F(x, y)$ . Similarly, we can show that  $gy = F(y, x)$ .

Finally, suppose that (b) holds. Since  $\{gx_n\}$  is a non-decreasing sequence and  $gx_n \rightarrow x$  and as  $\{gy_n\}$  is a non-increasing sequence and  $gy_n \rightarrow y$ , we have  $ggx_n \preceq gx$  and  $ggy_n \succeq gy$  for all  $n$ . Since  $F$  and  $g$  are compatible and  $g$  is continuous, from (17), (19) and (20) we have

$$\lim_{n \rightarrow \infty} ggx_n = gx = \lim_{n \rightarrow \infty} gF(x_n, y_n) = \lim_{n \rightarrow \infty} F(gx_n, gy_n) \quad (21)$$

and

$$\lim_{n \rightarrow \infty} ggy_n = gy = \lim_{n \rightarrow \infty} gF(y_n, x_n) = \lim_{n \rightarrow \infty} F(gy_n, gx_n). \quad (22)$$

We have

$$\begin{aligned} d(gx, F(x, y)) &\leq d(gx, ggx_{n+1}) + d(ggx_{n+1}, F(x, y)) \\ &= d(gx, ggx_{n+1}) + d(gF(x_n, y_n), F(x, y)) \\ &= d(gx, ggx_{n+1}) + d(F(gx_n, gy_n), F(x, y)) \\ &\leq d(gx, ggx_{n+1}) + \alpha d(ggx_n, gx) + \beta d(ggy_n, gy) \\ &\quad + \gamma [d(F(gx_n, gy_n), ggx_n) + d(F(x, y), gx)] \\ &\quad + \lambda [d(F(gx_n, gy_n), gx) + d(F(x, y), ggx_n)] \\ &\leq d(gx, ggx_{n+1}) + \alpha d(ggx_n, gx) + \beta d(ggy_n, gy) \\ &\quad + \gamma [d(F(gx_n, gy_n), ggx_n) + d(F(x, y), gx)] \\ &\quad + \lambda [d(F(gx_n, gy_n), gx) + d(F(x, y), gx) + d(gx, ggx_n)]. \end{aligned}$$

This implies

$$\begin{aligned} d(gx, F(x, y)) &\leq \frac{1}{1 - \gamma - \lambda} (d(gx, ggx_{n+1}) + \alpha d(ggx_n, gx) + \beta d(ggy_n, gy) \\ &\quad + \gamma d(F(gx_n, gy_n), ggx_n) + \lambda [d(F(gx_n, gy_n), gx) + d(gx, ggx_n)]) \quad (23) \end{aligned}$$

Let  $\theta \ll c$ . By (21), (22), there exist  $n_0 \in \mathbb{N}$  such that

$$\begin{aligned} d(gx, ggx_n) &\ll \frac{c(1 - \gamma - \lambda)}{1 + \alpha + \beta + \gamma + 2\lambda}, \quad d(ggy_n, gy) \ll \frac{c(1 - \gamma - \lambda)}{1 + \alpha + \beta + \gamma + 2\lambda}, \\ d(F(gx_n, gy_n), ggx_n) &\ll \frac{c(1 - \gamma - \lambda)}{1 + \alpha + \beta + \gamma + 2\lambda} \\ \text{and } d(F(gx_n, gy_n), gx) &\ll \frac{c(1 - \gamma - \lambda)}{1 + \alpha + \beta + \gamma + 2\lambda} \end{aligned}$$

for all  $n > n_0$ . Thus, from (23), we have  $d(gx, F(x, y)) \ll c$  for all  $n > n_0$ . Therefore,  $gx = F(x, y)$ .

Similarly, one can show that  $gy = F(y, x)$ . Thus we have proved that  $F$  and  $g$  have a coupled coincidence point. ■

**COROLLARY 21.** *Let  $(X, \preceq)$  be a partially ordered set and suppose there is a metric  $d$  such that  $(X, d)$  is a complete cone metric space. Let  $F : X \times X \rightarrow X$  is such  $F$  has the mixed monotone property and there exist non-negative constants  $\alpha, \beta, \gamma$  and  $\lambda$  satisfying  $\alpha + \beta + 2\gamma + 2\lambda < 1$  such that*

$$d(F(x, y), F(u, v)) \leq \alpha d(x, u) + \beta d(y, v) + \gamma [d(F(x, y), x) + d(F(u, v), u)] + \lambda [d(F(x, y), u) + d(F(u, v), x)] \quad (24)$$

for all  $x, y, u, v \in X$  with  $x \preceq u$  and  $y \succeq v$ . Suppose either

(a)  $F$  is continuous or

(b)  $X$  has the following property:

(i) if  $\{x_n\}$  is a non-decreasing sequence and  $\lim_{n \rightarrow \infty} x_n = x$  then  $x_n \preceq x$  for all  $n$ ,

(ii) if  $\{y_n\}$  is a non-increasing sequence and  $\lim_{n \rightarrow \infty} y_n = y$  then  $y \preceq y_n$  for all  $n$ .

If there exist  $x_0, y_0 \in X$  such that  $x_0 \preceq F(x_0, y_0)$  and  $y_0 \succeq F(y_0, x_0)$  then  $F$  has a coupled fixed point.

**REMARK 22.** Theorems 2.2 and 2.3 in [14], Theorems 2.1 and 2.2 in [10] are special cases of Corollary 22.

**THEOREM 23.** *In addition to the hypotheses of Theorem 20, suppose that for every  $(x, y), (z, t) \in X \times X$  there exists a  $(u, v) \in X \times X$  such that  $(gu, gv)$  is comparable to  $(gx, gy)$  and  $(gz, gt)$ . Then  $F$  and  $g$  have a unique coupled common fixed point.*

*Proof.* Suppose  $(x, y)$  and  $(z, t)$  are coupled coincidence points of  $F$  and  $g$ , that is,  $gx = F(x, y)$ ,  $gy = F(y, x)$ ,  $gz = F(z, t)$  and  $gt = F(t, z)$ . We shall show that  $gx = gz$  and  $gy = gt$ . By the assumption, there exists  $(u, v) \in X \times X$  that  $(gu, gv)$  is comparable to  $(gx, gy)$  and  $(gz, gt)$ .

Since  $F(X \times X) \subseteq g(X)$ , we define sequences  $\{u_n\}, \{v_n\}$  as follows

$$u_0 = u, v_0 = v, gu_{n+1} = F(u_n, v_n) \text{ and } gv_{n+1} = F(v_n, u_n),$$

for all  $n$ . Since  $(gu, gv)$  is comparable with  $(gx, gy)$ , we may assume that  $(gx, gy) \preceq (gu, gv) = (gu_0, gv_0)$ .

By using the mathematical induction, it is easy to prove that

$$(gx, gy) \preceq (gu_n, gv_n), \text{ for all } n. \quad (25)$$

From (6) and (25), we have

$$d(gx, gu_{n+1}) = d(F(x, y), F(u_n, v_n))$$

$$\begin{aligned}
&\leq \alpha d(gx, gu_n) + \beta d(gy, gv_n) + \gamma [d(F(x, y), gx) + d(F(u_n, v_n), gu_n)] \\
&\quad + \lambda [d(F(x, y), gu_n) + d(F(u_n, v_n), gx)] \\
&\quad + \lambda [d(gx, gu_n) + d(gu_{n+1}, gx)] \\
&\leq \alpha d(gx, gu_n) + \beta d(gy, gv_n) + \gamma [d(gu_{n+1}, gx) + d(gx, gu_n)] \\
&\quad + \lambda [d(gx, gu_n) + d(gu_{n+1}, gx)].
\end{aligned}$$

This implies

$$d(gx, gu_{n+1}) \leq \frac{\alpha + \gamma + \lambda}{1 - \gamma - \lambda} d(gx, gu_n) + \frac{\beta}{1 - \gamma - \lambda} d(gy, gv_n). \quad (26)$$

Similarly, from (6) and (25), we also have

$$d(gy, gv_{n+1}) \leq \frac{\alpha + \gamma + \lambda}{1 - \gamma - \lambda} d(gy, gv_n) + \frac{\beta}{1 - \gamma - \lambda} d(gx, gu_n). \quad (27)$$

Summing up (26) and (27), we obtain

$$\begin{aligned}
d(gx, gu_{n+1}) + d(gy, gv_{n+1}) &\leq \frac{\alpha + \beta + \gamma + \lambda}{1 - \gamma - \lambda} [d(gx, gu_n) + d(gy, gv_n)] \\
&\leq k^2 [d(gx, gu_{n-2}) + d(gy, gv_{n-2})] \\
&\quad \vdots \\
&\leq k^{n+1} [d(gx, gu_0) + d(gy, gv_0)]
\end{aligned}$$

where  $k = \frac{\alpha + \beta + \gamma + \lambda}{1 - \gamma - \lambda} < 1$ . This implies

$$d(gx, gu_{n+1}) \leq k^{n+1} [d(gx, gu_0) + d(gy, gv_0)],$$

for all  $n$ . Let  $\theta \ll c$  be given. Then there is a neighborhood of  $\theta$

$$N_\delta(\theta) = \{y \in E : \|y\| \leq \delta\},$$

where  $\delta > 0$ , such that  $c + N_\delta(\theta) \subseteq \text{Int}P$ . Since  $k < 1$ , there is an  $N_1 \in \mathbb{N}$  such that

$$\| -k^{n+1} [d(gx, gu_0) + d(gy, gv_0)] \| < \delta.$$

Then

$$-k^{n+1} [d(gx, gu_0) + d(gy, gv_0)] \in N_\delta(\theta).$$

for all  $n > N_1$ . Hence

$$c - k^{n+1} [d(gx, gu_0) + d(gy, gv_0)] \in c + N_\delta(\theta) \subseteq \text{Int}P.$$

Therefore,

$$k^{n+1} [d(gx, gu_0) + d(gy, gv_0)] \ll c,$$

for all  $n > N_1$ . That means  $d(gx, gu_{n+1}) \ll c$ , for all  $n > N_1$ . Thus,  $gu_n \rightarrow gx$  as  $n \rightarrow \infty$ . Similarly, one can show that  $gv_n \rightarrow gy$ ,  $gu_n \rightarrow gz$  and  $gv_n \rightarrow gt$  as  $n \rightarrow \infty$ . By the uniqueness of limits, we have  $gx = gz$  and  $gy = gt$ .

Since  $gx = F(x, y)$  and  $gy = F(y, x)$ , by the compatibility of  $F$  and  $g$ , it is easy to find that

$$ggx = gF(x, y) = F(gx, gy) \text{ and } ggy = gF(y, x) = F(gy, gx).$$

Denote  $gx = p$  and  $gy = q$ , then  $gp = F(p, q)$ ,  $gq = F(q, p)$ . Thus  $(p, q)$  is a coupled coincidence of  $F$  and  $g$ . Hence  $gx = gp$  and  $gy = gq$ . Therefore,

$$p = gp = F(p, q) \text{ and } q = gq = F(q, p).$$

This means that  $(p, q)$  is a coupled common fixed point of  $F$  and  $g$ .

Suppose  $(a, b)$  is another coupled common fixed point of  $F$  and  $g$ . Then from the previous argument,  $p = gp = ga = a$  and  $q = gq = gb = b$ . ■

We end the paper with a simple example which can be applied to Theorem 20 but not to Theorem 15.

EXAMPLE 20. Let  $X = \mathbb{R}$ ,  $E = C_{\mathbb{R}}^1[0, 1]$  and  $P = \{\phi \in E : \phi \geq 0\}$ . Define  $d : X \times X \rightarrow E$  by  $d(x, y) = |x - y|\phi$  for all  $x, y \in X$ , where  $\phi : [0, 1] \rightarrow \mathbb{R}$  such that  $\phi(t) = e^t$ . It is clear that  $(X, d)$  is a complete cone metric space. On the set  $X$ , we consider the following order relation

$$x, y \in X, \quad x \preceq y \Leftrightarrow x = y \text{ or } (x, y) = (0, 1).$$

Let  $F : X \times X \rightarrow X$  be given by

$$F(x, y) = \begin{cases} 1, & \text{if } x, y \text{ are comparable,} \\ 0, & \text{otherwise.} \end{cases}$$

and  $g : X \rightarrow X$  be given by  $gx = x$ , for all  $x \in X$ . It is easy to see that all the conditions of Theorem 2.5 are satisfied with  $\alpha, \beta, \gamma, \delta \geq 0$  and  $\alpha + \beta + 2\gamma + 2\delta < 1$ . Moreover,  $(1, 1)$  is a coupled coincidence point of  $F$  and  $g$ .

However, the condition (5) in Theorem 15 is not satisfied. In fact, suppose (5) holds. Take  $x = 1, y = 0, u = 1/2$  and  $v = 0$ ; we have

$$\begin{aligned} \phi &= d(F(1, 0), F(1/2, 0)) \\ &= d(F(x, y), F(u, v)) \\ &\leq a_1 d(gx, gu) + a_2 d(F(x, y), gx) + a_3 d(gy, gv) \\ &\quad + a_4 d(F(u, v), gu) + a_5 d(F(x, y), gu) + a_6 d(F(u, v), gx) \\ &= a_1 d(g1, g1/2) + a_2 d(F(1, 0), g1) + a_3 d(g0, g0) \\ &\quad + a_4 d(F(1/2, 0), g1/2) + a_5 d(F(1, 0), g1/2) + a_6 d(F(1/2, 0), g1) \\ &= 1/2 a_1 \phi + 1/2 a_4 \phi + 1/2 a_5 \phi + a_6 \phi \\ &< \phi, \end{aligned}$$

which is a contradiction. Thus we cannot apply Theorem 15 to this example.

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## REFERENCES

- [1] M. Abbas, M. Ali Khan, S. Radenović, *Common coupled fixed point theorems in cone metric spaces for  $w$ -compatible mappings*, Appl. Math. Comput. **217** (2010), 195–202.
- [2] M. Abbas, B.E. Rhoades, *Fixed and periodic point results in cone metric spaces*, Appl. Math. Lett. **22** (2009), 511–515.
- [3] M. Abbas, G. Jungck, *Common fixed point results of noncommuting mappings without continuity in cone metric spaces*, J. Math. Anal. Appl. **341** (2008), 418–420.
- [4] M. Abbas, B.E. Rhoades, T. Nazir, *Common fixed points for four maps in cone metric spaces*, Appl. Math. Comput. **216** (2010), 80–86.
- [5] I. Altun, B. Damjanović, D. Djorić, *Fixed point and common fixed point theorems on ordered cone metric spaces*, Appl. Math. Lett. **23** (2010), 310–316.
- [6] M. Arshad, A. Azam, P. Vetro, *Some common fixed point results in cone metric spaces*, Fixed Point Theory Appl. **2009** (2009) Article ID 493965, 11 pp.
- [7] R.P. Agarwal, M.A. El-Gebeily, D. O'Regan, *Generalized contractions in partially ordered metric spaces*, Appl. Anal. **87** (2008), 109–116.
- [8] A. Amini-Harandi, H. Emami, *A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations*, Nonlinear Anal. **72** (2010), 2238–2242.
- [9] I. Altun, H. Simsek, *Some fixed point theorems on ordered metric spaces and application*, Fixed Point Theory Appl. **2010** (2010), 17 pages. Article ID 621469.
- [10] T.G. Bhashkar, V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal. **65** (2006), 825–832.
- [11] B.S. Choudhury, A. Kundu, *A coupled coincidence point result in partially ordered metric spaces for compatible mappings*, Nonlinear Anal. **73** (2010), 2524–2531.
- [12] C. Di Bari, P. Vetro,  *$\phi$ -pairs and common fixed points in cone metric spaces*, Rend. Circ. Mat. Palermo. **57** (2008), 279–285.
- [13] C. Di Bari, P. Vetro, *Weakly  $\phi$ -pairs and common fixed points in cone metric spaces*, Rend. Circ. Mat. Palermo. **58** (2009), 125–132.
- [14] H.S. Ding, L. Li, *Coupled fixed point theorems in partially ordered cone metric spaces*, Filomat **25** (2011), 137–149.
- [15] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, 1985.
- [16] L.G. Huang, X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. **332** (2007), 1467–1475.
- [17] R.H. Haghi, Sh. Rezapour, N. Shahzad, *Some fixed point generalizations are not real generalizations*, Nonlinear Anal. **74** (2011), 1799–1803.
- [18] R.H. Haghi, Sh. Rezapour, *Fixed points of multifunctions on regular cone metric spaces*, Expo. Math. **28** (2009), 71–77.
- [19] J. Harjani, B. Lopez, K. Sadarangani, *Fixed point theorems for mixed monotone operators and applications to integral equations*, Nonlinear Anal. **74** (2011), 1749–1760.
- [20] D. Ilić, V. Rakočević, *Common fixed points for maps on cone metric space*, J. Math. Anal. Appl. **341** (2008), 876–882.
- [21] S. Janković, Z. Kadelburg, S. Radenović, B.E. Rhoades, *Assad-Kirk type fixed point theorems for a pair of nonself mappings on cone metric spaces*, Fixed Point Theory Appl. **2009** (2009), Article ID 761086, 16 pp.
- [22] G. Jungck, S. Radenović, S. Radojević, V. Rakočević, *Common fixed point theorems for weakly compatible pairs of cone metric spaces*, Fixed Point Theory Appl. **2009** (2009), Article ID 643840, 13 pp.
- [23] Z. Kadelburg, S. Radenović, B. Ristić, *Strict contractive conditions and common fixed point theorems in cone metric spaces*, Fixed Point Theory Appl. **2009** (2009), Article ID 173833, 14 pp.
- [24] Z. Kadelburg, S. Radenović, V. Rakočević, *Remarks on "Quasi-contraction on a cone metric space"*, Appl. Math. Lett. **22** (2009), 1647–1679.

- [25] E. Karapinar, *Couple fixed point theorems for nonlinear contractions in cone metric spaces*, Comput. Math. Appl. **59** (2010), 3656–3668.
- [26] D. Klim, D. Wardowski, *Dynamic processes and fixed points of set-valued nonlinear contractions in cone metric spaces*, Nonlinear Anal. **71** (2009), 5170–5175.
- [27] V. Lakshmikantham, Lj. Ćirić, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric space*, Nonlinear Anal. **70** (2009), 4341–4349.
- [28] N.V. Luong, N.X. Thuan, *Coupled fixed points in partially ordered metric spaces and application*, Nonlinear Anal. **74** (2011), 983–992.
- [29] J.J. Nieto, R.L. Pouso, R. Rodríguez-López, *Fixed point theorems in ordered abstract sets*, Proc. Amer. Math. Soc. **135** (2007), 2505–2517.
- [30] J.J. Nieto, R. Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order **22** (2005), 223–239.
- [31] D. O'Regan, A. Petrusel, *Fixed point theorems for generalized contractions in ordered metric spaces*, J. Math. Anal. Appl. **341** (2008), 1241–1252.
- [32] A.C.M. Ran, M.C.B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc. **132** (2004), 1435–1443.
- [33] S. Radenović, *Common fixed points under contractive conditions in cone metric spaces*, Comput. Math. Appl. **58** (2009), 1273–1278.
- [34] S. Radenović, B.E. Rhoades, *Fixed point theorem for two non-self mappings in cone metric spaces*, Comput. Math. Appl. **57** (2009), 1701–1707.
- [35] Sh. Rezapour, R.H. Haghi, N. Shahzad, *Some notes on fixed points of quasi-contraction maps*, Appl. Math. Lett. **23** (2010), 498–502.
- [36] S. Rezapour, R. Hambarani, *Some notes on the paper “Cone metric spaces and fixed point theorems of contractive mappings”*, J. Math. Anal. Appl. **345** (2008), 719–724.
- [37] Sh. Rezapour, H. Khandani, S.M. Vaezpour, *Efficacy of cones on topological vector spaces and application to common fixed points of multifunctions*, Rend. Circ. Mat. Palermo **59** (2010), 185–197.
- [38] Sh. Rezapour, R.H. Haghi, *Two results about fixed point of multifunctions*, Bull. Iranian Math. Soc. **36** (2010), 279–287.
- [39] Sh. Rezapour, R.H. Haghi, *Fixed point of multifunctions on cone metric spaces*, Numer. Funct. Anal. Opt. **30** (2009), 825–832.
- [40] F. Sabetghadam, H.P. Masiha, A.H. Sanatpour, *Some coupled fixed point theorems in cone metric space*, Fixed Point Theory Appl. **2009** (2009), Article ID 125426, 8 pp.
- [41] B. Samet, *Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces*, Nonlinear Anal. **72** (2010), 4508–4517.
- [42] W. Shatanawi, *Partially ordered cone metric spaces and coupled fixed point results*, Comput. Math. Appl. **60** (2010), 2508–2515.
- [43] Wei-Shih Du, *A note on cone metric fixed point theory and its equivalence*, Nonlinear Anal. **72** (2009), 2259–2261.

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