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SIGNED TOTAL DISTANCE k-DOMATIC NUMBERS OF GRAPHS

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Abstract. In this paper we initiate the study of signed total distance k-domatic numbers in graphs and we present its sharp upper bounds.

1. Introduction

In this paper, k is a positive integer, and G is a finite simple graph without isolated vertices and with vertex set $V = V(G)$ and edge set $E = E(G)$. For a vertex $v \in V(G)$, the *open k-neighborhood* $N_{k,G}(v)$ is the set $\{u \in V(G) \mid u \neq v \text{ and } d(u,v) \leq k\}$. The *open k-neighborhood* $N_{k,G}(S)$ of a set $S \subseteq V$ is the set $\bigcup_{v\in S} N_{k,G}(v)$. The *$k$-degree* of a vertex v is defined as $\deg_{k,G}(v) = |N_{k,G}(v)|$. The minimum and maximum k-degree of a graph G are denoted by $\delta_k(G)$ and $\Delta_k(G)$, respectively. If $\delta_k(G) = \Delta_k(G)$, then the graph G is called *distance-k-regular*. The *k-th power* G^k of a graph G is the graph with vertex set $V(G)$ where two different vertices u and v are adjacent if and only if the distance $d(u,v)$ is at most k in G. Now we observe that $N_{k,G}(v) = N_{1,G^k}(v) = N_{G^k}(v)$, $\deg_{k,G}(v) = \deg_{1,G^k}(v) = \deg_{G^k}(v)$, $\delta_k(G) = \delta_1(G^k) = \delta(G^k)$ and $\Delta_k(G) = \Delta_1(G^k) = \Delta(G^k)$. If $k = 1$, then we also write $\deg_G(v)$, $N_G(v)$, $\delta(G)$ for $\deg_{1,G}(v)$, $N_{1,G}(v)$, $\delta_1(G)$ etc. Consult [7] for the notation and terminology which are not defined here.

For a real-valued function $f : V(G) \longrightarrow \mathbb{R}$, the weight of f is $w(f) = \sum_{v\in V} f(v)$. For $S \subseteq V$, we define $f(S) = \sum_{v\in S} f(v)$. So $w(f) = f(V)$. A *signed total distance k-dominating function* (STDkD function) is a function $f : V(G) \to \{-1, 1\}$ satisfying $\sum_{u\in N_{k,G}(v)} f(u) \geq 1$ for every $v \in V(G)$. The minimum of the values of $\sum_{v\in V(G)} f(v)$ taken over all signed total distance k-dominating functions f is called the *signed total distance k-domination number* and is denoted by $\gamma^t_{k,s}(G)$. Then the function assigning $+1$ to every vertex of G is a STDkD function, called the function ϵ, of weight n. Thus $\gamma^t_{k,s}(G) \leq n$ for every graph of order n. Moreover, the weight of every STDkD function different from ϵ

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is at most $n-2$ and more generally, $\gamma^t_{k,s}(G) \equiv n \pmod 2$. Hence $\gamma^t_{k,s}(G) = n$ if and only if ϵ is the unique STDkD function of G. In the special case when $k = 1$, $\gamma^t_{k,s}(G)$ is the signed total domination number $\gamma^t_s(G)$ investigated in [8] and has been studied by several authors (see for example [2]). The signed total distance 2-domination number of graphs was introduced by Zelinka [9]. By these definitions, we easily obtain

$$\gamma^t_{k,s}(G) = \gamma^t_s(G^k). \tag{1}$$

A set $\{f_1, f_2, \ldots, f_d\}$ of signed total distance k-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq 1$ for each $v \in V(G)$, is called a *signed total distance k-dominating family* on G. The maximum number of functions in a signed total distance k-dominating family on G is the *signed total distance k-domatic number* of G, denoted by $d^t_{k,s}(G)$. The signed total distance k-domatic number is well-defined and $d^t_{k,s}(G) \geq 1$ for all graphs G, since the set consisting of any one STDkD function, for instance the function ϵ, forms a STDkD family of G. A *$d^t_{k,s}$-family* of a graph G is a STDkD family containing $d^t_{k,s}(G)$ STDkD functions. The signed total distance 1-domatic number $d^t_{1,s}(G)$ is the usual signed total domatic number $d^t_s(G)$ which was introduced by Henning in [3] and has been studied by several authors (see for example [5]). Obviously,

$$d^t_{k,s}(G) = d^t_s(G^k). \tag{2}$$

Observation 1. Let G be a graph of order n without isolated vertices. If $\gamma^t_{k,s}(G) = n$, then ϵ is the unique STDkD function of G and so $d^t_{k,s}(G) = 1$.

We first study basic properties and sharp upper bounds for the signed total distance k-domatic number of a graph. Some of them generalize the result obtained for the signed total domatic number.

In this paper we make use of the following results.

Proposition A. *Let G be a graph of order n and minimum degree $\delta(G) \geq 1$. Then $\gamma^t_s(G) = n$ if and only if for each $v \in V(G)$, there exists a vertex $u \in N_G(v)$ such that* $\deg_G(u) = 1$ *or* $\deg_G(u) = 2$.

Proof. Assume that $\gamma^t_s(G) = n$ and there exists a vertex v every neighbor of which has degree at least 3. Then the function f that assigns to v the value -1 and to all other vertices the value 1 is a signed total dominating function of G. This leads to the contradiction $\gamma^t_s(G) \leq n-2$. Hence at least one neighbor of v is of degree 1 or 2. On the other hand, if every vertex of v has a neighbor of degree 1 or 2, then ϵ is the unique signed total dominating function of G, and so $\gamma^t_s(G) = n$. ■

The special case of Proposition A that G is a tree can be found in [2], the proof is almost the same.

Proposition B. [3] *The signed total domatic number $d^t_s(G)$ of a graph G, without isolated vertex, is an odd integer.*

Proposition C. [3] *If G is a graph without isolated vertices, then $1 \leq d^t_s(G) \leq \delta(G)$.*

Proposition D. [4, 6] *Let G be a graph with $\delta(G) \geq 1$, and let v be a vertex of even degree $\deg_G(v) = 2t$ with an integer $t \geq 1$. Then $d_s^t(G) \leq t$ when t is odd and $d_s^t(G) \leq t-1$ when t is even.*

Proposition E. [3] *Let $k \geq 1$ be an integer, and let K_n be the complete graph of order n. For $n \geq 2$, we have*

$$\gamma_{k,s}^t(K_n) = \gamma_s^t(K_n) = \begin{cases} 3 & \text{if } n \equiv 1 \pmod 2 \\ 2 & \text{otherwise.} \end{cases} \tag{3}$$

Proposition F. [3] *If K_n is the complete graph of order $n \geq 2$, then*

$$d_s^t(K_n) = \begin{cases} \lfloor \frac{n+1}{3} \rfloor - \lceil \frac{n}{3} \rceil + \lfloor \frac{n}{3} \rfloor & \text{if } n \text{ is odd,} \\ \frac{n}{2} - \lceil \frac{n+2}{4} \rceil + \lfloor \frac{n+2}{4} \rfloor & \text{if } n \text{ is even.} \end{cases} \tag{4}$$

Since $N_{k,K_n}(v) = N_{K_n}(v)$ for each vertex $v \in V(K_n)$ and each positive integer k, each signed total dominating function of K_n is a signed total distance k-dominating function of K_n and vice versa. Using Proposition F, we obtain

$$d_{k,s}^t(K_n) = d_s^t(K_n) = \begin{cases} \lfloor \frac{n+1}{3} \rfloor - \lceil \frac{n}{3} \rceil + \lfloor \frac{n}{3} \rfloor & \text{if } n \text{ is odd,} \\ \frac{n}{2} - \lceil \frac{n+2}{4} \rceil + \lfloor \frac{n+2}{4} \rfloor & \text{if } n \text{ is even.} \end{cases}$$

More generally, the following result is valid.

Observation 2. Let $k \geq 1$ be an integer, and let G be a graph of order n without isolated vertices. If $\operatorname{diam}(G) \leq k$, then $\gamma_{k,s}^t(G) = \gamma_s^t(K_n)$ and $d_{k,s}^t(G) = d_s^t(K_n)$.

The next result is immediate by Observation 2, Propositions E and F.

Corollary 3. *If $k \geq 2$ is an integer, and G is a graph of order n with* $\operatorname{diam}(G) = 2$ *and $\delta(G) \geq 1$, then*

$$\gamma_{k,s}^t(G) = \begin{cases} 3 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even,} \end{cases}$$

and

$$d_{k,s}^t(G) = \begin{cases} \lfloor \frac{n+1}{3} \rfloor - \lceil \frac{n}{3} \rceil + \lfloor \frac{n}{3} \rfloor & \text{if } n \text{ is odd,} \\ \frac{n}{2} - \lceil \frac{n+2}{4} \rceil + \lfloor \frac{n+2}{4} \rfloor & \text{if } n \text{ is even.} \end{cases}$$

Corollary 4. *Let $k \geq 2$ be an integer, and let G be a graph of order n with $\delta(G) \geq 1$. If* $\operatorname{diam}(G) \neq 3$, *then $\gamma_{k,s}^t(G) = \gamma_s^t(K_n)$ and $d_{k,s}^t(G) = d_s^t(K_n)$ or $\gamma_{k,s}^t(\overline{G}) = \gamma_s^t(K_n)$ and $d_{k,s}^t(\overline{G}) = d_s^t(K_n)$.*

Proof. If $\operatorname{diam}(G) \leq 2$, then it follows from Observation 2 that $\gamma_{k,s}^t(G) = \gamma_s^t(K_n)$ and $d_{k,s}^t(G) = d_s^t(K_n)$. If $\operatorname{diam}(G) \geq 3$, then the hypothesis $\operatorname{diam}(G) \neq 3$ implies that $\operatorname{diam}(G) \geq 4$. Now, according to a result of Bondy and Murty [1, page 14], we deduce that $\operatorname{diam}(\overline{G}) \leq 2$. Applying again Observation 2, we obtain $\gamma_{k,s}^t(\overline{G}) = \gamma_s^t(K_n)$ and $d_{k,s}^t(\overline{G}) = d_s^t(K_n)$. ∎

Corollary 5. *If $k \geq 3$ is an integer and G a graph of order n with $\delta(G) \geq 1$, then $\gamma^t_{k,s}(G) = \gamma^t_s(K_n)$ and $d^t_{k,s}(G) = d^t_s(K_n)$ or $\gamma^t_{k,s}(\overline{G}) = \gamma^t_s(K_n)$ and $d^t_{k,s}(\overline{G}) = d^t_s(K_n)$.*

Proposition 6. *Let $k \geq 1$ be an integer, and let G be a graph of order n and minimum degree $\delta(G) \geq 1$.*

If $k = 1$, then $\gamma^t_{k,s}(G) = n$ if and only if for each $v \in V(G)$, there exists a vertex $u \in N_G(v)$ such that $\deg_G(u) = 1$ or $\deg_G(u) = 2$.

If $k \geq 2$, then $\gamma^t_{k,s}(G) = n$ if and only if all components of G are of order 2 or 3.

Proof. In the case $k = 1$, Proposition A implies the desired result.

Assume now that $k \geq 2$. If all components of G are of order 2 or 3, then it is easy to see that ϵ is the unique STDkD function of G and thus $\gamma^t_{k,s}(G) = n$.

Conversely, assume that $\gamma^t_{k,s}(G) = n$. Suppose to the contrary that G has a component G_1 of order $n(G_1) \geq 4$. If $\operatorname{diam}(G_1) \geq 3$, then assume that $x_1x_2\ldots x_m$ is a longest path in G_1. It is straightforward to verify that the function $f : V(G) \to \{-1, 1\}$ defined by $f(x_1) = -1$ and $f(x) = 1$ otherwise is a signed total distance k-dominating function of G which is a contradiction. If $\operatorname{diam}(G_1) \leq 2$, then Proposition E, Observation 2 and Corollary 3 show that $\gamma^t_{k,s}(G_1) \leq 3 < 4 \leq n(G_1)$ and consequently $\gamma^t_{k,s}(G) < n$. This contradiction completes the proof. ■

2. Basic properties of the signed total distance *k*-domatic number

In this section we present basic properties of $d^t_{k,s}(G)$ and sharp bounds on the signed total distance k-domatic number of a graph.

Proposition 7. *Let G be a graph with $\delta(G) \geq 1$. The signed total distance k-domatic number of G is an odd integer.*

Proof. According to the identity (2), we have $d^t_{k,s}(G) = d^t_s(G^k)$. In view of Proposition B, $d^t_s(G^k)$ is odd and thus $d^t_{k,s}(G)$ is odd, and the proof is complete. ■

Theorem 8. *If G is a graph with $\delta(G) \geq 1$, then*

$$1 \leq d^t_{k,s}(G) \leq \delta_k(G).$$

Moreover, if $d^t_{k,s}(G) = \delta_k(G)$, then for each function of any $d^t_{k,s}-$family $\{f_1, f_2, \cdots, f_d\}$ and for all vertices v of minimum k-degree $\delta_k(G)$, $\sum_{u\in N_{k,G}(v)} f_i(u) = 1$ and $\sum_{i=1}^{d} f_i(u) = 1$ for every $u \in N_{k,G}(v)$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a STDkD family of G such that $d = d^t_{k,s}(G)$, and let v be a vertex of minimum k-degree $\delta_k(G)$. Then $|N_{k,G}(v)| = \delta_k(G)$ and

$$\begin{aligned} 1 \leq d = \sum_{i=1}^{d} 1 &\leq \sum_{i=1}^{d} \sum\nolimits_{u\in N_{k,G}(v)} f_i(u) \\ &= \sum_{u\in N_{k,G}(v)} \sum_{i=1}^{d} f_i(u) \leq \sum_{u\in N_{k,G}(v)} 1 = \delta_k(G). \end{aligned}$$

If $d^t_{k,s}(G) = \delta_k(G)$, then the two inequalities occurring in the proof become equalities, which gives the two properties given in the statement. ∎

THEOREM 9. *Let $k \geq 1$ be an integer, and let G be a graph with $\delta(G) \geq 1$. If G contains a vertex v of even k-degree $\deg_{k,G}(v) = 2t$ with an integer $t \geq 1$, then $d^t_{k,s}(G) \leq t$ when t is odd and $d^t_{k,s}(G) \leq t-1$ when t is even.*

Proof. Since $\deg_{k,G}(v) = \deg_{G^k}(v) = 2t$, Proposition D and (2) imply that $d^t_{k,s}(G) = d^t_s(G^k) \leq t$ when t is odd and $d^t_{k,s}(G) = d^t_s(G^k) \leq t-1$ when t is even. ∎

Restricting our attention to graphs G of even minimum k-degree, Theorem 9 leads to a considerable improvement of the upper bound of $d^t_{k,s}(G)$ given in Theorem 8.

COROLLARY 10. *If $k \geq 1$ is an integer, and G is a graph of even minimum k-degree $\delta_k(G) \geq 1$, then $d^t_{k,s}(G) \leq \delta_k(G)/2$ when $\delta_k(G) \equiv 2 \pmod 4$ and $d^t_{k,s}(G) \leq \delta_k(G)/2 - 1$ when $\delta_k(G) \equiv 0 \pmod 4$.*

THEOREM 11. *Let G be a graph of order n with signed total distance k-domination number $\gamma^t_{k,s}(G)$ and signed total distance k-domatic number $d^t_{k,s}(G)$. Then*

$$\gamma^t_{k,s}(G) \cdot d^t_{k,s}(G) \leq n.$$

Moreover, if $\gamma^t_{k,s}(G) \cdot d^t_{k,s}(G) = n$, then for each STDkD family $\{f_1, f_2, \cdots, f_d\}$ on G with $d = d^t_{k,s}(G)$, each function f_i is a $\gamma^t_{k,s}$-function and $\sum_{i=1}^d f_i(v) = 1$ for all $v \in V$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a STDkD family on G such that $d = d^t_{k,s}(G)$ and let $v \in V$. Then

$$\begin{aligned} d \cdot \gamma^t_{k,s}(G) &= \sum_{i=1}^{d} \gamma^t_{k,s}(G) \leq \sum_{i=1}^{d} \textstyle\sum_{v\in V} f_i(v) \\ &= \sum_{v\in V} \sum_{i=1}^{d} f_i(v) \leq \sum_{v\in V} 1 = n. \end{aligned}$$

If $\gamma^t_{k,s}(G) \cdot d^t_{k,s}(G) = n$, then the two inequalities occurring in the proof become equalities. Hence for the $d^t_{k,s}$-family $\{f_1, f_2, \cdots, f_d\}$ on G and for each i, $\sum_{v\in V} f_i(v) = \gamma^t_{k,s}(G)$, thus each function f_i is a $\gamma^t_{k,s}$-function, and $\sum_{i=1}^d f_i(v) = 1$ for all v. ∎

The next corollary is a consequence of Theorem 11 and Proposition 7, and it improves Observation 1.

COROLLARY 12. *If $\gamma^t_{k,s}(G) > \frac{n}{3}$, then $d^t_{k,s}(G) = 1$.*

The upper bound on the product $\gamma^t_{k,s}(G) \cdot d^t_{k,s}(G)$ leads to a bound on the sum.

THEOREM 13. *If G is a graph of order n with minimum degree $\delta(G) \geq 1$, then*

$$\gamma^t_{k,s}(G) + d^t_{k,s}(G) \leq n+1,$$

with equality if and only if $d^t_{k,s}(G) = 1$ and $\gamma^t_{k,s}(G) = n$.

Proof. According to Theorem 11, we obtain

$$\gamma_{k,s}^t(G) + d_{k,s}^t(G) \le \frac{n}{d_{k,s}^t(G)} + d_{k,s}^t(G). \tag{6}$$

In view of Theorem 8, we have $1 \le d_{k,s}^t(G) \le n$. Using theses inequalities, and the fact that the function $g(x) = x + n/x$ is decreasing for $1 \le x \le \sqrt{n}$ and increasing for $\sqrt{n} \le x \le n$ inequality (6) leads to

$$\gamma_{k,s}^t(G) + d_{k,s}^t(G) \le \frac{n}{d_{k,s}^t(G)} + d_{k,s}^t(G) \le \max\{g(1), g(n)\} = n + 1,$$

and the desired bound is proved.

If $d_{k,s}^t(G) = 1$ and $\gamma_{k,s}^t(G) = n$, then obviously $\gamma_{k,s}^t(G) + d_{k,s}^t(G) = n + 1$.

Conversely, assume that $\gamma_{k,s}^t(G) + d_{k,s}^t(G) = n + 1$. In view of Theorem 8, we observe that $d_{k,s}^t(G) \le \delta_k(G) \le n - 1$. If $n = 2$, then we deduce that $d_{k,s}^t(G) = 1$. If we assume in the case $n \ge 3$ that $2 \le d_{k,s}^t(G)$, then we obtain as above that

$$\begin{aligned}\gamma_{k,s}^t(G) + d_{k,s}^t(G) &\le \frac{n}{d_{k,s}^t(G)} + d_{k,s}^t(G) \le \max\{g(2), g(n-1)\} \\ &= \max\left\{\frac{n}{2} + 2, \frac{n}{n-1} + n - 1\right\} < n + 1,\end{aligned}$$

a contradiction to the assumption $\gamma_{k,s}^t(G) + d_{k,s}^t(G) = n + 1$. It follows that $d_{k,s}^t(G) = 1$ in each case and hence $\gamma_{k,s}^t(G) = n$. This completes the proof. ∎

Corollary 14. *Let $k \ge 1$ be an integer, and let G be a graph of order n and minimum degree $\delta(G) \ge 1$.*

If $k = 1$, then $\gamma_{k,s}^t(G) + d_{k,s}^t(G) = n+1$ if and only if for each $v \in V(G)$, there exists a vertex $u \in N_G(v)$ such that $\deg_G(u) = 1$ or $\deg_G(u) = 2$.

If $k \ge 2$, then $\gamma_{k,s}^t(G) + d_{k,s}^t(G) = n+1$ if and only if all components of G are of order 2 or 3.

Proof. If $k = 1$ and for each $v \in V(G)$, there exists a vertex $u \in N_G(v)$ such that $\deg_G(u) = 1$ or $\deg_G(u) = 2$, then Proposition A yields $\gamma_{k,s}^t(G) = n$. Thus, by Observation 1, $d_{k,s}^t(G) = 1$ and so $\gamma_{k,s}^t(G) + d_{k,s}^t(G) = n + 1$. If $k \ge 2$ and all components of G are of order 2 or 3, then it follows from Proposition 6 that $\gamma_{k,s}^t(G) = n$ and therefore $\gamma_{k,s}^t(G) + d_{k,s}^t(G) = n + 1$.

Conversely, assume that $\gamma_{k,s}^t(G) + d_{k,s}^t(G) = n + 1$. Theorem 13 implies that $d_{k,s}^t(G) = 1$ and hence $\gamma_{k,s}^t(G) = n$. Now Proposition 6 leads to the desired result, and the proof is complete. ∎

If $2 \le d_{k,s}^t(G)$, then Theorem 13 shows that $\gamma_{k,s}^t(G) + d_{k,s}^t(G) \le n$. In the next corollary we will improve this bound slightly.

Corollary 15. *Let G be a graph of order $n \ge 3$ with $\delta(G) \ge 1$. If $2 \le d_{k,s}^t(G)$, then*

$$\gamma_{k,s}^t(G) + d_{k,s}^t(G) \le n - 1.$$

Proof. Since $d^t_{k,s}(G) \geq 2$, Theorem 13 implies that $\gamma^t_{k,s}(G) + d^t_{k,s}(G) \leq n$. Now suppose to the contrary that $\gamma^t_{k,s}(G) + d^t_{k,s}(G) = n$. It follows from Theorem 7 that $d^t_{k,s}(G)$ is odd, a contradiction to the fact that, as seen in the introduction, $\gamma^t_{k,s}(G) \equiv n \pmod 2$. ■

COROLLARY 16. *Let G be a graph of order n with $\delta(G) \geq 1$, and let $k \geq 1$ be an integer. If $\min\{\gamma^t_{k,s}(G), d^t_{k,s}(G)\} \geq a$, with $2 \leq a \leq \sqrt{n}$, then*

$$\gamma^t_{k,s}(G) + d^t_{k,s}(G) \leq a + \frac{n}{a}.$$

Proof. Since $\min\{\gamma^t_{k,s}(G), d^t_{k,s}(G)\} \geq a \geq 2$, it follows from Theorem 11 that $a \leq d^t_{k,s}(G) \leq \frac{n}{a}$. Applying the inequality (6), we obtain

$$\gamma^t_{k,s}(G) + d^t_{k,s}(G) \leq d^t_{k,s}(G) + \frac{n}{d^t_{k,s}(G)}.$$

The bound results from the facts that the function $g(x) = x + n/x$ is decreasing for $1 \leq x \leq \sqrt{n}$ and increasing for $\sqrt{n} \leq x \leq n$. ■

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