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ON A NEW CLASS OF HARMONIC UNIVALENT FUNCTIONS

Waggas Galib Atshan and Abbas Kareem Wanas

Abstract. We define a new class of harmonic univalent functions of the form $f = h + \overline{g}$ in the open unit disk U. Also we study some properties of this class, like coefficient bounds, extreme points, convex combination, distortion bounds, integral operator and convolution property.

1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain $\mathbb{C}$, if both u and v are real harmonic in $\mathbb{C}$. In any simply connected domain $D \subset \mathbb{C}$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ in D (see Clunie and Sheil-Small [2]).

Denote by $M_{\mathcal{H}}$ the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense preserving in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. So $f = h + \overline{g} \in M_{\mathcal{H}}$ is normalized by $f(0) = f_z(0) - 1 = 0$. For $f = h + \overline{g} \in M_{\mathcal{H}}$, we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ g(z) = \sum_{n=1}^{\infty} b_n z^n,\ |b_1| < 1. \tag{1.1}$$

Also denote by $W_{\mathcal{H}}$ the subclass of $M_{\mathcal{H}}$ containing all functions $f = h + \overline{g}$ where h and g are given by

$$h(z) = z - \sum_{n=2}^{\infty} a_n z^n,\ g(z) = -\sum_{n=1}^{\infty} b_n z^n,\ (a_n \ge 0, b_n \ge 0, |b_1| < 1). \tag{1.2}$$

We denote by $KM_{\mathcal{H}}(\gamma, \alpha, \beta)$ the class of all functions of the form (1.1) that satisfy the condition

$$\operatorname{Re}\left\{\frac{zf''(z) + f'(z)}{f'(z) + \gamma z f''(z)}\right\} > \beta \left|\frac{zf''(z) + f'(z)}{f'(z) + \gamma z f''(z)} - 1\right| + \alpha, \tag{1.3}$$

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where $0 \leq \alpha < 1, \beta \geq 0, 0 \leq \gamma < 1$ and $z \in U$.

Let $KW_{\mathcal{H}}(\gamma, \alpha, \beta)$ be the subclass of $KM_{\mathcal{H}}(\gamma, \alpha, \beta)$, where

$$KW_{\mathcal{H}}(\gamma, \alpha, \beta) = W_{\mathcal{H}} \cap KM_{\mathcal{H}}(\gamma, \alpha, \beta).$$

When $\beta = 0$, the class reduces to $KW_{\mathcal{H}}(\gamma, \alpha, 0) = C(\gamma, \alpha)$, which was studied by Mostafa [6] for analytic part.

Such type of study was carried out by various authors for another classes, like Frasin [4], Yalcin [9], Gencel and Yalcin [5], Cotirla [3], Porwal et al. [7] and Seker [8].

2. Coefficient bounds

First we determine the sufficient condition for $f = h + \overline{g}$ to be in the class $KM_{\mathcal{H}}(\gamma, \alpha, \beta)$.

THEOREM 2.1. *Let $h + \overline{g}$ with h and g given by (1.1). If*

$$\sum_{n=2}^{\infty} n\{(n-1)[\beta(1-\gamma) - \alpha\gamma] + n - \alpha\}(|a_n| + |b_n|) \leq (1-\alpha)(1-|b_1|), \tag{2.1}$$

where $0 \leq \alpha < 1, \beta \geq 0, 0 \leq \gamma < 1$, then f is harmonic univalent in U and $f \in KM_{\mathcal{H}}(\gamma, \alpha, \beta)$.

Proof. For proving $f \in KM_{\mathcal{H}}(\gamma, \alpha, \beta)$, we must show that (1.3) holds true. It is sufficient to show that

$$\operatorname{Re}\left\{\frac{zf''(z) + f'(z)}{f'(z) + \gamma z f''(z)}(1 + \beta e^{i\theta}) - \beta e^{i\theta}\right\} > \alpha \quad (-\pi \leq \theta \leq \pi),$$

or equivalently

$$\operatorname{Re}\left\{\frac{(1 + \beta e^{i\theta})(zf''(z) + f'(z)) - \beta e^{i\theta}(f'(z) + \gamma z f''(z))}{f'(z) + \gamma z f''(z)}\right\} > \alpha. \tag{2.2}$$

If we put

$$A(z) = (1 + \beta e^{i\theta})(zf''(z) + f'(z)) - \beta e^{i\theta}(f'(z) + \gamma z f''(z))$$

and $B(z) = f'(z) + \gamma z f''(z)$, we only need to prove that

$$|A(z) + (1-\alpha)B(z)| - |A(z) - (1+\alpha)B(z)| \geq 0.$$

But

$$\begin{aligned} |A(z) + (1-\alpha)B(z)| = \Big|&(1 + \beta e^{i\theta})\Big(\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1} + \sum_{n=1}^{\infty} n(n-1)b_n(\overline{z})^{n-1} \\ &+ 1 + \sum_{n=2}^{\infty} n a_n z^{n-1} + \sum_{n=1}^{\infty} n b_n(\overline{z})^{n-1}\Big) - \beta e^{i\theta}\Big(1 + \sum_{n=2}^{\infty} n a_n z^{n-1} \\ &+ \sum_{n=1}^{\infty} n b_n(\overline{z})^{n-1} + \sum_{n=2}^{\infty} \gamma n(n-1)a_n z^{n-1} + \sum_{n=1}^{\infty} \gamma n(n-1)b_n(\overline{z})^{n-1}\Big) \\ &+ (1-\alpha)\Big(1 + \sum_{n=2}^{\infty} n a_n z^{n-1} + \sum_{n=1}^{\infty} n b_n(\overline{z})^{n-1} + \sum_{n=2}^{\infty} \gamma n(n-1)a_n z^{n-1} \end{aligned}$$

$$+\sum_{n=1}^{\infty}\gamma n(n-1)b_n(\overline{z})^{n-1}\Big)\Big|$$

$$=\Big|(2-\alpha)+\sum_{n=2}^{\infty}n\left((n-1)(\beta e^{i\theta}(1-\gamma)+(1-\alpha)\gamma)+n+1-\alpha\right)a_n z^{n-1}$$

$$+\sum_{n=1}^{\infty}n\left((n-1)(\beta e^{i\theta}(1-\gamma)+(1-\alpha)\gamma)+n+1-\alpha\right)b_n(\overline{z})^{n-1}\Big|.$$

Also

$$|A(z)-(1+\alpha)B(z)|=\Big|(1+\beta e^{i\theta})\Big(\sum_{n=2}^{\infty}n(n-1)a_n z^{n-1}+\sum_{n=1}^{\infty}n(n-1)b_n(\overline{z})^{n-1}$$

$$+1+\sum_{n=2}^{\infty}na_n z^{n-1}+\sum_{n=1}^{\infty}nb_n(\overline{z})^{n-1}\Big)-\beta e^{i\theta}\Big(1+\sum_{n=2}^{\infty}na_n z^{n-1}$$

$$+\sum_{n=1}^{\infty}nb_n(\overline{z})^{n-1}+\sum_{n=2}^{\infty}\gamma n(n-1)a_n z^{n-1}+\sum_{n=1}^{\infty}\gamma n(n-1)b_n(\overline{z})^{n-1}\Big)$$

$$-(1+\alpha)\Big(1+\sum_{n=2}^{\infty}na_n z^{n-1}+\sum_{n=1}^{\infty}nb_n(\overline{z})^{n-1}+\sum_{n=2}^{\infty}\gamma n(n-1)a_n z^{n-1}$$

$$+\sum_{n=1}^{\infty}\gamma n(n-1)b_n(\overline{z})^{n-1}\Big)\Big|$$

$$=\Big|-\alpha+\sum_{n=2}^{\infty}n\left((n-1)(\beta e^{i\theta}(1-\gamma)-(1+\alpha)\gamma)+n-(1+\alpha)\right)a_n z^{n-1}$$

$$+\sum_{n=1}^{\infty}n\left((n-1)(\beta e^{i\theta}(1-\gamma)-(1+\alpha)\gamma)+n-(1+\alpha)\right)b_n(\overline{z})^{n-1}\Big|.$$

Then

$$|A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)|$$

$$\geq 2(1-\alpha)-\sum_{n=2}^{\infty}2n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)|a_n||z|^{n-1}$$

$$-\sum_{n=1}^{\infty}2n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)|b_n||z|^{n-1}$$

$$>2\Big\{(1-\alpha)-\sum_{n=2}^{\infty}n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)|a_n|$$

$$-\sum_{n=1}^{\infty}n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)|b_n|\Big\}>0.$$

The harmonic univalent function

$$f(z)=z+\sum_{n=2}^{\infty}\frac{x_n}{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)}z^n$$

$$+\sum_{n=1}^{\infty}\frac{\overline{y}_n}{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)}(\overline{z})^n, \tag{2.3}$$

where $\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=1-\alpha$, show that the coefficient bound given by (2.1) is sharp. The functions of the form (2.3) are in the class $KM_{\mathcal{H}}(\gamma,\alpha,\beta)$, because

$$\sum_{n=2}^{\infty}n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)\frac{|x_n|}{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)}$$

$$+\sum_{n=1}^{\infty} n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)\frac{|y_n|}{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)}$$
$$=\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=1-\alpha.$$

The restriction placed in Theorem 2.1 on the moduli of the coefficients of $f=h+\overline{g}$ enables us to conclude for arbitrary rotation of the coefficients of f that the resulting functions would still be harmonic univalent and $f\in KM_{\mathcal{H}}(\gamma,\alpha,\beta)$. ■

In the following theorem, it is shown that the condition (2.1) is also necessary for functions in $KW_{\mathcal{H}}(\gamma,\alpha,\beta)$.

THEOREM 2.2. *Let $f=h+\overline{g}$ with h and g be given by* (1.2). *Then $f\in KW_{\mathcal{H}}(\gamma,\alpha,\beta)$ if and only if*

$$\sum_{n=2}^{\infty} n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)a_n + \sum_{n=1}^{\infty} n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)b_n\le 1-\alpha \quad (2.4)$$

where $0\le\alpha<1$, $\beta\ge 0$, $0\le\gamma<1$.

Proof. Since $KW_{\mathcal{H}}(\gamma,\alpha,\beta)\subset KM_{\mathcal{H}}(\gamma,\alpha,\beta)$, we only need to prove the "only if" part of the theorem. Assume that $f\in KW_{\mathcal{H}}(\gamma,\alpha,\beta)$. Then by (1.3), we have

$$\operatorname{Re}\left\{\frac{zf''(z)+f'(z)}{f'(z)+\gamma zf''(z)}(1+\beta e^{i\theta})-\beta e^{i\theta}\right\}>\alpha.$$

This is equivalent to

$$\operatorname{Re}\left\{\frac{(1+\beta e^{i\theta})(zf''(z)+f'(z))-\beta e^{i\theta}(f'(z)+\gamma zf''(z))}{f'(z)+\gamma zf''(z)}\right\}$$
$$=\operatorname{Re}\Big\{(1-\alpha)-\sum_{n=2}^{\infty} n((n-1)(\beta e^{i\theta}(1-\gamma)-\alpha\gamma)+n-\alpha)a_n z^{n-1}$$
$$-\sum_{n=1}^{\infty} n((n-1)(\beta e^{i\theta}(1-\gamma)-\alpha\gamma)+n-\alpha)b_n(\overline{z})^{n-1}\Big\}$$
$$\times\Big\{1-\sum_{n=2}^{\infty} n(1+\gamma(n-1))a_n z^{n-1}-\sum_{n=1}^{\infty} n(1+\gamma(n-1))b_n(\overline{z})^{n-1}\Big\}^{-1}\ge 0. \quad (2.5)$$

The above required condition (2.5) must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0\le z=r<1$, we must have

$$\operatorname{Re}\Big\{(1-\alpha)-\Big[\sum_{n=2}^{\infty}(n(n-\alpha)-n(n-1)\alpha\gamma)a_n r^{n-1}+$$
$$\sum_{n=1}^{\infty}(n(n-\alpha)-n(n-1)\alpha\gamma)b_n r^{n-1}\Big]$$
$$-\beta e^{i\theta}\Big[\sum_{n=2}^{\infty} n(n-1)(1-\gamma)a_n r^{n-1}+\sum_{n=1}^{\infty} n(n-1)(1-\gamma)b_n r^{n-1}\Big]\Big\}$$
$$\times\Big\{1-\sum_{n=2}^{\infty} n(1+\gamma(n-1))a_n r^{n-1}-\sum_{n=1}^{\infty} n(1+\gamma(n-1))b_n r^{n-1}\Big\}^{-1}\ge 0.$$

Since $\operatorname{Re}(-e^{i\theta}) \geq -|e^{i\theta}| = -1$, and letting $r \to 1^-$, the above inequality reduces to

$$\frac{1-\alpha-\sum\limits_{n=2}^{\infty} n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)a_n}{1-\sum\limits_{n=2}^{\infty} n(1+\gamma(n-1))a_n-\sum\limits_{n=1}^{\infty} n(1+\gamma(n-1))b_n} - \frac{\sum\limits_{n=1}^{\infty} n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)b_n}{1-\sum\limits_{n=2}^{\infty} n(1+\gamma(n-1))a_n-\sum\limits_{n=1}^{\infty} n(1+\gamma(n-1))b_n} \geq 0.$$

This gives (2.4) and the proof is complete. ■

3. Extreme points

THEOREM 3.1. *Let f be given by (1.2). Then $f \in KW_{\mathcal{H}}(\gamma,\alpha,\beta)$ if and only if f can be expressed as*

$$f(z) = \sum_{n=1}^{\infty} (\mu_n h_n(z) + \delta_n g_n(z)),\ (z \in U) \tag{3.1}$$

where $h_1(z) = z$,

$$h_n(z) = z - \frac{1-\alpha}{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)} z^n,\ (n = 2,3,\cdots)$$

and

$$g_n(z) = z - \frac{1-\alpha}{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)} (\overline{z})^n,\ (n = 1,2,\cdots),$$

$\sum_{n=1}^{\infty}(\mu_n+\delta_n) = 1,\ (\mu_n \geq 0, \delta_n \geq 0)$.

In particular, the extreme points of $KW_{\mathcal{H}}(\gamma,\alpha,\beta)$ are $\{h_n\}$ and $\{g_n\}$.

Proof. Assume that f can be expressed by (3.1). Then, we have

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} (\mu_n h_n(z) + \delta_n g_n(z)) \\ &= \sum_{n=1}^{\infty} (\mu_n+\delta_n) z - \sum_{n=2}^{\infty} \frac{1-\alpha}{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)} \mu_n z^n \\ &\quad - \sum_{n=1}^{\infty} \frac{1-\alpha}{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)} \delta_n (\overline{z})^n \\ &= z - \sum_{n=2}^{\infty} \frac{1-\alpha}{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)} \mu_n z^n \\ &\quad - \sum_{n=1}^{\infty} \frac{1-\alpha}{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)} \delta_n (\overline{z})^n. \end{aligned}$$

Therefore

$$\sum_{n=2}^{\infty} n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)\frac{1-\alpha}{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)}\mu_n$$
$$+\sum_{n=1}^{\infty} n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)\frac{1-\alpha}{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)}\delta_n$$
$$=(1-\alpha)\left(\sum_{n=1}^{\infty}(\mu_n+\delta_n)-\mu_1\right)=(1-\alpha)(1-\mu_1)\leq 1-\alpha.$$

So $f\in KW_{\mathcal{H}}(\gamma,\alpha,\beta)$.

Conversely, let $f\in KW_{\mathcal{H}}(\gamma,\alpha,\beta)$, by putting

$$\mu_n=\frac{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)}{1-\alpha}a_n,\ (n=2,3,\cdots)$$

and

$$\delta_n=\frac{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)}{1-\alpha}b_n,\ (n=1,2,\cdots).$$

We define $\mu_1=1-\sum\limits_{n=2}^{\infty}\mu_n-\sum_{n=1}^{\infty}\delta_n$.

Then note that $0\leq\mu_n\leq 1$ $(n=2,3,\cdots)$, $0\leq\delta_n\leq 1E$ $(n=1,2,\cdots)$. Hence

$$\begin{aligned}
f(z)&=z-\sum_{n=2}^{\infty}a_nz^n-\sum_{n=1}^{\infty}b_n(\overline{z})^n\\
&=z-\sum_{n=2}^{\infty}\frac{1-\alpha}{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)}\mu_nz^n\\
&\quad-\sum_{n=1}^{\infty}\frac{1-\alpha}{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)}\delta_n(\overline{z})^n\\
&=z-\sum_{n=2}^{\infty}(z-h_n(z))\mu_n-\sum_{n=1}^{\infty}(z-g_n(z))\delta_n\\
&=\left(1-\sum_{n=2}^{\infty}\mu_n-\sum_{n=2}^{\infty}\delta_n\right)z+\sum_{n=2}^{\infty}\mu_nh_n(z)+\sum_{n=1}^{\infty}\delta_ng_n(z)\\
&=\mu_1h_1(z)+\sum_{n=2}^{\infty}\mu_nh_n(z)+\sum_{n=1}^{\infty}\delta_ng_n(z)=\sum_{n=1}^{\infty}(\mu_nh_n(z)+\delta_ng_n(z)),
\end{aligned}$$

that is the required representation. ■

4. Convex combination

THEOREM 4.1. *The class $KW_{\mathcal{H}}(\gamma,\alpha,\beta)$ is closed under convex combinations.*

Proof. For $j=1,2,3,\cdots$, let $f_j\in KW_{\mathcal{H}}(\gamma,\alpha,\beta)$, where f_j is given by

$$f_j(z)=z-\sum_{n=2}^{\infty}\alpha_{n,j}z^n-\sum_{n=1}^{\infty}b_{n,j}(\overline{z})^n.$$

Then by (2.4), we have

$$\sum_{n=2}^{\infty} n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)\alpha_{n,j}$$
$$+\sum_{n=1}^{\infty} n((b-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)b_{n,j}\leq 1-\alpha. \quad (4.1)$$

For $\sum_{j=1}^{\infty} t_j = 1, 0 \leq t_j \leq 1$, the convex combination of f_j's may be written as

$$\sum_{j=1}^{\infty} t_j f_j(z) = z - \sum_{n=2}^{\infty} \Big(\sum_{j=1}^{\infty} t_j a_{n,j} \Big) z^n - \sum_{n=1}^{\infty} \Big(\sum_{j=1}^{\infty} t_j b_{n,j} \Big) (\overline{z})^n.$$

Then by (4.1), we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)\Big(\sum_{j=1}^{\infty} t_j a_{n,j} \Big) \\
&\qquad + \sum_{n=1}^{\infty} n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)\Big(\sum_{j=1}^{\infty} t_j b_{n,j} \Big) \\
&\quad = \sum_{j=1}^{\infty} t_j \Big\{ \sum_{n=2}^{\infty} n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha) a_{n,j} \\
&\qquad + \sum_{n=1}^{\infty} n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha) b_{n,j} \Big\} \leq \sum_{j=1}^{\infty} t_j (1-\alpha) = 1-\alpha.
\end{aligned}$$

Therefore

$$\sum_{j=1}^{\infty} t_j f_j(z) \in KW_{\mathcal{H}}(\gamma, \alpha, \beta).$$

This completes the proof. ∎

5. Distortion bounds

THEOREM 5.1. *Let $f \in KW_{\mathcal{H}}(\gamma, \alpha, \beta)$. Then for $|z| = r < 1$, we have*

$$|f(z)| \geq (1-b_1)r - \frac{(1-\alpha)(1-b_1)}{2(\beta(1-\gamma)-\alpha(1+\gamma)+2)} r^2 \tag{5.1}$$

and

$$|f(z)| \leq (1+b_1)r + \frac{(1-\alpha)(1-b_1)}{2(\beta(1-\gamma)-\alpha(1+\gamma)+2)} r^2. \tag{5.2}$$

Proof. Assume that $f \in KW_{\mathcal{H}}(\gamma, \alpha, \beta)$. Then by (2.4), we get

$$\begin{aligned}
|f(z)| &= \Big| z - \sum_{n=2}^{\infty} a_n z^n - \sum_{n=1}^{\infty} b_n (\overline{z})^n \Big| \geq (1-b_1)r - \sum_{n=2}^{\infty} (a_n + b_n) r^n \\
&\geq (1-b_1)r - \sum_{n=2}^{\infty} (a_n + b_n) r^2 \\
&= (1-b_1)r - \frac{1}{2(\beta(1-\gamma)-\alpha(1+\gamma)+2)} \\
&\quad \times \sum_{n=2}^{\infty} 2(\beta(1-\gamma)-\alpha(1+\gamma)+2)(a_n+b_n) r^2 \\
&\geq (1-b_1)r - \frac{1}{2(\beta(1-\gamma)-\alpha(1+\gamma)+2)} \\
&\quad \times \sum_{n=2}^{\infty} n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)(a_n+b_n) r^2
\end{aligned}$$

$$\geq (1-b_1)r - \frac{1}{2(\beta(1-\gamma)-\alpha(1+\gamma)+2)}[(1-\alpha)-(1-\alpha)b_1]r^2$$
$$= (1-b_1)r - \frac{(1-\alpha)(1-b_1)}{2(\beta(1-\gamma)-\alpha(1+\gamma)+2)}r^2.$$

Relation (5.2) can be proved by using similar statements. So the proof is complete. ■

6. Integral operator

DEFINITION 6.1. [1] The Bernardi operator is defined by

$$L_c(k(z)) = \frac{c+1}{z^c}\int_0^z \varepsilon^{c-1}k(\varepsilon)\,d\varepsilon,\ c\in\mathbb{N}=\{1,2,\cdots\}. \tag{6.1}$$

If $k(z)=z+\sum_{n=2}^{\infty} e_n z^n$, then

$$L(c(k(z)) = z + \sum_{n=2}^{\infty}\frac{c+1}{c+n}\,e_n z^n. \tag{6.2}$$

REMARK 6.1. If $f=h+\overline{g}$, where

$$h(z)=z-\sum_{n=2}^{\infty}a_n z^n,\ g(z)=-\sum_{n=1}^{\infty}b_n z^n\ (a_n\geq 0, b_n\geq 0),$$

then

$$L_c(f(z)) = L_c(h(z)) + \overline{L_c(g(z))}. \tag{6.3}$$

THEOREM 6.1. *If $f\in KW_{\mathcal{H}}(\gamma,\alpha,\beta)$, then $L_c(f)$ $(c\in\mathbb{N})$ is also in $KW_{\mathcal{H}}(\gamma,\alpha,\beta)$.*

Proof. By (6.2) and (6.3), we get

$$L_c(f(z)) = L_c\Big(z-\sum_{n=2}^{\infty}a_n z^n - \sum_{n=1}^{\infty}b_n(\overline{z})^n\Big)$$
$$= z - \sum_{n=2}^{\infty}\frac{c+1}{c+n}a_n z^n - \sum_{n=1}^{\infty}\frac{c+1}{c+n}b_n(\overline{z})^n.$$

Since $f\in KW_{\mathcal{H}}(\gamma,\alpha,\beta)$, then by Theorem 2.2, we have

$$\sum_{n=2}^{\infty}\frac{n(n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)}{1-\alpha}a_n + \sum_{n=1}^{\infty}\frac{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)}{1-\alpha}b_n \leq 1.$$

Since $c\in\mathbb{N}=\{1,2,\cdots\}$, then $\frac{c+1}{c+n}\leq 1$, therefore

$$\sum_{n=2}^{\infty}\frac{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)}{1-\alpha}\left(\frac{c+1}{c+n}\right)a_n$$
$$+\sum_{n=1}^{\infty}\frac{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)}{1-\alpha}\left(\frac{c+1}{c+n}\right)b_n$$
$$\leq \sum_{n=2}^{\infty}\frac{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)}{1-\alpha}a_n$$

$$+\sum_{n=1}^{\infty}\frac{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)}{1-\alpha}b_n\leq 1,$$

and this gives the result. ∎

7. Convolution property

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n-\sum_{n=1}^{\infty}b_n(\overline{z})^n,$$

and

$$F(z)=z-\sum_{n=2}^{\infty}c_nz^n-\sum_{n=1}^{\infty}d_n(\overline{z})^n,$$

we define the convolution of two harmonic functions f and F as

$$(f*F)(z)=f(z)*F(z)=z-\sum_{n=2}^{\infty}a_nc_nz^n-\sum_{n=1}^{\infty}b_nd_n(\overline{z})^n.$$

THEOREM 7.1. *For* $0\leq\eta\leq\alpha<1$, *let* $f\in KW_{\mathcal{H}}(\gamma,\alpha,\beta)$ *and* $F\in KW_{\mathcal{H}}(\gamma,\eta,\beta)$. *Then*

$$f*F\in KW_{\mathcal{H}}(\gamma,\alpha,\beta)\subset KW_{\mathcal{H}}(\gamma,\eta,\beta).$$

Proof. It is easy to see that $KW_{\mathcal{H}}(\gamma,\alpha,\beta)\subset KW_{\mathcal{H}}(\gamma,\eta,\beta)$.

Since $f\in KW_{\mathcal{H}}(\gamma,\alpha,\beta)$ and $F\in KW_{\mathcal{H}}(\gamma,\eta,\beta)$, then by Theorem 2.2, we have

$$\sum_{n=2}^{\infty}\frac{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)}{1-\alpha}a_n+\sum_{n=1}^{\infty}\frac{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)}{1-\alpha}b_n\leq 1\quad(7.1)$$

and

$$\sum_{n=2}^{\infty}\frac{n((n-1)(\beta(1-\gamma)-\eta\gamma)+n-\eta)}{1-\eta}c_n+\sum_{n=1}^{\infty}\frac{n((n-1)(\beta(1-\gamma)-\eta\gamma)+n-\eta)}{1-\eta}d_n\leq 1.\quad(7.2)$$

From (7.2), we get the following inequalities

$$c_n<\frac{1-\eta}{n((n-1)(\beta(1-\gamma)-\eta\gamma)+n-\eta)}\ (n=2,3,\cdots),\ c_1=1$$

$$d_n<\frac{1-\eta}{n((n-1)(\beta(1-\gamma)-\eta\gamma)+n-\eta)}\ (n=2,3,\cdots),\ 0\leq d_1<1.$$

Therefore

$$\sum_{n=2}^{\infty}\frac{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)}{1-\alpha}a_nc_n$$

$$
\begin{aligned}
&+\sum_{n=1}^{\infty}\frac{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)}{1-\alpha}b_nd_n\\
&=b_1d_1+\sum_{n=2}^{\infty}\frac{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)}{1-\alpha}a_nc_n\\
&+\sum_{n=2}^{\infty}\frac{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)}{1-\alpha}b_nd_n\\
&<b_1+\sum_{n=2}^{\infty}\frac{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)(1-\eta)}{n((n-1)(\beta(1-\gamma)-\eta\gamma)+n-\eta)(1-\alpha)}a_n\\
&+\sum_{n=2}^{\infty}\frac{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)(1-\eta)}{n((n-1)(\beta(1-\gamma)-\eta\gamma)+n-\eta)(1-\alpha)}b_n\\
&=\sum_{n=2}^{\infty}\frac{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)(1-\eta)}{n((n-1)(\beta(1-\gamma)-\eta\gamma)+n-\eta)(1-\alpha)}a_n\\
&+\sum_{n=1}^{\infty}\frac{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)(1-\eta)}{n((n-1)(\beta(1-\gamma)-\eta\gamma)+n-\eta)(1-\alpha)}b_n\\
&\leq\sum_{n=2}^{\infty}\frac{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)}{1-\alpha}a_n\\
&+\sum_{n=1}^{\infty}\frac{n((n-1)(\beta(1-\gamma)-\alpha\gamma)+n-\alpha)}{1-\alpha}b_n\leq 1.
\end{aligned}
$$

Then $f*F\in KW_{\mathcal{H}}(\gamma,\alpha,\beta)\subset KW_{\mathcal{H}}(\gamma,\eta,\beta)$, and the proof is complete. ■

REFERENCES

[**1**] S.D. Bernardi, *Convex and starlike univalent functions*, Trans. Amer. Math. Soc. **135** (1969), 429–446.

[**2**] J. Clunie, T. Sheil-Small, *Harmonic univalent functions*, Ann. Acad. Aci. Fenn. Ser. A. I. Math. **9** (1984), 3–25.

[**3**] L. Cotirla, *A new class of harmonic multivalent functions defined by an integral operator*, Acta Univ. Apulensis, **21** (2010), 55–63.

[**4**] B.A. Frasin, *On the analytic part of harmonic univalent functions*, Bull. Korean Math. Soc. **42** (2005), 563–569.

[**5**] T. Gencel, S. Yalcin, *A special class of harmonic univalent functions*, Sarajevo J. Math. **4(17)** (2008), 167–206.

[**6**] A.O. Mostafa, *A study on starlike and convex properties for hypergeometric functions*, J. Ineq. Pure Appl. Math. **10** (2009), Article 87, 8 pages.

[**7**] S. Porwal, K.K. Dixit, S.B. Joshi, *Convolution of Salagean-type harmonic univalent functions*, J. Math. **43** (2011), 69–73.

[**8**] B. Seker, *Sâlâgean-type harmonic univalent functions*, Inter. J. Physical Sci. **6** (2011), 801–807.

[**9**] S. Yalcin, *On certain harmonic univalent functions defined by salagean derivatives*, Soochow J. Math. **31** (2005), 321–331.

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Department of Mathematics, College of Computer Science and Mathematics, University of Al-Qadisiya, Diwaniya, Iraq

E-mail: waggashnd@gmail.com, waggas_hnd@yahoo.com, k.abbaswmk@yahoo.com