МАТЕМАТИЧКИ ВЕСНИК
66, 2 (2014), 133–139
June 2014

оригинални научни рад
research paper

GENERALIZATIONS OF PRIMAL IDEALS OVER COMMUTATIVE SEMIRINGS

Malik Bataineh and Ruba Malas

Abstract. In this article we generalize some definitions and results from ideals in rings to ideals in semirings. Let R be a commutative semiring with identity. Let $\phi\colon \vartheta(R) \to \vartheta(R) \cup \{\emptyset\}$ be a function, where $\vartheta(R)$ denotes the set of all ideals of R. A proper ideal $I \in \vartheta(R)$ is called ϕ-prime ideal if $ra \in I - \phi(I)$ implies $r \in I$ or $a \in I$. An element $a \in R$ is called ϕ-prime to I if $ra \in I - \phi(I)$ (with $r \in R$) implies that $r \in I$. We denote by $p(I)$ the set of all elements of R that are not ϕ-prime to I. I is called a ϕ-primal ideal of R if the set $P = p(I) \cup \phi(I)$ forms an ideal of R. Throughout this work, we define almost primal and ϕ-primal ideals, and we also show that they enjoy many of the properties of primal ideals.

1. Introduction

The concept of semiring was introduced by H. S. Vandiver in 1935. The concept of primal ideals in a commutative ring R was introduced and studied in [6] (see also [7]). The set of elements of R that are not prime to I is denoted by $S(I)$; while the set of elements of R that are not weakly prime to I is denoted by $W(I)$. A proper ideal I of R is said to be a primal if $S(I)$ forms an ideal; therefore, 0 is not necessarily a primal, such an ideal is always a prime ideal. If I is a primal ideal of R, then ($P =$ the set of elements of R that are not prime to I) is a prime ideal of R, called the adjoint prime ideal P of I. In this case we also say that I is a P-primal ideal (see [4]). Also, a proper ideal I of R is called a weakly primal if the set $P = W(I) \cup \{0\}$ forms an ideal; this ideal is called the weakly adjoint ideal P of I and is always a weakly prime ideal [5, Proposition 4]. Weakly primal ideals in a commutative semiring were introduced and studied by S. Ebrahimi Atani in [4]. Bhatwadekar and Sharma (2005) defined a proper ideal I of an integral domain R to be almost prime if for $a, b \in R$ with $ab \in I - I^2$, either $a \in I$ or $b \in I$ holds. This definition can obviously be made for any commutative ring R. Prime ideals play a central role in the commutative ring theory. Let $\phi\colon \vartheta(R) \to \vartheta(R) \cup \{\emptyset\}$ be a function, where $\vartheta(R)$ denotes the set of all ideals of R. D. D. Anderson and M. Bataineh (2008) defined a proper ideal I of a commutative ring R to be a ϕ-prime

2010 Mathematics Subject Classification: 13A15, 16Y60
Keywords and phrases: Primal ideal; ϕ-prime ideal; weakly primal ideal; ϕ-primal ideal.

ideal if for $a, b \in R$ with $ab \in I - \phi(I)$ implies $a \in I$ or $b \in I$. M. Henriksen (1958) called an ideal I of a semiring R a k-ideal, whenever $x, x + y \in I$ implies $y \in I$.

Throughout the paper, R will be a commutative semiring with identity; however, in most places the existence of an identity plays no role. By a proper ideal I of R, we mean an ideal $I \in \vartheta(R)$ with $I \neq R$. Let us generalize the definition of a primal ideal; a proper ideal I of R is an almost primal (resp., P_ϕ-primal) if the set $P = p(I) \cup I^2$ (resp., $P = p(I) \cup \phi(I)$) forms an ideal of R, where $p(I)$ is the set of all elements in R that are not almost prime (resp., ϕ-prime) to I. This ideal is called the almost adjoint ideal P of I. Given two functions $\psi_1, \psi_2 \colon \vartheta(R) \to \vartheta(R) \cup \{\emptyset\}$, we define $\psi_1 \leq \psi_2$ if $\psi_1(I) \subseteq \psi_2(I)$ for each $I \in \vartheta(R)$.

Example 1.1. Let R be a commutative semiring. Define the following functions $\phi_\alpha \colon \vartheta(R) \to \vartheta(R) \cup \{\emptyset\}$ and the corresponding ϕ_α-primal ideals, by

$\phi_\varnothing$	$\phi(J) = \emptyset$	a ϕ-primal ideal is a primal.
ϕ_0	$\phi(J) = 0$	a ϕ-primal ideal is a weakly primal.
ϕ_2	$\phi(J) = J^2$	a ϕ-primal ideal is an almost primal.
ϕ_n $(n \geq 2)$	$\phi(J) = J^n$	a ϕ-primal ideal is a n-almost primal.
ϕ_ω	$\phi(J) = \bigcap_{n\in\mathbb{N}} J^n$	a ϕ-primal ideal is a ω-primal.
ϕ_1	$\phi(J) = J$	a ϕ-primal ideal is any ideal.

Observe that $\phi_\varnothing \leq \phi_0 \leq \phi_\omega \leq \cdots \leq \phi_{n+1} \leq \phi_n \leq \cdots \leq \phi_2$.

Remark 1.2. Let I be a P_ϕ-primal ideal in a commutative semiring R and $a \in I - \phi(I)$. As $a \cdot 1_R \in I - \phi(I)$ with $1_R \notin I$, a is not a ϕ-prime to I; and therefore, $I \subseteq P$.

2. Results

In this section, we give several characterizations of ϕ-primal ideals.

Theorem 2.1. *Let I be a proper k-ideal of a commutative semiring R, $\phi\colon \vartheta(R) \to \vartheta(R) \cup \{\emptyset\}$ be a function and P be a proper ideal. Then the following are equivalent.*

(i) I is a P_ϕ-primal.

(ii) For every $a \notin P - \phi(I)$, $(I :_R a) = I \cup (\phi(I) :_R a)$, and for $a \in P - \phi(I)$, $I \cup (\phi(I) :_R a) \subsetneq (I :_R a)$.

(iii) If $a \notin P - \phi(I)$, then $(I :_R a) = A$ or $(I :_R a) = (\phi(I) :_R a)$, and if $a \in P - \{\phi(I)\}$, then $I \subsetneq (I :_R a)$ and $(\phi(I) :_R a) \subsetneq (I :_R a)$.

Proof. *(i)* $\Rightarrow$ *(ii)* Let I be a P_ϕ-primal ideal of R. Then $P - \phi(I)$ is exactly the set of all elements in R that are not ϕ-prime to I. Suppose $a \notin P - \phi(I)$, and $b \in (I :_R a)$. If $ab \notin \phi(I)$ as a is ϕ- prime to I, then $b \in I$. If $ab \in \phi(I)$, then $b \in (\phi(I) :_R a)$. As the reverse containment holds for any ideal I, we get the equality. Now suppose that $a \in P - \phi(I)$. Then, there exists $s \in R - I$ such that $as \in I - \phi(I)$; hence $s \in (I :_R a) - (I \cup (\phi(I) :_R a))$.

$(ii) \Rightarrow (iii)$ Let $a \notin P - \phi(I)$. By [2, Lemma 2.1 and Lemma 2.2], if an ideal of R is a union of two k-ideals then it is equal to one of them. Moreover, if $a \in P - \phi(I)$, then by (ii), we have $I \subsetneq (I :_R a)$ and $(\phi(I) :_R a) \subsetneq (I :_R a)$.

$(iii) \Rightarrow (i)$ Since $P - \phi(I)$ is the set of all elements in R that are not ϕ-prime to I, we conclude that I is P-almost primal. ■

Theorem 2.2. *Let R be a semiring. Then every k-ϕ-prime ideal of R is a ϕ-primal.*

Proof. Let A be a k-ϕ-prime ideal of R. Assume that $A \neq \phi(A)$. It is enough to show that $A - \phi(A)$ is exactly the set of all elements in R that are not ϕ-prime to A. Let $a \in A - \phi(A)$. Then $a \cdot 1_R \in A - \phi(A)$ with $1_R \notin A$, gives that a is not ϕ-prime to A. On the other hand, if $a \notin A - \phi(A)$ and $a \in \phi(A)$, then a is a ϕ-prime to A. If $a \notin A$, let $b \in R$ be such that $ab \in A - \phi(A)$, which implies $b \in A$. Thus $a \notin A - \phi(A)$ is ϕ-prime to A. This shows that A is a ϕ-primal ideal of R. ■

Theorem 2.3. *Let I be a k-ideal of a semiring R. If I is a P_ϕ-primal ideal of R, then P is a ϕ-prime ideal of R.*

Proof. Suppose that $a, b \notin P$; we show that either $ab \in \phi(P)$ or $ab \notin P$. Assume that $ab \notin \phi(P)$. Let $rab \in I - \phi(I)$ for some $r \in R$. Then, Theorem 2.1 gives that $ra \in (I :_R b) = I \cup (\phi(I) : b)$ where $ra \notin (\phi(I) :_R b)$; hence $ra \in I - \phi(I)$. Thus, $r \in (I :_R a) = I \cup (\phi(I) : a)$, and so $r \in I$. Therefore, ab is ϕ-prime to I and $ab \notin P$ as required. ■

Proposition 2.4. *Let R be a commutative semiring and J a proper ideal of R.*

(i) Let $\psi_1, \psi_2 \colon \vartheta(R) \to \vartheta(R) \cup \{\emptyset\}$ be functions with $\psi_1 \leq \psi_2$. Then if J ψ_1-primal, then J is ψ_2-primal.

(ii) J primal $\Rightarrow$ J weakly primal $\Rightarrow$ J ω-primal $\Rightarrow$ J $(n+1)$-almost primal $\Rightarrow$ J n-almost primal $(n \geq 2)$ $\Rightarrow$ J almost primal.

Proof. (i) Assume that J is P-ψ_1 primal ideal; we will show that J is a P-ψ_2 primal ideal in R. By Theorem 2.3, P is ψ_1-prime and hence ψ_2-prime. It is enough to show that P-$\psi_2(J)$ is exactly the set of all elements in R that are not ψ_2-prime to J. Let $x \in P - \psi_2(J)$ and $y \in R$ be such that $xy \in J - \psi_2(J)$. Since $\psi_1 \leq \psi_2$, we get P-$\psi_2(J) \subseteq P - \psi_1(J)$. So, x is not ψ_1-prime to J, and thus $y \notin J$. Hence, x is not ψ_2-prime to J. Now let $x \in R$ be not ψ_2-prime to J, then there exists $y_0 \in R$ with $xy_0 \in J - \psi_2(J)$ and $y_0 \notin J$. Now, we claim that x is not ψ_1-prime to J. Suppose not, that is for all $y \in R$ with $xy \in J - \psi_1(J)$ implies $y \in J$; which is a contradiction, since $xy_0 \in J - \psi_2(J)$ with $y_0 \notin J$ so $x \in P$.

(ii) follows by (i). ■

We next give a further general conditions for ϕ-primal ideals to be primal ideals.

Theorem 2.5. *Let R be a commutative semiring and $\phi\colon \vartheta(R) \to \vartheta(R) \cup \{\emptyset\}$ a function. Let P and I be k-ideals of R. If I is a P_ϕ-primal ideal of R that is not primal, then $I^2 \subseteq \phi(I)$. Moreover, if P is a prime ideal of R with $I^2 \nsubseteq \phi(I)$, then I is a P-primal ideal in R.*

Proof. Suppose that $I^2 \nsubseteq \phi(I)$ and P is a prime ideal of R. It suffices to show that P is exactly the set of elements that are not prime to I. If $a \in P$, then a is not ϕ-prime to I, so a is not prime to I. Now, assume that a is not prime to I. Then, there exists $r \in R - I$ such that $ra \in I$. If $ra \in I - \phi(I)$, then a is not ϕ-prime to I; hence $a \in P$. So, assume that $ra \in \phi(I)$. If $aI \nsubseteq \phi(I)$, then there exists $r_0 \in I$ such that $ar_0 \notin \phi(I)$; $a(r + r_0) \in I - \phi(I)$ with $r + r_0 \notin I$. Since I is a k-ideal, hence a is not ϕ-prime to I, and we have $a \in P$. So, we can assume that $aI \subseteq \phi(I)$. If $rI \nsubseteq \phi(I)$, then there exists $c \in I$ such that $rc \notin \phi(I)$ and so $(a + c)r \in I - \phi(I)$ with $r \notin I$, which gives that $a + c \in P$, $a \in P$. So we can assume $rI \subseteq \phi(I)$. As $I^2 \nsubseteq \phi(I)$, there exists $a_0, b_0 \in I$ such that $a_0b_0 \in I$ with $a_0b_0 \notin \phi(I)$. Hence $(a + a_0)(r + b_0) \in I - \phi(I)$ with $r + b_0 \notin I$, which implies that $a + a_0 \in P$. ∎

An ideal I of a semiring R is called a partitioning ideal (= Q-ideal) (see [4]) if there exists a subset Q of R such that

(1) $R = \bigcup\{q + I : q \in Q\}$, and

(2) $(q_1 + I) \cap (q_2 + I) \neq \emptyset$, for any q_1, $q_2 \in Q$ if and only if $q_1 = q_2$.

Let I be a Q-ideal of a semiring R, and $R/I = \{q + I : q \in Q\}$. Then R/I forms a semiring under the binary operations $\oplus$ and $\odot$, which are defined as follows: $(q_1 + I) \oplus (q_2 + I) = q_3 + I$ for a unique element $q_3 \in Q$ satisfying that $q_1 + q_2 + I \subseteq q_3 + I$

$(q_1 + I) \odot (q_2 + I) = q_4 + I$, for a unique element $q_4 \in Q$ satisfying that $q_1q_2 + I \subseteq q_4 + I$. This semiring R/I is called the quotient semiring of R by I.

Let J be an ideal of R and $\phi\colon \vartheta(R) \to \vartheta(R) \cup \{\emptyset\}$ be a function. As in [1], we define $\phi_J\colon \vartheta(R/J) \to \vartheta(R/J) \cup \{\emptyset\}$ by $\phi_J(I/J) = (\phi(I) + J)/J$ for every ideal $I \in \vartheta(R)$ with $J \subseteq I$ (and $\phi_J(I/J) = \emptyset$ if $\phi(I) = \emptyset$).

Theorem 2.6. *Let J be a Q-ideal of a semiring R, I a proper k-ideal of R and J a ϕ-prime ideal of R with $J \subseteq \phi(I)$. Then I is a ϕ-primal if and only if I/J is a ϕ_J-primal of R/J.*

Proof. Assume that I is a P_ϕ-primal. It is easy to show P/J is ϕ_J-prime. Let $a + J \in P/J - \phi_J(I/J) = P/J - (\phi(I) + J)/J$, where $a \in P \cap Q$ [3, Proposition 2.2]. Then $a \notin \phi(I)$, and hence a is not ϕ-prime to I, and so there exists $r \in R - I$ such that $ra \in I - \phi(I)$. If $ra \in J - \phi(J)$, then J ϕ-prime gives that $r \in J$, which is a contradiction since $r \notin I$. Thus $ra \notin J - \phi(J)$. There is an element $q_1 \in Q$ such that $r \in q_1 + J$, so $r = q_1 + c$ for some $c \in J$; hence, $aq_1 \notin J - \phi(I)$. It follows that $(q_1 + J) \odot (a + J) \in I/J - \phi_J(I/J)$ with $q_1 + J \notin I/J$, which implies that $a + J$ is not ϕ_J-prime to I/J. Now assume that $b + J$ is not ϕ_J-prime to I/J, where $b \in Q$. Then there exists $c + J \in R/J - I/J$ such that $(c + J) \odot (b + J) = q_2+$

$J \in I/J - \phi_J(I/J)$ where $q_2 \in Q \cap I$ and $q_2 \notin \phi(I)$ is a unique element such that $bc + J \subseteq q_2 + J$; hence $cb \in I - \phi(I)$ with $c \notin I$. So, $b \notin \phi(I)$ is not ϕ-prime to I. Therefore, $b + J \in P/J - \phi_J(I/J)$.

On the other hand, let I/J be a P/J-ϕ_J-primal ideal of R/J; we show that I is a P-ϕ-primal. Let $a \in P - \phi(I)$; we can assume that $a \notin J$, so there is an element $q_3 \in Q$ such that $a \in q_3 + J$ which can be written as $a = q_3 + d$, for some $d \in J$. As J is a ϕ-prime ideal and $q_3 + J \in P/J - \phi_J(I/J)$, there exists $r + J \in R/J - I/J$ such that $(q_3 + J) \oplus (r + J = q_4 + J \in I/J - \phi_J(I/J)$, where q_4 is a unique element $\in I \cap Q$ such that $q_3 r + J \subseteq q_4 + I$, $ra \in I - \phi(I)$ with $r \notin I$. Thus, a is not ϕ-prime to I. Now assume that a is not ϕ-prime to I (so $a \notin \phi(I)$). Without loss of generality, we assume that $a \notin I$, and then there is an element $r \in R - I$ such that $ra \in I - \phi(I)$. So there are elements q_5, $q_6 \in Q$ such that $a \in q_5 + J$ and $r \in q_6 + J$; so $a = q_5 + e$ and $r = q_6 + f$ for some $e, f \in J$, which leads to $ef \in I - \phi(I)$. Therefore, J is a ϕ-prime ideal which gives $q_7 + J = (q_5 + J) \odot (q_6 + J) \in I/J - \phi_J(I/J)$, where q_7 is a unique element $\in Q \cap I$ such that $q_5 q_6 + J \subseteq q_7 + J$ with $(q_6 + J) \notin I/J$. Consequently, $a + J = q_5 + e + J \in P/J - \phi_J(I/J)$, since I/J is a P/J-ϕ_J-primal ideal of R/J and then $a \in P$. ■

Note that if P is a ϕ-primal but not primal, then by Theorem 2.5, $P^2 \subseteq \phi(P)$. Moreover, if $\phi \leq \phi_2$, then $P^2 \subseteq \phi(P) \subseteq P^2$; so $\phi(P) = P^2$. In particular, If P is a weakly primal but not a primal, then $P^2 = 0$. Now if $\phi \leq \phi_3$, then $P^2 = \phi(P) \subseteq P^3$; so $P^2 = P^3$; and hence P is idempotent. We next move to construct a ϕ-primal ideal J where $\phi_\omega \leq \phi$.

THEOREM 2.7. *Let T and S be commutative semirings and I a P-weakly primal ideal of T. Then $J = I \times S$ is a ϕ-primal ideal of $R = T \times S$ for each ϕ with $\phi_\omega \leq \phi$.*

Proof. Assume that I is a primal; then it is clear that J is also a primal. Suppose that I is a P-weakly primal but not a primal. Then $I^2 = 0$, $J^2 = 0 \times S$; and hence $\phi_\omega(J) = 0 \times S$. Let us show that J is $P \times S_{\phi_\omega}$-primal of $T \times S$. It is enough to show that $P \times S - \phi_\omega(J)$ is exactly the set of all elements that are not a ϕ_ω-prime to J. Let (x_1, x_2) be not ϕ_ω-prime to J; then there exists $(y_1, y_2) \in T \times S$ such that $(x_1, x_2)(y_1, y_2) \in J - \phi_\omega(J)$ with $(y_1, y_2) \notin J$. Now, $(x_1, x_2)(y_1, y_2) \in I \times S - \{0\} \times S$; $(x_1 y_1, x_2 y_2) \in I - \{0\} \times S$. So, $x_1 y_1 \in I - \{0\}$ with $y_1 \notin I$. As I is a P-weakly primal, $x_1 \in P - \{0\}$, and then $(x_1, x_2) \in P \times S - \phi_\omega(J)$ is not ϕ_ω-prime to J. On the other hand, assume that $(x_1, y_1) \in P \times S - \phi_w(J)$, where $x_1 \in P - \{0\}$ is not ϕ_ω-prime to J. As $x_1 \in P - \{0\}$, x_1 is not weakly prime to I then there exists $r \in T - I$ such that $x_1 r \in I - \{0\}$. Thus, $(r, 1) \in T \times S - I \times S$; and then $(x_1, x_2)(r, 1) \in I - \{0\} \times S = I \times S - \{0\} \times S = J - \phi_\omega(J)$. Therefore, (x_1, y_1) is not ϕ_ω-prime to J. ■

The semiring of fractions is defined in [4] as follows: let R be a semiring, and S be the set of all multiplicatively cancellable elements of R ($1 \in S$). Define a relation $\sim$ on $R \times S$ as follows: for $(a, s), (b, t) \in R \times S$, $(a, s) \sim (b, t)$ if and only if $at = bs$. Then $\sim$ is an equivalence relation on $R \times S$. For $(a, s) \in R \times S$, let us denote the equivalence classes of $\sim$ by $\frac{a}{s}$, and denote the set of all equivalence classes of

$\sim$ by R_S. Then R_S is a semiring under the operations for which $\frac{a}{s} + \frac{b}{t} = \frac{at+sb}{st}$ and $(\frac{a}{s})(\frac{b}{t}) = \frac{ab}{st}$ for all $a, b \in R$ and $s, t \in S$. This new semiring R_S is called the semiring of fractions of R with respect to S; and its zero element is $\frac{0}{1}$. Its multiplicative identity element is $\frac{1}{1}$ and each element of S has a multiplicative inverse in R_S.

Throughout the paper, S will be the set of all multiplicatively cancellable elements of a semiring R. Now suppose that I is an ideal of a semiring R. The ideal generated by I in R_S, that is, the set of all finite sums $s_1a_1 + \cdots + s_na_n$ where $a_i \in R_S$ and $s_i \in I$, is called the extension of I to R_S, and is denoted by IR_S. Again, if J is an ideal of R_S, then the contraction of J in R, $J \cap R = \{r \in R : r/1 \in J\}$ is clearly an ideal of R.

Let $\phi\colon \vartheta(R) \to \vartheta(R) \cup \{\emptyset\}$ be a function. Define $\phi_S\colon \vartheta(R_S) \to \vartheta(R_S) \cup \{\emptyset\}$ by $\phi_S(J) = \phi(J \cap R)R_S$ (and $\phi_S(J) = \emptyset$ if $\phi(J \cap R) = \emptyset$).

Proposition 2.8. *Let R be a commutative semiring, and let $\phi\colon \vartheta(R) \to \vartheta(R) \cup \{\emptyset\}$ be a function. Assume P is an ideal of R and S is the set of all multiplicatively cancellable elements in R such that $P \cap S = \emptyset$. If P is a ϕ-prime ideal of R and $\phi(P)R_S \subseteq \phi_S(PR_S)$, then PR_S is a ϕ_S-prime of R_S. Moreover, if $PR_S \neq \phi(P)R_S$, then $PR_S \cap R = P$.*

Proof. Let $\frac{x}{s} \cdot \frac{y}{t} \in PR_S - \phi_S(PR_S)$, and let $xyu \in P$ for some $u \in S$, and for any $w \in S$, $xyw \notin \phi_S(PR_S) \cap R$. If $xyw \in \phi(P)$, then $\frac{x}{s}.\frac{y}{t} \in \phi(P)R_S \subseteq \phi_S(PR_S)$, which is a contradiction. So $x(yu) \in P - \phi(P)$. As P is ϕ-prime, we have $x \in P$ or $yu \in P$. Hence, $\frac{x}{s} \in PR_S$ or $\frac{y}{t} \in PR_S$. Assume $x \in PR_S \cap R$; so there exists $s \in S$ with $xs \in P$. If $xs \notin \phi(P)$, then $xs \in P - \phi(P)$, which implies that $x \in P$, and if $xs \in \phi(P)$, then $x \in \phi(P)R_S \cap R$, and so $PR_S \cap R \subseteq P \cup (\phi(P)R_S \cap R)$. Thus, $PR_S \cap R \subseteq P$ or $PR_S \cap R \subseteq \phi(P)R_S \cap R$. By this and the second case, we conclude that $PR_S = \phi(P)R_S$. ■

Note that A is a ϕ-primal if $P = A$. Since $\phi_S(IR_S) = \phi(IR_S \cap R)R_S$, then we have $I \subseteq IR_S \cap R$.

Lemma 2.9. *Let $I$and A be k-deals of a commutative semiring R. Then,*

(i) If I is a P_ϕ-primal ideal of R with $P \cap S = \emptyset$, and $\phi(P)R_S \subseteq \phi_S(PR_S)$ such that $\frac{a}{s} \in IR_S$ and $\frac{a}{s} \notin \phi_S(IR_S)$, then $a \in I - \phi(I)$.

(ii) If A is a ϕ-primal ideal of R with $A \cap S = \emptyset$, and $\phi(A)R_S \subseteq \phi_S(AR_S)$ such that $\frac{a}{s} \in AR_S$ and $\frac{a}{s} \notin \phi_S(AR_S)$, then $a \in A - \phi(A)$.

Proof. (i) Assume that I is a P_ϕ-primal ideal of R and $\frac{a}{s} \in IR_S - \phi_S(IR_S)$. Let $a \notin I - \phi(I)$. If $a \in \phi(I)$, then $\frac{a}{s} \in \phi(I)R_S \subseteq \phi_S(IR_S)$, which is a contradiction. So $a \notin \phi(I)$. If $a \notin I$, then there exists $t \in S$ such that $at \in I$ and for any $w \in S$, $aw \notin \phi(I)$. Thus $at \in I - \phi(I)$ with $a \notin I$, which contradicts with $S \cap P = \emptyset$.

(ii) Follows from (i) and Theorem 2.2. ■

Theorem 2.10. *Let I be a k-ideal of a semiring R. If I is a P_ϕ-primal, S is the set of multiplicatively cancellable elements in R, $P \cap S = \emptyset$, and $\phi(P)R_S \subseteq \phi_S(PR_S)$. Then IR_S is a $PR_S - \phi_S$-primal ideal of R_S.*

Proof. By Theorem 2.3 and Proposition 2.8, PR_S is a ϕ_S-prime ideal of R_S. So it is enough to show that $PR_S - \phi_S(IR_S)$ is exactly the set of all elements in R_S that are not ϕ_S-prime to IR_S. Assume that $\frac{r}{s}$ is not ϕ_S-prime to IR_S; then there exists $\frac{x}{t} \in R_S - IR_S$ such that $\frac{r}{s} \cdot \frac{x}{t} \in IR_S - \phi_S(IR_S)$. By Lemma 2.9, $rx \in I - \phi(I)$ with $x \notin I$, and so r is not ϕ-prime to I; that is $r \in P$. Hence, $\frac{r}{s} \in PR_S - \phi_S(IR_S)$. On the other hand, let $\frac{x}{s} \in PR_S - \phi_S(IR_S)$, $xu \in P$ for some $u \in S$ and $xw \notin \phi_S(IR_S) \cap R$ for all $w \in S$; then $xw \notin \phi(I)$; otherwise, $xw \in \phi(I)$ so $\frac{x}{s} \in \phi(I)R_S \subseteq \phi_S(IR_S)$ since $I \subseteq P$; and hence $xu \in P - \phi(I)$, xu is not ϕ-prime to I. So there exists $y \in R - I$ such that $xuy \in I - \phi(I)$; $\frac{xy}{s} = \frac{x}{s} . \frac{y}{1} \in IR_S - \phi_S(IR_S)$ with $\frac{y}{1} \notin IR_S$. ■

REFERENCES

[**1**] D. D. Anderson and M. Bataineh, *Generalizations of prime ideals*, Comm. Algebra **36** (2008), 686–696.

[**2**] S. Ebrahimi Atani, *On k-weakly primary ideals over semirings*, Sarajevo J. Math. **3** (2007), 9–13.

[**3**] S. Ebrahimi Atani, *The ideal theory in quotients of commutative semirings*, Glas. Mat. Ser. III **42** (2007), 301–308.

[**4**] S. Ebrahimi Atani, *On primal and weakly primal ideals over commutative semirings*, Glas. Mat. **43** (2008), 13–23.

[**5**] S. Ebrahimi Atani and A. Y. Darani, *On weakly primal ideals (I)*, Demonstratio Math. **40** (2007), 23–32.

[**6**] L. Fuchs, *On primal ideals*, Proc. Amer. Math. Soc. **1** (1950), 1–6.

[**7**] L. Fuchs and E. Mosteig, *Ideal theory in Prufer domains*, J. Algebra **252** (2002), 411–430.

(received 29.06.2012; in revised form 14.08.2013; available online 15.12.2013)

Mathematics Department, Jordan University of Science and Technology, Irbid 22110, Jordan

E-mail: msbataineh@just.edu.jo