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ON AN INEQUALITY OF PAUL TURÁN

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Abstract. Let $P(z)$ be a polynomial and $P'(z)$ its derivative. In this paper, we shall obtain certain compact generalizations and sharp refinements of some results of Govil, Malik, Turán and others concerning the maximum modulus of $P(z)$ and $P'(z)$ on the unit circle $|z| = 1$, which also yields a number of other interesting results for various choices of parameters.

1. Introduction and statement of results

Let $P(z)$ be a polynomial of degree n and $P'(z)$ be its derivative. It was shown by Tuŕan [8] that if $P(z)$ has all its zeros in $|z| \leqslant 1$, then

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1}$$

The inequality (1) is sharp with equality for the polynomial $P(z) = (z + 1)^n$.

As an extension of (1), Malik [5] showed that if $P(z)$ has all its zeros in $|z| \leqslant k$, where $k \leqslant 1$, then

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{1+k} \max_{|z|=1} |P(z)|. \tag{2}$$

The estimate (2) is sharp with equality for the polynomial $P(z) = (z + k)^n$.

The inequality (1) has been refined by Aziz and Dawood [1] who under the same hypothesis proved that

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{2} \{\max_{|z|=1} |P(z)| + \min_{|z|=1} |P(z)|\}. \tag{3}$$

The result is best possible and equality in (3) holds for $P(z) = \alpha z^n + \beta$, where $|\beta| \leqslant |\alpha|$.

In the literature, there exists some extensions and generalizations of inequalities (1), (2) and (3) (for reference see [4] and [7]). Aziz and Shah [2] have generalized the inequality (1) by proving the following result.

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THEOREM A. *If $P(z)$ is a polynomial of degree n having all its zeros in the disk $|z| \leqslant k \leqslant 1$ with s-fold zero at the origin, $0 < s \leqslant n$, then*

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n+ks}{1+k} \max_{|z|=1} |P(z)|.$$

The result is sharp and the extremal polynomial is $P(z) = z^s(z+k)^{n-s}$.

Recently, Aziz and Zargar [3] have obtained the following refinement of Theorem A.

THEOREM B. *If $P(z)$ is a polynomial of degree n, having all its zeros in the disk $|z| \leqslant k$, $k \leqslant 1$ with t-fold zero at the origin, $0 < t \leqslant n$, then*

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n+kt}{1+k} \max_{|z|=1} |P(z)| + \frac{n-t}{(1+k)k^t} \min_{|z|=k} |P(z)|. \tag{4}$$

The result is sharp and equality in (4) holds for the polynomial $P(z) = z^t(z+k)^{n-t}$.

In this paper, we shall first present the following generalization of Theorem B (which is obtained as a special case for $R = 1$).

THEOREM 1. *If $P(z)$ is a polynomial of degree n having all its zeros in the disk $|z| \leqslant k$, $k \leqslant 1$ with t-fold zero at the origin, $0 \leqslant t \leqslant n$, then for every $R \geqslant k$,*

$$\max_{|z|=R} |P'(z)| \geqslant \frac{nR+kt}{R(R+k)} \max_{|z|=R} |P(z)| + \frac{R^{t-1}}{k^t} \left(\frac{nR+kt}{R+k} - t \right) \min_{|z|=k} |P(z)|.$$

The result is best possible and equality holds for the polynomial $P(z) = z^t(z+k)^{n-t}$.

The following result follows by taking $R = k$ in Theorem 1.

COROLLARY 1. *If $P(z)$ is a polynomial of degree n having all its zeros in the disk $|z| \leqslant k$, $0 < k \leqslant 1$ with t-fold zero at the origin, $0 \leqslant t \leqslant n$, then*

$$\max_{|z|=k} |P'(z)| \geqslant \frac{1}{2k} \{ (n+t) \max_{|z|=k} |P(z)| + (n-t) \min_{|z|=k} |P(z)| \}. \tag{5}$$

The result is best possible with equality for the polynomial $P(z) = z^t(z+k)^{n-t}$.

Note that the inequality (3) follows from (5) by taking $k = 1$ and $t = 0$.

We next present the following generalization of Theorem 1 which includes Theorem B as a special case.

THEOREM 2. *If $P(z)$ is a polynomial of degree n having all its zeros in the disk $|z| \leqslant k$, $0 < k \leqslant 1$ with t-fold zero at the origin, $0 \leqslant t \leqslant n$, then for $r \leqslant R$, $rR \geqslant k^2$,*

$$\max_{|z|=R} |P'(z)| \geqslant \frac{R^{t-1}}{r^t} \frac{nR+kt}{R+k} \left(\frac{R+k}{r+k} \right)^{n-t} \max_{|z|=r} |P(z)| + \frac{R^{t-1}}{k^t} \left(\frac{nR+kt}{R+k} - t \right) \min_{|z|=k} |P(z)|. \tag{6}$$

The result is best possible and equality in (6) holds for the polynomial $P(z) = cz^t(z+k)^{n-t}$, $c \neq 0$.

Finally, we present the following compact generalization of inequalities (4) and (5), which is an improvement of Theorem 2 and yields a number of other interesting results for various choices of parameters t, r and R.

THEOREM 3. *If* $P(z)$ *is a polynomial of degree* n *having all its zeros in the disk* $|z| \leqslant k$, $0 < k \leqslant 1$ *with* t*-fold zero at the origin,* $0 \leqslant t \leqslant n$, *then for* $r \leqslant R$, $rR \geqslant k^2$,

$$\max_{|z|=R} |P'(z)| \geqslant \left(\frac{R+k}{r+k}\right)^{n-t} \Bigg[\frac{R^{t-1}}{r^t}\frac{nR+kt}{R+k}\max_{|z|=r}|P(z)| + \frac{R^{t-1}}{k^t}\Big(\frac{nR+kt}{R+k} - t\Big(\frac{r+k}{R+k}\Big)^{n-t}\Big)\min_{|z|=k}|P(z)|\Bigg]. \quad (7)$$

The result is best possible and equality in (7) holds for the polynomial $P(z) = cz^t(z+k)^{n-t}$, $c \neq 0$.

Since $n \geqslant t$ and $R \geqslant r$, we see that

$$\frac{nR+kt}{R+k} \geqslant t \geqslant t\left(\frac{r+k}{R+k}\right)^{n-t}.$$

This implies

$$\frac{nR+kt}{R+k} - t\left(\frac{r+k}{R+k}\right)^{n-t} \geqslant 0.$$

Using this fact in (7), the following result immediately follows from Theorem 3.

COROLLARY 2. *If* $P(z)$ *is a polynomial of degree* n *having all its zeros in the disk* $|z| \leqslant k$, $0 < k \leqslant 1$ *with* t*-fold zero at the origin,* $0 \leqslant t \leqslant n$, *then for* $r \leqslant R$, $rR \geqslant k^2$,

$$\max_{|z|=R}|P'(z)| \geqslant \left(\frac{R+k}{r+k}\right)^{n-t}\left[\frac{R^{t-1}}{r^t}\frac{nR+kt}{R+k}\max_{|z|=r}|P(z)|\right]. \quad (8)$$

The result is sharp and equality in (8) holds for the polynomial $P(z) = z^t(z+k)^{n-t}$.

If we take $t = 0$ in Theorem 3, we obtain

COROLLARY 3. *If* $P(z)$ *is a polynomial of degree* n *having all its zeros in the disk* $|z| \leqslant k$, $0 < k \leqslant 1$, *then for* $r \leqslant R$, $Rr \geqslant k^2$,

$$\max_{|z|=R}|P'(z)| \geqslant \left(\frac{R+k}{r+k}\right)^n\left[\frac{n}{R+k}\max_{|z|=r}|P(z)| + \frac{n}{R+k}\min_{|z|=k}|P(z)|\right].$$

The following result follows by taking $r = 1$ in Theorem 3.

Corollary 4. *If $P(z)$ is a polynomial of degree n having all its zeros in the disk $|z| \leqslant k$, $0 < k \leqslant 1$ with t-fold zero at the origin, $0 \leqslant t \leqslant n$, then for $k \leqslant R$,*

$$\max_{|z|=R} |P'(z)| \geqslant \left(\frac{R+k}{1+k}\right)^{n-t} \Bigg[R^{t-1} \frac{nR+kt}{R+k} \max_{|z|=1} |P(z)| + \frac{R^{t-1}}{k^t}\Big(\frac{nR+kt}{R+k} - t\Big(\frac{1+k}{R+k}\Big)^{n-t}\Big) \min_{|z|=k} |P(z)|\Bigg].$$

For the proofs of Theorems 2 and 3, we need the following lemma, which may be of independent interest.

Lemma. *If $P(z)$ is a polynomial of degree n, having all its zeros in $|z| \leqslant k$, $k > 0$ with t-fold zero at the origin, then for $|z| = 1$, $rR \geqslant k^2$ and $r \leqslant R$,*

$$|P(rz)| \leqslant \frac{r^t}{R^t} \left(\frac{r+k}{R+k}\right)^{n-t} |P(Rz)|. \tag{9}$$

Equality in (9) holds for the polynomial $P(z) = z^t(z+k)^{n-t}$.

Proof. Since $P(z)$ has all of its zeros in $|z| \leqslant k$ and t-fold zero at the origin, we can write

$$P(z) = z^t H(z), \tag{10}$$

where $H(z)$ is a polynomial of degree $n-t$ having all of its zeros in $|z| \leqslant k$, so that

$$H(z) = c \prod_{j=1}^{n-t} (z - R_j e^{i\theta_j}),$$

where $R_j \leqslant k$, $j = 1, 2, \ldots, n-t$. This implies that for each θ, $0 \leqslant \theta < 2\pi$,

$$\left|\frac{H(re^{i\theta})}{H(Re^{i\theta})}\right| = \prod_{j=1}^{n-t} \left|\frac{re^{i(\theta-\theta_j)} - R_j}{Re^{i(\theta-\theta_j)} - R_j}\right|. \tag{11}$$

Now for $R \geqslant r$, $Rr \geqslant R_j^2$ and for each θ, $0 \leqslant \theta < 2\pi$, it can be easily verified that

$$\left|\frac{re^{i(\theta-\theta_j)} - R_j}{Re^{i(\theta-\theta_j)} - R_j}\right|^2 \leqslant \left(\frac{r+R_j}{R+R_j}\right)^2.$$

Since $R_j \leqslant k$ for all $j = 1, 2, \ldots, n-t$, it follows from (11) that if $r \leqslant R$ and $rR \geqslant k^2$, then

$$\left|\frac{H(re^{i\theta})}{H(Re^{i\theta})}\right| \leqslant \left(\frac{r+k}{R+k}\right)^{n-t}.$$

Using (10), it follows that

$$\left|\frac{P(re^{i\theta})}{P(Re^{i\theta})}\right| = \frac{r^t}{R^t} \left|\frac{H(re^{i\theta})}{H(Re^{i\theta})}\right| \leqslant \frac{r^t}{R^t} \left(\frac{r+k}{R+k}\right)^{n-t}.$$

Hence, for $R \geqslant r$, $Rr \geqslant k^2$ and for each θ, $0 \leqslant \theta < 2\pi$, we have

$$|P(re^{i\theta})| \leqslant \frac{r^t}{R^t} \left(\frac{r+k}{R+k}\right)^{n-t} |P(Re^{i\theta})|$$

wherefrom the desired result follows immediately. ∎

2. Proofs of the theorems

Proof of Theorem 1. Let $m = \min_{|z|=k} |P(z)|$. Then $m \leqslant |P(z)|$ for $|z| = k$ gives $m|\frac{z}{k}|^t \leqslant |P(z)|$ for $|z| = k$. Since all the zeros of $P(z)$ lie in $|z| \leqslant k \leqslant 1$ with t-fold zero at the origin, it follows (by Rouché's Theorem for $m > 0$) that for every complex number α such that $|\alpha| < 1$, the polynomial $G(z) = P(z) + \frac{\alpha m}{k^t} z^t$ has all of its zeros in $|z| \leqslant k$ with t-fold zero at the origin. Hence, the polynomial $F(z) = G(Rz)$ has all of its zeros in $|z| \leqslant \frac{k}{R} \leqslant 1$, with t-fold zero at the origin, so that we can write

$$F(z) = z^t H(z), \tag{12}$$

where $H(z)$ is a polynomial of degree $n - t$, having all of its zeros in $|z| \leqslant \frac{k}{R} \leqslant 1$. From (12), we have

$$\frac{zF'(z)}{F(z)} = t + \frac{zH'(z)}{H(z)}. \tag{13}$$

If $z_1, z_2, \ldots, z_{n-t}$ are the zeros of $H(z)$, then $|z_j| \leqslant \frac{k}{R} \leqslant 1$ for all $j = 1, 2, \ldots, n-t$, and from (13), we obtain

$$\operatorname{Re}\left\{\frac{e^{i\theta}F'(e^{i\theta})}{F(e^{i\theta})}\right\} = t + \operatorname{Re}\left\{\frac{e^{i\theta}H'(e^{i\theta})}{H(e^{i\theta})}\right\} = t + \sum_{j=1}^{n-t} \operatorname{Re}\left(\frac{1}{1 - z_j e^{-i\theta}}\right) \tag{14}$$

for points $e^{i\theta}$, $0 \leqslant \theta < 2\pi$ which are not zeros of $H(z)$.

Now, if $|w| \leqslant \frac{k}{R} \leqslant 1$, then it can be easily verified that $\operatorname{Re}\left(\frac{1}{1-w}\right) \geqslant \frac{1}{1+\frac{k}{R}}$. Using this fact in (14), we see that

$$\left|\frac{F'(e^{i\theta})}{F(e^{i\theta})}\right| \geqslant \operatorname{Re}\left\{e^{i\theta}\frac{F'(e^{i\theta})}{F(e^{i\theta})}\right\} \geqslant t + \frac{n-t}{1+\frac{k}{R}} = \frac{tk+nR}{R+k}$$

for points $e^{i\theta}$, $0 \leqslant \theta < 2\pi$ which are not zeros of $H(z)$. This implies that

$$|F'(e^{i\theta})| \geqslant \frac{tk+nR}{R+k}|F(e^{i\theta})| \tag{15}$$

for points $e^{i\theta}$, $0 \leqslant \theta < 2\pi$, other than zeros of $F(z)$. Since (15) is trivially true for points $e^{i\theta}$ which are the zeros of $F(z)$, it follows that

$$|F'(z)| \geqslant \frac{tk+nR}{R+k}|F(z)| \quad \text{for} \quad |z| = 1. \tag{16}$$

Replacing $F(z)$ by $G(Rz)$ in (16), we get

$$|G'(Rz)| \geqslant \frac{tk+nR}{R(R+k)}|G(Rz)| \quad \text{for} \quad |z| = 1. \tag{17}$$

Using that $G(z) = P(z) + \frac{\alpha m}{k^t} z^t$, it follows that

$$\left|P'(Rz) + \frac{\alpha m t R^{t-1}}{k^t} z^{t-1}\right| \geqslant \frac{tk+nR}{R(R+k)}\left|P(Rz) + \frac{\alpha m R^t}{k^t} z^t\right| \tag{18}$$

for $|z| = 1$ and for every α, $|\alpha| < 1$. Choosing the argument on the RHS of (18) such that

$$\left|P(Rz) + \frac{\alpha m R^t}{k^t} z^t\right| = |P(Rz)| + \frac{|\alpha| m R^t}{k^t} \quad \text{for} \quad |z| = 1,$$

from (18), we obtain

$$|P'(Rz)| + \frac{mtR^{t-1}}{k^t}|\alpha| \geqslant \frac{tk+nR}{R(R+k)}\left\{|P(Rz)| + \frac{|\alpha| mR^t}{k^t}\right\}$$

for $|z|=1$ and $|\alpha|<1$. Letting $|\alpha|\to 1$, we conclude that

$$|P'(Rz)| \geqslant \frac{tk+nR}{R(R+k)}|P(Rz)| + \frac{R^{t-1}}{k^t}\left\{\frac{tk+nR}{R+k} - t\right\} m \tag{19}$$

for $|z|=1$, which gives

$$\max_{|z|=R}|P'(z)| \geqslant \frac{tk+nR}{R(R+k)}\max_{|z|=R}|P(z)| + \frac{R^{t-1}}{k^t}\left\{\frac{tk+nR}{R+k} - t\right\}\min_{|z|=k}|P(z)|.$$

This completes the proof of Theorem 1. ■

Proof of Theorem 2. Proceeding similarly as in the proof of Theorem 1, it follows from (19) that

$$|P'(Rz)| \geqslant \frac{tk+nR}{R(R+k)}|P(Rz)| + \frac{R^{t-1}}{k^t}\left\{\frac{tk+nR}{R+k} - t\right\} m$$

for $|z|=1$. Applying the above Lemma, it follows that

$$|P'(Rz)| \geqslant \frac{tk+nR}{R(R+k)}\frac{R^t}{k^t}\left(\frac{R+k}{r+k}\right)^{n-t}|P(rz)| + \frac{R^{t-1}}{k^t}\left\{\frac{tk+nR}{R+k} - t\right\} m$$

for $|z|=1$. This implies that

$$\max_{|z|=R}|P'(z)| \geqslant \frac{R^{t-1}}{r^t}\frac{tk+nR}{R+k}\left(\frac{R+k}{r+k}\right)^{n-t}\max_{|z|=r}|P(z)| + \frac{R^{t-1}}{k^t}\left(\frac{tk+nR}{R+k} - t\right)\min_{|z|=k}|P(z)|,$$

which completes the proof of Theorem 2. ■

Proof of Theorem 3. We proceed similarly as in the proof of Theorem 1. It follows from (17) that

$$|G'(Rz)| \geqslant \frac{tk+nR}{R(R+k)}|G(Rz)| \quad \text{for} \quad |z|=1.$$

Now, applying the above Lemma to $G(z)$, we get

$$|G'(Rz)| \geqslant \frac{tk+nR}{R(R+k)}\frac{R^t}{r^t}\left(\frac{R+k}{r+k}\right)^{n-t}|G(rz)| \quad \text{for} \quad |z|=1, \tag{20}$$

where $r\leqslant R$ and $rR\geqslant k^2$. Since $G(z)=P(z)+\frac{\alpha m}{k^t}z^t$, it follows from (20) that

$$\left|P'(Rz) + \frac{\alpha mtR^{t-1}}{k^t}z^{t-1}\right| \geqslant \frac{tk+nR}{R(R+k)}\frac{R^t}{r^t}\left(\frac{R+k}{r+k}\right)^{n-t}\left|P(rz) + \frac{\alpha mt}{k^t}(rz)^t\right| \tag{21}$$

for $|z|=1$ and for every α with $|\alpha|<1$. Choosing the argument of α such that

$$\left|P(rz) + \frac{\alpha mt}{k^t}(rz)^t\right| = |P(rz)| + |\alpha|\frac{m}{k^t}r^t \quad \text{for} \quad |z|=1,$$

it follows from (21) that

$$|P'(Rz)| \geqslant \frac{tK+nR}{R(R+k)}\left(\frac{R+k}{r+k}\right)^{n-t}\frac{R^t}{r^t}|P(rz)| + \frac{|\alpha|}{k^t}R^{t-1}\left[\frac{tk+nR}{R+k}\left(\frac{R+k}{r+k}\right)^{n-t}-t\right]m$$

for $|z|=1$. Letting $|\alpha|\to 1$, we get

$$|P'(Rz)| \geqslant \frac{tk+nR}{R(R+k)}\left(\frac{R+k}{r+k}\right)^{n-t}\frac{R^t}{r^t}|P(rz)| + \frac{R^{t-1}}{k^t}\left(\frac{R+k}{r+k}\right)^{n-t}\left[\frac{tk+nR}{R+k}-t\left(\frac{r+k}{R+k}\right)^{n-t}\right]m$$

for $|z|=1$. This implies that

$$\max_{|z|=R}|P'(z)| \geqslant \left(\frac{R+k}{r+k}\right)^{n-t}\left\{\frac{R^{t-1}(tk+nR)}{r^t(R+k)}\max_{|z|=r}|P(z)| + \frac{R^{t-1}}{k^t}\left[\frac{tk+nR}{R+k}-t\Big(\frac{r+k}{R+k}\Big)^{n-t}\right]\min_{|z|=k}|P(z)|\right\},$$

which proves the desired result. ∎

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