МАТЕМАТИЧКИ ВЕСНИК
66, 2 (2014), 155–164
June 2014

оригинални научни рад
research paper

APPROXIMATION OF FUNCTIONS BELONGING TO THE GENERALIZED LIPSCHITZ CLASS BY $C^1 \cdot N_p$ SUMMABILITY METHOD OF CONJUGATE SERIES OF FOURIER SERIES

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Abstract. In the present study, a new theorem on the degree of approximation of function $\tilde{f}$, conjugate to a periodic function f belonging to weighted $W(L_r, \xi(t))$-class using semi-monotonicity on the generating sequence $\{p_n\}$ has been established.

1. Introduction

In 1941, Alexits [1] (later Zygmund [22] and Zamansky [20], too) proved a very interesting result pertaining to the degree of approximation of conjugate functions. The degree of approximation of functions belonging to $\mathrm{Lip}\,\alpha$, $\mathrm{Lip}(\alpha, r)$, $\mathrm{Lip}(\xi(t), r)$ and $W(L_r, \xi(t))$-classes, $(r \geq 1)$ by Nörlund (N_p) matrices and general summability matrices has been proved by various investigators like Khan [6], Mohapatra and Sahney [15,16], Qureshi [18], Mohapatra and Chandra [12–14], Holland et al. [5], Das et al. [3], Mittal et al. [9-11], Chandra [2], Leindler [8], Rhoades et al. [19] and Nigam and Sharma [17]. Recently, Lal [7] has proved a theorem on the degree of approximation of function f belonging to weighted $W(L_r, \xi(t))$-class by $C^1 \cdot N_p$ summability method of its Fourier series of a 2π-periodic function f where $\xi(t)$ is a positive increasing function in t. Lal [7] has assumed monotonicity on the generating sequence $\{p_n\}$. The approximation of function $\tilde{f}$, conjugate to a periodic function $f \in W(L_r, \xi(t))$ $(r \geq 1)$ using product $C^1 \cdot N_p$-summability has not been studied so far. In this paper, we obtain a new theorem on the degree of approximation of function $\tilde{f}$, conjugate to a periodic function $f \in W(L_r, \xi(t))$-class using semi-monotonicity on the generating sequence $\{p_n\}$.

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with the sequence of n^{th} partial sums $\{s_n\}$. Let $\{p_n\}$ be a non-negative sequence of constants, real or complex, and let

2010 Mathematics Subject Classification: 40G05, 42B05, 42B08.

Keywords and phrases: Generalized Lipschitz $W(L_r, \xi(t))$-class of functions; conjugate Fourier series; degree of approximation; C^1 means; N_p means; product summability $C^1 \cdot N_p$ transform.

us write

$$P_n = \sum_{k=0}^{n} p_k \neq 0 \ \forall\, n \geq 0,\ p_{-1} = 0 = P_{-1} \text{ and } P_n \to \infty \text{ as } n \to \infty.$$

The sequence to sequence transformation $\tilde{t}_n^N = \sum_{\nu=0}^{n} \frac{p_{n-\nu}\tilde{s}_n}{P_n}$ defines the sequence $\{\tilde{t}_n^N\}$ of Nörlund means of the sequence $\{\tilde{s}_n\}$, generated by the sequence of coefficients $\{p_n\}$. The series $\sum_{n=0}^{\infty} a_n$ is said to be N_p summable to the sum s if $\lim_{n\to\infty} \tilde{t}_n^N$ exists and is equal to s. In the special case in which

$$p_n = \binom{n+\alpha-1}{\alpha-1} = \frac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma(\alpha)};\ (\alpha > 0),$$

the Nörlund summability N_p reduces to the familiar C^α summability.

The product of C^1 summability with a N_p summability defines $C^1 \cdot N_p$ summability. Thus the $C^1 \cdot N_p$ mean is given by $\tilde{t}_n^{CN}(f) = \frac{1}{n+1}\sum_{k=0}^{n} P_k^{-1} \sum_{\nu=0}^{k} p_{k-\nu}\tilde{s}_\nu(f)$.

If $\tilde{t}_n^{CN}(f) \to s$ as $n \to \infty$, then the infinite series $\sum_{n=0}^{\infty} a_n$ or the sequence $\{\tilde{s}_n\}$ is said to be $C^1 \cdot N_p$ summable to the sum s.

$$\begin{aligned}
\tilde{s}_n \to s &\implies N_p(\tilde{s}_n) = \tilde{t}_n^N = P_n^{-1} \sum_{\nu=0}^{n} p_{n-\nu}\tilde{s}_n \to s,\ \text{as } n \to \infty,\ N_p \text{ method is regular,}\\
&\implies C^1(N_p(\tilde{s}_n)) = \tilde{t}_n^{CN} \ \to s,\ \text{as } n \to \infty, C^1 \text{ method is regular,}\\
&\implies C^1 \cdot N_p \text{ method is regular.}
\end{aligned}$$

Let $f(x)$ be a 2π-periodic and Lebesgue integrable function. The Fourier series of $f(x)$ is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(f;x) \tag{1.1}$$

with n-th partial sums $s_n(f;x)$.

The conjugate series of Fourier series (1.1) is given by

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(f;x). \tag{1.2}$$

A function $f(x) \in \operatorname{Lip} \alpha$ if

$$|f(x+t) - f(x)| = O(|t|^{\,\alpha}) \text{ for } 0 \leq \alpha \leq 1,\ t \geq 0,$$

and $f(x) \in \operatorname{Lip}(\alpha, r)$ for $0 \leq x \leq 2\pi$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r \, dx \right)^{1/r} = O(|t|^\alpha),\ 0 \leq \alpha \leq 1,\ r \geq 1,\ t \geq 0,$$

$f(x) \in \operatorname{Lip}(\xi(t), r)$ if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r \, dx \right)^{1/r} = O(\xi(t)),\ r \geq 1,\ t \geq 0,$$

$f(x) \in W(L_r, \xi(t))$ [19] if

$$\omega_r(t; f) = \left(\int_0^{2\pi} |(f(x+t) - f(x)) \sin^{\beta}(x/2)|^r \, dx \right)^{1/r} = O(\xi(t)),$$

$\beta \geq 0,\ r \geq 1,\ t \geq 0$, where $\xi(t)$ is a positive increasing function of t.

If $\beta = 0$ then $W(L_r, \xi(t))$ reduces to the class $Lip(\xi(t), r)$, if $\xi(t) = t^{\alpha}$, $(0 \leq \alpha \leq 1)$ then $\mathrm{Lip}(\xi(t), r)$ class coincides with the class $\mathrm{Lip}(\alpha, r)$ and if $r \to \infty$ then $\mathrm{Lip}(\alpha, r)$ reduces to the class $\mathrm{Lip}\,\alpha$.

L_∞-norm of a function $f\colon R \to R$ is defined by $\|f\|_\infty = \sup\{|f(x)| : x \in R\}$. L_r-norm of f is defined by $\|f\|_r = \left(\int_0^{2\pi} |f(x)|^r \, dx \right)^{1/r}$, $r \geq 1$.

The degree of approximation of a function $f\colon R \to R$ by trigonometric polynomial t_n of order n under sup norm $\| \, \|_\infty$ is defined by [21]: $\|t_n - f\|_\infty = \sup\{|t_n - f(x)| : x \in R\}$, and $E_n(f)$ of a function $f \in L_r$ is given by

$$E_n(f) = \min_n \|t_n(f) - f(x)\|_r.$$

The conjugate function $\tilde{f}(x)$ is defined for almost every x by

$$\tilde{f}(x) = \frac{-1}{2\pi} \int_0^{\pi} \psi(t) \cot(t/2) \, dt = \lim_{h \to 0} \left(\frac{-1}{2\pi} \int_h^{\pi} \psi(t) \cot(t/2) \, dt \right).$$

We note that $\tilde{t}_n^N$ and $\tilde{t}_n^{CN}$ are also trigonometric polynomials of degree (or order) n and the series, conjugate to a Fourier series, is not necessarily a Fourier series [21]. Hence a separate study of conjugate series is desirable and attracted the attention of researchers.

Abel's Transformation: The formula

$$\sum_{k=m}^{n} u_k v_k = \sum_{k=m}^{n-1} U_k (v_k - v_{k+1}) - U_{m-1} v_m + U_n v_n, \tag{1.3}$$

where $0 \leq m \leq n$, $U_k = u_0 + u_1 + u_2 + \cdots + u_k$, if $k \geq 0$, $U_{-1} = 0$, which can be verified, is known as Abel's transformation and will be used extensively in what follows.

If $v_m, v_{m+1}, \ldots, v_n$ are non-negative and non-increasing, the left-hand side of (1.3) does not exceed $2v_m \max_{m-1 \leq k \leq n} |U_k|$ in absolute value. In fact,

$$\Big| \sum_{k=m}^{n} u_k v_k \Big| \leq \max |U_k| \Big\{ \sum_{k=m}^{n-1} (v_k - v_{k+1} + v_m + v_n \Big\} = 2v_m \max |U_k|. \tag{1.4}$$

We write throughout

$$\psi(t) = f(x+t) - f(x-t), \quad W_n = \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} P_k^{-1} \sum_{\nu=0}^{k} (\nu+1) \, |p_\nu - p_{\nu-1}|,$$

$$\tilde{J}(n,t) = \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} P_k^{-1} \sum_{\nu=0}^{k} p_\nu \frac{\cos(k - \nu + 1/2)t}{\sin(t/2)}, \tag{1.5}$$

$\tau = [1/t]$, where τ denotes the greatest integer not exceeding $1/t$. Furthermore, C denotes an absolute positive constant, not necessarily the same at each occurrence.

2. Main theorem

In this section we state our main result.

THEOREM 1. *Let $\tilde{f}$ be the conjugate to a 2π-periodic function f belonging to $W(L_r, \xi(t))$-class. Then its degree of approximation by $C^1 \cdot N_p$ means of conjugate series of Fourier series* (1.2) *is given by*

$$\|\tilde{t}_n^{CN}(f) - \tilde{f}(x)\|_r = O\Big((n+1)^{\beta+1/r}\xi\Big(\frac{1}{n+1}\Big)\Big), \tag{2.1}$$

provided $\{p_n\}$ satisfies

$$W_n < C, \tag{2.2}$$

and $\xi(t)$ satisfies the following conditions:

$$\{\xi(t/t)\} \text{ is non-increasing in } t, \tag{2.3}$$

$$\bigg(\int_0^{\pi/(n+1)} \Big(\frac{t\,|\psi(t)|}{\xi(t)}\Big)^r \sin^{\beta r}(t/2)\,dt\bigg)^{1/r} = O((n+1)^{-1}) \text{ and} \tag{2.4}$$

$$\bigg(\int_{\pi/(n+1)}^{\pi} \Big(\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\Big)^r dt\bigg)^{1/r} = O((n+1)^{\delta}), \tag{2.5}$$

where δ is an arbitrary number such that $s(\beta-\delta)-1 \ge 0$, $r^{-1}+s^{-1}=1$, $1 \le r \le \infty$; conditions (2.4) *and* (2.5) *hold uniformly in x.*

REMARK 1. $\xi(\frac{\pi}{n+1}) \le \pi\,\xi(\frac{1}{n+1})$, for $(\frac{\pi}{n+1}) \ge (\frac{1}{n+1})$.

REMARK 2. Condition $W_n < C$ implies $(n+1)p_n < CP_n$ [4].

REMARK 3. The product transform $C^1 \cdot N_p$ plays an important role in signal theory as a double digital filter [11] and theory of machines in Mechanical Engineering.

REMARK 4. The condition $1/\sin^{\beta}(t) = O(1/t^{\beta})$, $1/(n+1) \le t \le \pi$ used by Lal [7] is not valid since $\sin t \to 0$ as $t \to \pi$.

REMARK 5. There is a fatal error in the proof of Theorem 2 of Lal [7, p. 349]. In the calculation of $|I_1|$ the author of [7] obtains

$$\int_{\epsilon}^{1/(n+1)} \frac{dt}{t^{(1+\beta)s}} = \Big[\frac{t^{1-\beta s-s}}{1-\beta s-s}\Big] \text{ for some } 0 < \epsilon < \frac{1}{n+1};$$

note that $-\beta s - s + 1 < 0$. Therefore one has $\frac{1}{\beta s+s-1}\Big[\frac{1}{\epsilon^{\beta s+s-1}} - (n+1)^{\beta s+s-1}\Big]$, which need not be $O\big((n+1)^{\beta s+s-1}\big)$, since ϵ might be $O(1/n^{\gamma})$ for some $\gamma > 1$.

3. Lemmas

We need the following lemmas for the proof of our theorem.

LEMMA 1. $|\tilde{J}(n,t)| = O(\tau)$ *for* $0 < t \le \pi/(n+1)$.

Proof. For $0 < t \le \pi/(n+1)$, $\sin(t/2) \ge (t/\pi)$ and $|\cos nt| \le 1$, and we have

$$\begin{aligned}
|\tilde{J}(n,t)| &= \left|\frac{1}{2\pi(n+1)} \sum_{k=0}^{n} P_k^{-1} \sum_{\nu=0}^{k} p_\nu \frac{\cos(k-\nu+1/2)t}{\sin(t/2)}\right| \\
&\le \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} P_k^{-1} \sum_{\nu=0}^{k} p_\nu \frac{|\cos(k-\nu+1/2)t|}{|\sin(t/2)|} \\
&\le \frac{1}{2t(n+1)} \sum_{k=0}^{n} P_k^{-1} \sum_{\nu=0}^{k} p_\nu = \frac{1}{2t(n+1)} \sum_{k=0}^{n} P_k^{-1} P_k = O(\tau).
\end{aligned}$$

This completes the proof of Lemma 1. ∎

Lemma 2. *Let $\{p_n\}$ be a non-negative sequence satisfying* (2.2). *Then*

$$|\tilde{J}(n,t)| = O(\tau) + O\Big(\frac{\tau^2}{(n+1)}\Big)\Big(\sum_{k=\tau}^{n} P_k^{-1} \sum_{\nu=0}^{k-1} |\Delta p_\nu|\Big) \text{ uniformly in } 0 < t \le \pi. \tag{3.1}$$

Proof. We have

$$\begin{aligned}
\tilde{J}(n,t) &= \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} P_k^{-1} \sum_{\nu=0}^{k} p_\nu \frac{\cos(k-\nu+1/2)t}{\sin(t/2)} \\
&= \frac{1}{2\pi(n+1)} \left(\sum_{k=0}^{\tau-1} + \sum_{k=\tau}^{n}\right) P_k^{-1} \sum_{\nu=0}^{k} p_\nu \frac{\cos(k-\nu+1/2)t}{\sin(t/2)} \\
&= \tilde{J}_1(n,t) + \tilde{J}_2(n,t),\ (say),
\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}
|\tilde{J}_1(n,t)| &= \left|\frac{1}{2\pi(n+1)} \sum_{k=0}^{\tau-1} P_k^{-1} \sum_{\nu=0}^{k} p_\nu \frac{\cos(k-\nu+1/2)t}{\sin(t/2)}\right| \\
&\le \frac{1}{2\pi(n+1)} \sum_{k=0}^{\tau-1} P_k^{-1} \sum_{\nu=0}^{k} p_\nu \frac{|\cos(k-\nu+1/2)t|}{|\sin(t/2)|} \\
&\le \frac{1}{2t(n+1)} \sum_{k=0}^{\tau-1} P_k^{-1} \sum_{\nu=0}^{k} p_\nu = O\Big(\tfrac{\tau^2}{(n+1)}\Big),
\end{aligned} \tag{3.3}$$

and using Abel's transformation and $\sin(t/2) \ge (t/\pi)$, for $0 < t \le \pi$, we get

$$\begin{aligned}
|\tilde{J}_2(n,t)| &= \left|\frac{1}{2\pi(n+1)} \sum_{k=\tau}^{n} P_k^{-1} \sum_{\nu=0}^{k} p_\nu \frac{\cos(k-\nu+1/2)t}{\sin(t/2)}\right| \\
&\le \frac{1}{2t(n+1)} \sum_{k=\tau}^{n} P_k^{-1} \left\{\sum_{\nu=0}^{k-1} |\Delta p_\nu| \left|\left(\sum_{\gamma=0}^{\nu} \cos(k-\gamma+1/2)t\right)\right|\right. \\
&\quad \left. + \left|\left(\sum_{\gamma=0}^{k} \cos(k-\gamma+1/2)t\right)\right| p_k\right\} \\
&= \frac{O(t^{-1})}{2t(n+1)} \left(\sum_{k=\tau}^{n} P_k^{-1} \sum_{\nu=0}^{k-1} |\Delta p_\nu| + \sum_{k=\tau}^{n} P_k^{-1} p_k\right),
\end{aligned}$$

by virtue of the fact that $\sum_{k=\lambda}^{\mu} \exp(-ikt) = O(t^{-1})$, $0 \le \lambda \le k \le \mu$. Hence,

$$\begin{aligned}|\tilde{J}_2(n,t)| &= O\Big(\frac{\tau^2}{(n+1)}\Big)\Big(\sum_{k=\tau}^{n} P_k^{-1}\sum_{\nu=0}^{k-1}|\Delta p_\nu| + \sum_{k=\tau}^{n} P_k^{-1} p_k \tfrac{(k+1)}{(k+1)}\Big)\\ &= O\Big(\frac{\tau^2}{(n+1)}\Big)\Big(\sum_{k=\tau}^{n} P_k^{-1}\sum_{\nu=0}^{k-1}|\Delta p_\nu| + \frac{(n+1)}{\tau}\Big),\\ |\tilde{J}(n,t)| &= O(\tau) + O\Big(\frac{\tau^2}{(n+1)}\Big)\Big(\sum_{k=\tau}^{n} P_k^{-1}\sum_{\nu=0}^{k-1}|\Delta p_\nu|\Big), \qquad (3.4)\end{aligned}$$

in view of Remark 2. Combining (3.2)–(3.4) yields (3.1). This completes the proof of Lemma 2. ∎

4. Proof of Theorem 1

Let $\tilde{s}_n(f;x)$ denote the partial sum of series (1.2). We have

$$\tilde{s}_n(f;x) - \tilde{f}(x) = \frac{1}{2\pi}\int_0^{\pi} \psi(t)\frac{\cos(n+1/2)t}{\sin(t/2)}\,dt.$$

Denoting $C^1 \cdot N_p$ means of $\tilde{s}_n(f;x)$ by $\tilde{t}_n^{CN}(f)$, we write

$$\begin{aligned}\tilde{t}_n^{CN}(f) - \tilde{f}(x) &= \int_0^{\pi} \psi(t)\frac{1}{2\pi(n+1)}\sum_{k=0}^{n} P_k^{-1}\sum_{\nu=0}^{k} p_\nu \frac{\cos(k-\nu+1/2)t}{\sin(t/2)}\,dt\\ &= \int_0^{\pi}\psi(t)\tilde{J}(n,t)\,dt\\ &= \Big[\int_0^{\pi/(n+1)} + \int_{\pi/(n+1)}^{\pi}\Big]\psi(t)\tilde{J}(n,t)\,dt\\ &= I_1 + I_2 \text{ say.} \qquad (4.1)\end{aligned}$$

Clearly,

$$|\psi(x+t) - \psi(t)| \le |f(u+x+t) - f(u+x)| + |f(u-x-t) - f(u-x)|.$$

Hence, by Minkowski's inequality, we have

$$\begin{aligned}&\Big(\int_0^{2\pi} |(\psi(x+t) - \psi(t))\sin^{\beta}(x/2)|^r\,dx\Big)^{1/r}\\ &\quad\le \Big(\int_0^{2\pi} |(f(u+x+t) - f(u+x))\sin^{\beta}(x/2)|^r\,dx\Big)^{1/r}\\ &\qquad + \Big(\int_0^{2\pi} |(f(u-x-t) - f(u-x))\sin^{\beta}(x/2)|^r\,dx)^{1/r}\\ &\quad= O(\xi(t)).\end{aligned}$$

Then $f \in W(L_r, \xi(t)) \Longrightarrow \psi(t) \in W(L_r, \xi(t))$.

Using Hölder's inequality, $\psi(t) \in W(L_r, \xi(t))$, condition (2.4), $\sin(t/2) \geq (t/\pi)$, for $0 < t \leq \pi$, Lemma 1, Remark 2 and Second Mean Value Theorem for integrals, we have

$$\begin{aligned}
|I_1| &\leq \left[\int_0^{\pi/(n+1)} \Big(\frac{t|\psi(t)\,\sin^\beta(t/2)|}{\xi(t)}\Big)^r dt\right]^{1/r} \left[\int_0^{\pi/(n+1)} \Big(\frac{\xi(t)|\tilde{J}(n,t)|}{t\sin^\beta(t/2)}\Big)^s dt\right]^{1/s} \\
&= O\Big(\frac{1}{n+1}\Big)\left[\int_0^{\pi/(n+1)} \Big(\frac{\xi(t)}{t^{2+\beta}}\Big)^s dt\right]^{1/s} \\
&= O\Big\{\Big(\frac{1}{n+1}\Big)\xi\Big(\frac{\pi}{n+1}\Big)\Big\}\left[\int_0^{\pi/(n+1)} \Big(\frac{1}{t^{2+\beta}}\Big)^s dt\right]^{1/s} \\
&= O\Big((n+1)^{\beta+1/r}\xi\Big(\frac{1}{n+1}\Big)\Big), \quad r^{-1}+s^{-1}=1. \qquad (4.2)
\end{aligned}$$

Using Lemma 2, we have

$$\begin{aligned}
|I_2| &= O\left[\int_{\pi/(n+1)}^{\pi} \frac{|\psi(t)|}{t}\,dt\right] + O\left[\int_{\pi/(n+1)}^{\pi} \frac{|\psi(t)|}{t(n+1)}\Big(\tau\sum_{k=\tau}^{n} P_k^{-1}\sum_{\nu=0}^{k-1}|\Delta p_\nu|\Big)\,dt\right] \\
&= O(I_{21}) + O(I_{22}).
\end{aligned}$$

Using Hölder's inequality, conditions (2.3) and (2.5), $|\sin t| \leq 1$, $\sin(t/2) \geq (t/\pi)$, for $0 < t \leq \pi$, Remark 2 and Second Mean Value Theorem for integrals, we have

$$\begin{aligned}
|I_{21}| &\leq \left[\int_{\pi/(n+1)}^{\pi} \Big(\frac{t^{-\delta}|\psi(t)|\sin^\beta(t/2)}{\xi(t)}\Big)^r dt\right]^{1/r} \left[\int_{\pi/(n+1)}^{\pi} \Big(\frac{\xi(t)}{t^{-\delta+1}\sin^\beta(t/2)}\Big)^s dt\right]^{1/s} \\
&= O\big((n+1)^\delta\big)\left[\int_{\pi/(n+1)}^{\pi} \Big(\frac{\xi(t)}{t^{-\delta+1+\beta}}\Big)^s dt\right]^{1/s} \\
&= O\big((n+1)^\delta\big)\left[\int_{1/\pi}^{(n+1)/\pi} \Big(\frac{\xi(1/y)}{y^{\delta-1-\beta}}\Big)^s \frac{dy}{y^2}\right]^{1/s} \\
&= O\Big((n+1)^\delta\frac{\xi\big(\frac{\pi}{n+1}\big)}{\pi/(n+1)}\Big)\left[\int_{1/\pi}^{(n+1)/\pi} \frac{dy}{y^{(\delta-\beta)s+2}}\right]^{1/s} \\
&= O\Big((n+1)^{\delta+1}\xi\Big(\frac{1}{n+1}\Big)\Big)\Big(\frac{(n+1)^{(\beta-\delta)s-1}-(\pi)^{(-\beta+\delta)s+1}}{(\beta-\delta)s-1}\Big)^{1/s} \\
&= O\Big((n+1)^{\beta+1/r}\xi\Big(\frac{1}{n+1}\Big)\Big), \quad r^{-1}+s^{-1}=1. \qquad (4.3)
\end{aligned}$$

Similarly as above, using conditions (2.2), (2.3) and (2.5), $|\sin t| \leq 1$, $\sin(t/2) \geq (t/\pi)$, for $0 < t \leq \pi$, Remark 2 and Second Mean Value Theorem for integrals, we have

$$\begin{aligned}
|I_{22}| &\leq \left[\int_{\pi/(n+1)}^{\pi} \Big(\frac{t^{-\delta}|\psi(t)|\sin^\beta(t/2)}{\xi(t)}\Big)^r dt\right]^{1/r} \\
&\quad\times \left[\int_{\pi/(n+1)}^{\pi} \Big(\frac{\xi(t)}{t^{-\delta+1}\sin^\beta(t/2)}\frac{1}{n+1}\Big(\tau\sum_{k=\tau}^{n} P_k^{-1}\sum_{\nu=0}^{k-1}|\Delta p_\nu|\Big)\Big)^s dt\right]^{1/s}
\end{aligned}$$

$$= O\big((n+1)^{\delta-1}\big)\left[\int_{\pi/(n+1)}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+1+\beta}}\left(\tau\sum_{k=\tau}^{n}P_k^{-1}\sum_{\nu=0}^{k-1}|\Delta p_\nu|\right)\right)^s dt\right]^{1/s}$$

$$= O\big((n+1)^{\delta-1}\big)\left[\int_{\pi/(n+1)}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+1+\beta}}\left(\sum_{k=\tau}^{n}P_k^{-1}\sum_{\nu=0}^{k-1}(\nu+1)|\Delta p_\nu|\right)\right)^s dt\right]^{1/s}$$

$$= O\big((n+1)^{\delta-1}\big)\left[\int_{\pi/(n+1)}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+1+\beta}}\left(\sum_{k=0}^{n}P_k^{-1}\sum_{\nu=0}^{k}(\nu+1)|\Delta p_\nu|\right)\right)^s dt\right]^{1/s}$$

$$= O\big((n+1)^{\delta-1}\big)\left[\int_{\pi/(n+1)}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+1+\beta}}\,W_n\,2\pi\,(n+1)\right)^s dt\right]^{1/s}$$

$$= O\big((n+1)^{\delta}\big)\left[\int_{\pi/(n+1)}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+1+\beta}}\right)^s dt\right]^{1/s}$$

$$= O\big((n+1)^{\delta}\big)\left[\int_{1/\pi}^{(n+1)/\pi}\left(\frac{\xi(1/y)}{y^{\delta-1-\beta}}\right)^s \frac{dy}{y^2}\right]^{1/s}$$

$$= O\Big((n+1)^{\beta+1/r}\xi\Big(\frac{1}{n+1}\Big)\Big), \quad r^{-1}+s^{-1}=1. \tag{4.4}$$

Collecting (4.1)–(4.4), we have

$$|\tilde{t}_n^{CN}(f)-\tilde{f}(x)| = O\Big((n+1)^{\beta+1/r}\xi\Big(\frac{1}{n+1}\Big)\Big). \tag{4.5}$$

Now, using the L_r-norm of a function, we get

$$\begin{aligned}\|\tilde{t}_n^{CN}(f)-\tilde{f}(x)\|_r &= \left[\int_0^{2\pi}|\tilde{t}_n^{CN}(f)-\tilde{f}(x)|^r dx\right]^{1/r}\\ &= O\left[\int_0^{2\pi}\Big((n+1)^{\beta+1/r}\xi\Big(\frac{1}{n+1}\Big)\Big)^r dx\right]^{1/r}\\ &= O\left((n+1)^{\beta+1/r}\xi\Big(\frac{1}{n+1}\Big)\Big(\int_0^{2\pi}dx\Big)^{1/r}\right)\\ &= O\Big((n+1)^{\beta+1/r}\xi\Big(\frac{1}{n+1}\Big)\Big).\end{aligned}$$

This completes the proof of Theorem 1. ■

5. Applications

The following corollaries can be derived from Theorem 1.

COROLLARY 1. *If $\xi(t) = t^\alpha$, $0 < \alpha \le 1$, then the class* Lip$(\xi(t), r)$, $r \ge 1$, *reduces to the class* Lip(α, r), $\frac{1}{r} < \alpha < 1$ *and the degree of approximation of a function $\tilde{f}(x)$, conjugate to a 2π-periodic function f belonging to the class* Lip(α, r), *is given by*

$$|\tilde{t}_n^{CN}(f)-\tilde{f}(x)| = O\big((n+1)^{-\alpha+1/r}\big). \tag{5.1}$$

Proof. Putting $\beta = 0$ in Theorem 1, we have

$$\|\tilde{t}_n^{CN}(f) - \tilde{f}(x)\|_r = \left[\int_0^{2\pi} |\tilde{t}_n^{CN}(f) - \tilde{f}(x)|^r dx\right]^{1/r} = O\Big((n+1)^{1/r}\xi\Big(\frac{1}{n+1}\Big)\Big)$$
$$= O\big((n+1)^{-\alpha+1/r}\big).$$

Thus we get

$$|\tilde{t}_n^{CN}(f) - \tilde{f}(x)| \le \left[\int_0^{2\,\pi} |\tilde{t}_n^{CN}(f) - \tilde{f}(x)|^r dx\right]^{1/r} = O\big((n+1)^{-\alpha+1/r}\big),$$

$r \ge 1$. This completes the proof of Corollary 1. ∎

Corollary 2. *If $\xi(t) = t^\alpha$ for $0 < \alpha < 1$, and $r \to \infty$ in Corollary 1, then $f \in \mathrm{Lip}\,\alpha$. In this case, using* (5.1) *we get that*

$$\|\tilde{t}_n^{CN}(f) - \tilde{f}(x)\|_\infty = O\big((n+1)^{-\alpha}\big).$$

Proof. For $r \to \infty$, we get

$$\|\tilde{t}_n^{CN}(f) - \tilde{f}(x)\|_\infty = \sup_{0\le x\le 2\pi} |\tilde{t}_n^{CN}(f) - \tilde{f}(x)| = O\big((n+1)^{-\alpha}\big).$$

This completes the proof of Corollary 2. ∎

Acknowledgement. The authors are highly thankful to the anonymous learned referees for their observations, careful reading, their critical remarks, valuable comments and several useful pertinent suggestions, which greatly helped us for the overall improvements and the better presentation of this paper significantly.

The second author is thankful to the "Ministry of Human Resource and Development" of India for financial support to carry out the above work.

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(received 25.08.2012; in revised form 28.06.2013; available online 10.09.2013)

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