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GROWTH AND OSCILLATION OF MEROMORPHIC SOLUTIONS OF LINEAR DIFFERENCE EQUATIONS

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Abstract. In this paper, we study the growth and the oscillation of solutions of linear difference equations with meromorphic coefficients. Also, we investigate the growth of difference polynomials generated by meromorphic solutions of some difference equations. We improve and generalize some results due to Z. X. Chen, I. Laine and C. C. Yang.

1. Introduction

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory [13, 14, 22]. In addition, we will use $\lambda(f)$ and $\overline{\lambda}(f)$ to denote respectively the exponents of convergence of the zero-sequence and distinct zeros of a meromorphic function f, $\rho(f)$ to denote the order of growth of f and $\tau(f)$ the type of f.

Recently, there has been an increasing renewed interest in complex difference equations and difference analogues of Nevanlinna theory [1, 3, 5–9, 11, 12, 15, 17, 18, 21, 23]. We firstly recall some existence results for meromorphic solutions of difference equations. The following two results have been proved by Shimomura [18] and Yanagihara [21], respectively.

Theorem A. [18] *For any nonconstant polynomial $P(y)$, the difference equation*

$$y(z+1) = P(y(z))$$

has a nontrivial entire solution.

Theorem B. [21] *For any nonconstant rational function $R(y)$, the difference equation*

$$y(z+1) = R(y(z))$$

has a nontrivial meromorphic solution in the complex plane.

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The following two results concerning both existence and growth restrictions for meromorphic solutions of linear difference equations, have been proved by Bank and Kaufman [2] and Whittaker [20], respectively.

THEOREM C. [2] *For any nonconstant rational function $R(z)$, the difference equation*

$$y(z+1) - y(z) = R(z)$$

has a nontrivial meromorphic solution $y(z)$ such that $T(r, y) = O(r)$.

THEOREM D. [20] *Let ρ be a real number, and let $\Psi(z)$ be a given entire function with order $\rho(\Psi) = \rho$. Then the equation*

$$F(z+1) = \Psi(z)F(z)$$

admits a meromorphic solution of order $\rho(F) \leq \rho + 1$.

In the recent paper [8], Chiang and Feng improved Theorem D by showing that $\rho(F) \leq \rho + 1$ can be replaced by $\rho(F) = \rho + 1$ (Corollary 9.3). In fact, they have investigated meromorphic solutions of the linear difference equation

$$a_n(z)f(z+n) + a_{n-1}(z)f(z+n-1) + \cdots + a_1(z)f(z+1) + a_0(z)f(z) = 0, \quad (1.1)$$

where $a_n(z), \ldots, a_0(z)$ are entire functions such that $a_n(z)a_0(z) \not\equiv 0$, and proved the following two results.

THEOREM E. [8] *Let $a_0(z), a_1(z), \ldots, a_n(z)$ be polynomials such that there exists an integer l, $0 \leq l \leq n$ such that*

$$\deg(a_l) > \max_{0 \leq j \leq n,\, j \neq l} \{\deg(a_j)\}.$$

If $f(z)$ is a meromorphic solution of (1.1)*, then $\rho(f) \geq 1$.*

THEOREM F. [8] *Let $a_0(z), a_1(z), \ldots, a_n(z)$ be entire functions such that there exists an integer l, $0 \leq l \leq n$ such that*

$$\rho(a_l) > \max_{0 \leq j \leq n,\, j \neq l} \{\rho(a_j)\}.$$

If $f(z)$ is a meromorphic solution of (1.1)*, then $\rho(f) \geq \rho(a_l) + 1$.*

In [15], I. Laine and C. C. Yang obtained the following theorem which is an improvement of the previous result.

THEOREM G. [15] *Let $a_0(z), a_1(z), \ldots, a_n(z)$ be entire functions of finite order such that among those having the maximal order $\rho = \max_{0 \leq j \leq n}\{\rho(a_j)\}$, one has exactly its type strictly greater than the others. Then for any meromorphic solution of* (1.1)*, we have $\rho(f) \geq \rho + 1$.*

Recently, Theorem E has been improved by Z. X. Chen as follows.

Theorem H. [6, Theorem 1.2] *Let* $a_0(z), a_1(z), \ldots, a_n(z)$ *be polynomials such that* $a_n(z)a_0(z) \not\equiv 0$ *and satisfy*

$$\deg(a_0 + a_1 + \cdots + a_n) = \max\{\deg a_j : j = 0, \ldots, n\} \geq 1.$$

Then every finite order meromorphic solution $f(z) \not\equiv 0$ *of the equation* (1.1) *satisfies* $\rho(f) \geq 1$, *and* $f(z)$ *assumes every non-zero value* $a \in \mathbb{C}$ *infinitely often and* $\lambda(f - a) = \rho(f)$.

2. Growth and oscillation of solutions

There arise many interesting questions such as:

Question 2.1. What can be said if we replace the condition "$a_0(z), \ldots, a_n(z)$ be entire functions" in Theorems F and G by "$a_0(z), \ldots, a_n(z)$ be meromorphic functions"?

Question 2.2. What about oscillation and fixed point of meromorphic solutions of (1.1) under the above condition?

The aim of this paper is to give an answer for the above questions, and we obtain the following results.

Theorem 2.1. *Let* $a_0(z), a_1(z), \ldots, a_n(z)$ *be meromorphic functions such that* $\lambda(\frac{1}{a_l}) < \rho(a_l) = \rho$ $(0 < \rho < \infty)$, $\tau(a_l) = \tau$ $(0 < \tau < \infty)$. *Suppose that*

$$\max\{\rho(a_j) : 0 \leq j \leq n, j \neq l\} \leq \rho \tag{2.1}$$

and

$$\sum_{\rho(a_j)=\rho} \tau(a_j) < \tau. \tag{2.2}$$

If $f(z)$ *is a meromorphic solution of* (1.1), *then* $\rho(f) \geq \rho(a_l) + 1$.

Recently, Z. X. Chen [6,7] have investigated the complex oscillation of entire solutions to homogeneous and nonhomogeneous linear difference equations, and obtained some relations of the exponent of convergence of zeros and the order of growth of entire solutions to complex linear difference equations. The following theorem is an extension of result obtained by S. A. Gao, Z. X. Chen and T. W. Chen in [10].

Theorem 2.2. *Let* $a_0(z)$, $a_1(z), \ldots,$ $a_n(z)$, $F(z)(\not\equiv 0)$ *be finite order meromorphic functions. If* f *is a meromorphic solution of the equation*

$$a_n\ f(z+n) + a_{n-1}f(z+n-1) + \cdots + a_1 f(z+1) + a_0 f(z) = F \tag{2.3}$$

with

$$\max\{\rho(a_j)\ (j = 0, \ldots, n),\ \rho(F)\} < \rho(f), \tag{2.4}$$

then $\lambda(f) = \rho(f)$.

Example 2.1. The function $f(z) = \frac{1}{\Gamma(z)} + 1$ with $\rho(f) = 1$ satisfies the difference equation

$$z(z+1)f(z+2) - zf(z+1) = z^2$$

and the assumption (2.4). Therefore $\lambda(f) = \rho(f) = 1$.

Example 2.2. The function $f(z) = e^z$ with $\rho(f) = 1$ satisfies

$$f(z+2) - ef(z+1) + f(z) = e^z.$$

It is clear that f does not satisfies the assumption (2.4) and we have $0 = \lambda(f) < \rho(f) = 1$.

Theorem 2.3. *Under the hypotheses of Theorem 2.1, let φ be a meromorphic function such that one of the following conditions holds:*

(i) *φ is not a solution of* (1.1) *with $\rho(\varphi) < \rho(f)$;*

(ii) *$\varphi \not\equiv 0$ and $\rho(\varphi) < \rho(a_l) + 1$.*

Then $\lambda(f - \varphi) = \rho(f)$.

Corollary 2.1. *Under the hypotheses of Theorem 2.3, we have $\lambda(f - z) = \rho(f)$.*

Example 2.3. The function $f(z) = e^{z^2}$ with $\rho(f) = 2$ satisfies the difference equation

$$e^{-4z-4}f(z+2) - f(z) = 0.$$

Set $\varphi(z) = z$. Then, obviously the hypotheses of Theorem 2.3 are satisfied. Therefore, $\lambda(e^{z^2} - z) = \rho(e^{z^2}) = 2$.

3. Growth of difference polynomials

It is natural to ask what can be said about the growth of difference polynomials generated by solutions of difference equations. In this section we consider the difference equation

$$f(z+2) + a(z)f(z+1) + b(z)f(z) = 0, \tag{3.1}$$

where $a(z)$ and $b(z)$ are meromorphic functions. We denote by

$$\begin{cases} f_j = f(z+j), \ j \geq 1 \text{ is an integer}, \\ \qquad f = f_0 = f(z). \end{cases} \tag{3.2}$$

It is known that (see [8, 9]) $\rho(f_j) = \rho(f)$ for all $j \in \mathbb{N}$. We define also

$$\Delta f = f(z+1) - f(z)$$

and

$$\Delta^n f = \Delta^{n-1}(\Delta f), \ n \geq 1 \text{ is an integer}.$$

In [9], Chiang and Feng proved the following inequality

$$\rho(\Delta^n f) \leq \rho(f),\ n \in \mathbb{N}$$

and the equality is not true always. By using (3.2) the equation (3.1) can be written as

$$f_2 + a(z)f_1 + b(z)f = 0. \tag{3.3}$$

The aim of this section is to find the relation between the growth of the solution f and the difference polynomial

$$g = ef_2 + df_1 + cf, \tag{3.4}$$

where $c(z), d(z)$ and $e(z)$ are meromorphic functions of finite order. Before we state our results, we define

$$\alpha = d - ea,\ \beta = c - eb, \tag{3.5}$$

$$\gamma = c_1 - e_1 b_1 - d_1 a + e_1 a_1 a,\ \delta = e_1 a_1 b - d_1 b, \tag{3.6}$$

$$h = \alpha\delta - \beta\gamma, \tag{3.7}$$

where $a_1(z) := a(z+1)$, $b_1(z) := b(z+1)$, $c_1(z) := c(z+1)$, $d_1(z) := d(z+1)$ and $e_1(z) := e(z+1)$. We obtain the following results.

Theorem 3.1. *Let a and b be meromorphic functions satisfying $\lambda(\frac{1}{b}) < \rho(b) < \infty$, $\rho(a) < \rho(b)$ and $0 < \tau(a) < \tau(b) < \infty$ if $\rho(a) = \rho(b) > 0$, and let c, d, e be meromorphic functions not all vanishing identically such that $h \not\equiv 0$ and*

$$\max\{\rho(c), \rho(d), \rho(e)\} < \rho(b) + 1.$$

If $f(z)$ is a meromorphic solution of (3.1), then the difference polynomial (3.4) satisfies

$$\rho(g) = \rho(f) \geq \rho(b) + 1.$$

Remark 3.1. In Theorem 3.1, if we do not have the condition $h \not\equiv 0$, then the conclusion of Theorem 3.1 cannot holds. For example, if $e(z) = 1$, $d(z) = a(z)$ and $c(z) = b(z)$, then $h \equiv 0$. It is clear that

$$\max\{\rho(c), \rho(d), \rho(e)\} < \rho(b) + 1$$

and $g = 0$, so $\rho(g) \neq \rho(f)$. Hence the condition $h \not\equiv 0$ in Theorem 3.1 is necessary.

Corollary 3.1. *Let a and b be meromorphic functions satisfying $\lambda(\frac{1}{b}) < \rho(b) < \infty$, $\rho(a) < \rho(b)$ and $0 < \tau(a) < \tau(b) < \infty$ if $\rho(a) = \rho(b) > 0$. If $f(z)$ is a meromorphic solution of (3.1), then*

$$\rho(\Delta f) = \rho(f).$$

Furthermore, if

$$h = b_1 - 3b - 2a - aa_1 - 2ba_1 - bb_1 - 1 \not\equiv 0,$$

then $\rho(\Delta^2 f) = \rho(f)$.

4. Some lemmas

Setting $p = 1$ in Lemma 3.5 in [4] (see, also Lemma 8 in [19]), we obtain

Lemma 4.1. *Let f be a meromorphic function with $\rho(f) = \rho$ $(0 < \rho < \infty)$ and $\tau(f) = \tau$ $(0 < \tau < \infty)$. Then for any given $\beta < \tau(f)$, there exists a subset E_1 of $[1, \infty)$ that has infinite logarithmic measure such that*

$$T(r, f) > \beta r^{\rho}$$

holds for all $r \in E_1$.

Lemma 4.2. [8] *Let η_1, η_2 be two arbitrary complex numbers such that $\eta_1 \neq \eta_2$ and let $f(z)$ be a finite order meromorphic function. Let σ be the order of $f(z)$, then for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O(r^{\sigma-1+\varepsilon}).$$

Using the properties of the order of growth and the definition of the type, we easily obtain the following result for which we omit the proof. For details see [16].

Lemma 4.3. [16] *Let f and g be meromorphic functions in the complex plane such that $0 < \rho(f), \rho(g) < \infty$ and $0 < \tau(f), \tau(g) < \infty$. Then we have:*

(i) *If $\rho(f) > \rho(g)$, then we obtain*

$$\tau(f+g) = \tau(fg) = \tau(f). \tag{4.1}$$

(ii) *If $\rho(f) = \rho(g)$ and $\tau(f) \neq \tau(g)$, then we get*

$$\rho(f+g) = \rho(fg) = \rho(f) = \rho(g). \tag{4.2}$$

5. Proof of the theorems and corollaries

Proof of Theorem 2.1. If $\rho(f) = \infty$, then the result is trivial. Next we suppose $\rho(f) < \infty$. We divide through equation (1.1) by $f(z+l)$ to get

$$a_l(z) = -\left(a_n(z)\frac{f(z+n)}{f(z+l)} + \cdots + a_1(z)\frac{f(z+1)}{f(z+l)} + a_0(z)\frac{f(z)}{f(z+l)}\right). \tag{5.1}$$

It follows that

$$\begin{aligned} T(r, a_l) &= m(r, a_l) + N(r, a_l) \\ &\leq \sum_{\substack{j \neq l \\ j=0}}^{n} m(r, a_j) + \sum_{\substack{j \neq l \\ j=0}}^{n} m\left(r, \frac{f(z+j)}{f(z+l)}\right) + N(r, a_l) + O(1). \end{aligned} \tag{5.2}$$

By Lemma 4.2, we have for sufficiently large r and any given $\varepsilon > 0$

$$m\left(r, \frac{f(z+j)}{f(z+l)}\right) = O(r^{\rho(f)-1+\varepsilon}),\ j = 0, \dots, n,\ j \neq l. \tag{5.3}$$

Let us choose σ such that $\lambda(1/a_l) < \sigma < \rho(a_l) = \rho$. Then we have for any given ε $(0 < \varepsilon < \rho - \sigma)$

$$N(r, a_l) \leq r^{\lambda(\frac{1}{a_l})+\varepsilon} < r^{\sigma+\varepsilon}. \tag{5.4}$$

Assume that $\max_{0\leq j\leq n, j\neq l}\{\rho(a_j)\} \leq \rho(a_l)$ and $\sum_{\rho(a_j)=\rho} \tau(a_j) < \tau(a_l)$. Then there exists a set $J_1 \subseteq \{0, 1, \dots, l-1, l+1, \dots, n\}$ such that for $j \in J_1$ we have $\rho(a_j) = \rho(a_l) = \rho$ and $\sum_{j\in J_1} \tau(a_j) < \tau(a_l) = \tau$ and for $i \in \{0, 1, \dots, l-1, l+1, \dots, n\} \setminus J_1$ we have $\rho(a_i) < \rho(a_l) = \rho$. Hence, we can choose α_1, α_2 satisfying $\sum_{j\in J_1} \tau(a_j) < \alpha_1 < \alpha_2 < \tau$ such that for any given ε $(0 < \varepsilon < \frac{\alpha_2-\alpha_1}{n})$, we have

$$T(r, a_j) \leq (\tau(a_j) + \varepsilon)r^{\rho},\ j \in J_1 \tag{5.5}$$

and

$$T(r, a_i) \leq r^{\rho_0},\ i \in \{0, 1, \dots, l-1, l+1, \dots, n\} \setminus J_1, \tag{5.6}$$

where $0 < \rho_0 < \rho$. By applying Lemma 4.1, there exists a subset I of $[1, \infty)$ that has infinite logarithmic measure such that for all $r \in I$, we have

$$T(r, a_l) > \alpha_2 r^{\rho}. \tag{5.7}$$

By using the assumptions (5.3)–(5.7), we obtain from (5.2) for any given ε $(0 < \varepsilon < \min\{\frac{\alpha_2-\alpha_1}{n}, \rho - \sigma\})$ and for all $r \in I$

$$\begin{aligned}\alpha_2 r^{\rho} &< \sum_{j\in J_1} (\tau(a_j) + \varepsilon)r^{\rho} + \sum_{i\in\{0,1,\dots,l-1,l+1,\dots,n\}\setminus J_1} r^{\rho_0} + O(r^{\rho(f)-1+\varepsilon}) + r^{\sigma+\varepsilon} \\ &< (\alpha_1 + \varepsilon n)r^{\rho} + nr^{\rho_0} + O(r^{\rho(f)-1+\varepsilon}) + r^{\sigma+\varepsilon}.\end{aligned}$$

It follows that

$$(\alpha_2 - \alpha_1 - \varepsilon n)r^{\rho} < nr^{\rho_0} + O(r^{\rho(f)-1+\varepsilon}) + r^{\sigma+\varepsilon}. \tag{5.8}$$

Since $0 < \varepsilon < \min\{\frac{\alpha_2-\alpha_1}{n}, \rho - \sigma\}$, we obtain from (5.8) that $\rho(a_l) = \rho \leq \rho(f) - 1$. ∎

Proof of Theorem 2.2. By (2.3) we have

$$\frac{1}{f(z)} = \frac{1}{F}\left(a_n \frac{f(z+n)}{f(z)} + \cdots + a_1 \frac{f(z+1)}{f(z)} + a_0\right). \tag{5.9}$$

Set $\max\{\rho(a_j)\ (j = 0, \dots, n), \rho(F)\} = \beta < \rho(f) = \rho$. Then, for any given ε $(0 < \varepsilon < \frac{\rho-\beta}{2})$, we have

$$\sum_{j=0}^{n} T(r, a_j) + T(r, F) \leq (n+2)\exp\{r^{\beta+\varepsilon}\} = o(1)\exp\{r^{\rho-\varepsilon}\}. \tag{5.10}$$

By (5.9), (5.10) and Lemma 4.2, we obtain

$$\begin{aligned} T(r,f) &= T\Big(r,\frac{1}{f}\Big) + O(1) = m\Big(r,\frac{1}{f}\Big) + N\Big(r,\frac{1}{f}\Big) + O(1) \\ &\leq N\Big(r,\frac{1}{f}\Big) + m\Big(r,\frac{1}{F}\Big) + \sum_{j=0}^{n} m(r,a_j) + \sum_{j=1}^{n} m\Big(r,\frac{f(z+j)}{f(z)}\Big) + O(1) \\ &\leq N\Big(r,\frac{1}{f}\Big) + T\Big(r,\frac{1}{F}\Big) + \sum_{j=0}^{n} T(r,a_j) + \sum_{j=1}^{n} m\Big(r,\frac{f(z+j)}{f(z)}\Big) + O(1) \\ &\leq N\Big(r,\frac{1}{f}\Big) + O(r^{\rho-1+\varepsilon}) + o(1)\exp\{r^{\rho-\varepsilon}\}. \end{aligned} \tag{5.11}$$

By (5.11), we obtain that $\rho(f) \leq \lambda(f)$ and since $\lambda(f) \leq \rho(f)$ for every meromorphic function, we deduce that $\lambda(f) = \rho(f)$. ■

Proof of Theorem 2.3. Set $w(z) = f(z) - \varphi(z)$.

(i) If $\rho(f) > \rho(\varphi)$, then we have $\rho(w) = \rho(f)$. Substituting w into equation (1.1), we obtain

$$\begin{aligned} &a_n w(z+n) + a_{n-1}w(z+n-1) + \cdots + a_1 w(z+1) + a_0 w(z) \\ &\quad = -(a_n\varphi(z+n) + a_{n-1}\varphi(z+n-1) + \cdots + a_1\varphi(z+1) + a_0\varphi(z)) \\ &\quad = A(z). \end{aligned}$$

Since φ is not a solution of (1.1), then $A \not\equiv 0$. By Theorem 2.1 we have

$$\rho(f) \geq \max_{0\leq j\leq n}\{\rho(a_j)\} + 1$$

which implies

$$\rho(w) = \rho(f) > \max\{\rho(A), \rho(a_j)\ (j=0,\ldots,n)\}. \tag{5.12}$$

Then, by Theorem 2.2 we have $\lambda(w) = \rho(w)$, i.e., $\lambda(f-\varphi) = \rho(f)$.

(ii) Suppose now that $\varphi \not\equiv 0$ and $\rho(\varphi) < \rho(a_l) + 1$. Since $\varphi \not\equiv 0$ and

$$\rho(\varphi) < \rho(a_l) + 1 = \max_{0\leq j\leq n}\{\rho(a_j)\} + 1 \leq \rho(f),$$

then $A \not\equiv 0$. By (5.12) and Theorem 2.2, we obtain $\lambda(w) = \rho(w)$, i.e., $\lambda(f-\varphi) = \rho(f)$. This completes the proof of Theorem 2.3. ■

Proof of Corollary 2.1. Setting $g(z) = f(z) - z$, it is clear that $\rho(g) = \rho(f)$ because $\rho(f) > \rho(z) = 0$. Substituting $f = g + z$ into equation (1.1), we obtain

$$\sum_{j=0}^{n} a_j(z)g(z+j) = -\sum_{j=0}^{n}(z+j)a_j(z).$$

In order to prove $\rho(f) = \lambda(f-z)$ we need to prove $\sum_{j=0}^{n}(z+j)a_j(z) \not\equiv 0$. Suppose that $\sum_{j=0}^{n}(z+j)a_j(z) \equiv 0$. Then, by the conditions (2.1), (2.2) and Lemma 4.3 we have

$$\rho\Big(\sum_{j=0}^{n}(z+j)a_j(z)\Big) = \rho(a_l) > 0$$

which is a contradiction. Hence, by applying Theorem 2.3 we obtain $\lambda(f-z) = \rho(f-z) = \rho(f)$. ■

Proof of Theorem 3.1. By using Theorem 2.1, we have $\rho(f) \geq \rho(b) + 1$. Substituting $f_2 = -af_1 - bf$ into $g = ef_2 + df_1 + cf$, we get

$$g = e(-af_1 - bf) + df_1 + cf = (d - ea)f_1 + (c - eb)f. \tag{5.13}$$

Then

$$\begin{aligned} g_1 &= (d_1 - e_1a_1)f_2 + (c_1 - e_1b_1)f_1 \\ &= (d_1 - e_1a_1)(-af_1 - bf) + (c_1 - e_1b_1)f_1 \\ &= (c_1 - e_1b_1 - (d_1 - e_1a_1)a)f_1 + (-d_1b + e_1a_1b)f \\ &= (c_1 - e_1b_1 - d_1a + e_1a_1a)f_1 + (e_1a_1b - d_1b)f. \end{aligned} \tag{5.14}$$

By using (3.5), (3.6) we can deduce from (5.13) and (5.14) that

$$\begin{cases} g = \alpha f_1 + \beta f, \\ g_1 = \gamma f_1 + \delta f, \end{cases}$$

which implies by using (3.7) and $h \not\equiv 0$ that

$$f = \frac{\alpha g_1 - \gamma g}{h}. \tag{5.15}$$

It is clear from (3.4) that $\rho(g) \leq \rho(f)$, and by (5.15) we have $\rho(f) \leq \rho(g)$. Hence, $\rho(f) = \rho(g)$. ∎

Proof of Corollary 3.1. Set $g = \Delta f(z) = f_1(z) - f(z)$. Then by (3.4) we have

$$e(z) = 0, \quad d(z) = 1 \text{ and } c(z) = -1.$$

By (3.5)–(3.7), we obtain

$$\alpha = d - ea = 1, \quad \beta = c - eb = -1,$$
$$\gamma = c_1 - e_1b_1 - d_1a + e_1a_1a = -1 - a, \quad \delta = e_1a_1b - d_1b = -b,$$

and

$$h = \alpha\delta - \beta\gamma = -a - b - 1.$$

Since $\rho(a) < \rho(b) < \infty$ and $0 < \tau(a) < \tau(b) < \infty$ if $\rho(a) = \rho(b) < \infty$, then by Lemma 4.3 we have $\rho(h) = \rho(b) > 0$, so $h \not\equiv 0$. By using Theorem 3.1, we have $\rho(g) = \rho(\Delta f) = \rho(f)$.

Now set $g = \Delta^2 f(z) = f_2(z) - 2f_1(z) + f(z)$. Then by (3.4) we have $e(z) = 1$, $d(z) = -2$ and $c(z) = 1$. By (3.5)–(3.7), we obtain

$$\alpha = -2 - a, \quad \beta = 1 - b,$$
$$\gamma = 1 - b_1 + 2a + a_1a, \quad \delta = a_1b + 2b,$$

and

$$h = \alpha\delta - \beta\gamma = b_1 - 3b - 2a - aa_1 - 2ba_1 - bb_1 - 1.$$

Since $h \not\equiv 0$, then by using Theorem 3.1, we have $\rho(g) = \rho(\Delta^2 f) = \rho(f)$. ∎

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