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ELLIPTIC TRANSMISSION PROBLEM IN DISJOINT DOMAINS

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Abstract. In this paper, we investigate an elliptic transmission problem in disjoint domains. A priori estimate for its weak solution in appropriate Sobolev-like space is proved. A finite difference scheme approximating this problem is proposed and analyzed. An estimate of the convergence rate, compatible with the smoothness of the input data (up to a slowly increasing logarithmic factor of the mesh size), is obtained.

1. Introduction

In applications, especially in engineering, often are encountered composite or layered structure, where the properties of individual layers can vary considerably from the properties of the surrounding material. Layers can be structural, thermal, electromagnetic or optical, etc. Mathematical models of energy and mass transfer in domains with layers lead to so called transmission problems. In this paper we consider a class of non-standard elliptic transmission problems in disjoint domains [11]. As a model example it is taken an area consisting of two non-adjacent rectangles. In each subarea was given a boundary problem of elliptic type, where the interaction between their solutions is described by nonlocal integral conjugation conditions [9].

2. Formulation of the problem

As a model example,we consider the following boundary-value problem (BVP): Find functions $u^1(x_1,x_2)$ and $u^2(x_1,x_2)$ that satisfy the system of elliptic equations:

$$L^k u^k = f^k(x_1,x_2) x = (x_1,x_2) \in \Omega^k \tag{1}$$

$$l^k u^k = \begin{cases} r^k u^{3-k}, & x \in \Gamma^k_{1,3-k}, \\ 0, & x \in \Gamma^k \setminus \Gamma^k_{1,3-k}, \end{cases} \tag{2}$$

where $k = 1,2$ and $\Omega^1 = (a_1,b_1)\times(c,d)$, $\Omega^2 = (a_2,b_2)\times(c,d)$, $-\infty < a_1 < b_1 < a_2 < b_2 < +\infty$ and $c < d$. We denote $\Gamma^k = \partial\Omega^k = \bigcup_{i,j=1}^{2} \Gamma^k_{ij}$, where

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$\Gamma^1_{11} = \{x = (x_1, x_2) \in \Gamma^1 \,|\, x_1 = a_1\}$, $\Gamma^1_{12} = \{x \in \Gamma^1 \,|\, x_1 = b_1\}$, $\Gamma^k_{21} = \{x \in \Gamma^k \,|\, x_2 = c\}$, $\Gamma^k_{22} = \{x \in \Gamma^k \,|\, x_2 = d\}$, $\Gamma^2_{11} = \{x = (x_1, x_2) \in \Gamma^2 \,|\, x_1 = a_2\}$, $\Gamma^2_{12} = \{x \in \Gamma^2 \,|\, x_1 = b_2\}$, $\Delta^k = \Gamma^k_{1,3-k} \times \Gamma^{3-k}_{1,k}$ $(k = 1, 2)$,

$$L^k u^k := -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(p^k_{ij}\,\frac{\partial u^k}{\partial x_j}\right) + q^k u^k, \tag{3}$$

$$l^k u^k := \sum_{i,j=1}^{2} p^k_{ij}\,\frac{\partial u^k}{\partial x_j}\,\cos(\nu, x_i) + \alpha^k u^k, \tag{4}$$

$$\big(r^k u^{3-k}\big)(x) := \int_{\Gamma^{3-k}_{1,k}} \beta^k(x, x')\, u^{3-k}(x')\, \mathrm{d}\Gamma^{3-k}, \tag{5}$$

and ν is the unit outward normal to Γ^k $(k = 1, 2)$.

Notice that boundary condition (2) on $\Gamma^k \setminus \Gamma^k_{1,3-k}$ reduces to a standard conormal boundary condition while on $\Gamma^k_{1,3-k}$ it can be considered as a conjugation condition of non-local Robin-Dirichlet type. For a special choice of α^k and β^k such conjugation conditions describe linearized radiative heat transfer in a system of two absolutely black bodies [2].

We assume that the standard conditions of regularity and ellipticity are satisfied:

$$p^k_{ij} = p^k_{ji} \in L^\infty(\Omega^k), \quad q^k \in L^\infty(\Omega^k), \quad \alpha^k \in L^\infty(\Gamma^k), \qquad \beta^k \in L^\infty(\Delta^k), \tag{6}$$

$$c^k_0 \sum_{i=1}^{2} \xi_i^2 \le \sum_{i,j=1}^{2} p^k_{ij}\xi_i\xi_j \le c^k_1 \sum_{i=1}^{2} \xi_i^2, \qquad 0 \le q^k(x), \quad \forall x \in \bar{\Omega}^k, \quad \forall \xi \in \mathbb{R}^2. \tag{7}$$

By C, c_i and c^k_i we denote positive constants, independent of the solution of the boundary-value problem and the mesh-sizes. In particular, C may take different values in the different formulas.

3. Existence and uniqueness of weak solutions

We introduce the product space

$$L = L_2(\Omega^1) \times L_2(\Omega^2) = \{v = (v^1, v^2) | v^k \in L_2(\Omega^k)\},$$

endowed with the inner product and associated norm

$$(u, v)_L = (u^1, v^1)_{L_2(\Omega^1)} + (u^2, v^2)_{L_2(\Omega^2)}, \quad \|v\|_L = (v, v)_L^{1/2},$$

where

$$(u^k, v^k)_{L_2(\Omega^k)} = \iint_{\Omega^k} u^k v^k \, dxdy, \quad k = 1, 2.$$

We also define the spaces

$$H^s = \{v = (v^1, v^2) |\ v^k \in H^s(\Omega^k)\}, \quad s = 1, 2, \ldots$$

endowed with the inner product and associated norm

$$(u,v)_{H^s} = (u^1,v^1)_{H^s(\Omega^1)} + (u^2,v^2)_{H^s(\Omega^2)}, \quad \|v\|_{H^s} = (v,v)_{H^s}^{1/2},$$

where $H^s(\Omega^k)$ are the standard Sobolev spaces [1]. Finally, with $u=(u^1,u^2)$ and $v=(v^1,v^2)$ we define the following bilinear form:

$$A(u,v) = \sum_{k=1}^{2}\Bigg(\iint_{\Omega^k}\Big(\sum_{i,j=1}^{2} p_{ij}^k \frac{\partial u^k}{\partial x_j}\frac{\partial v^k}{\partial x_i} + q^k u^k v^k\Big)\,\mathrm{d}x_1\,\mathrm{d}x_2 + \int_{\Gamma^k} \alpha^k u^k v^k\,\mathrm{d}\Gamma^k - \int_{\Gamma^k_{1,3-i}}\int_{\Gamma^{3-k}_{1,i}} \beta^k\, u^{3-k}\, v^k\,\mathrm{d}\Gamma^{3-k}\,\mathrm{d}\Gamma^k\Bigg). \quad (8)$$

Lemma 1. *Under the conditions* (6)*, the bilinear form A, defined by* (8)*, is bounded on $H^1\times H^1$. If in adition the conditions* (7) *are fulfilled, this form satisfies the Gårding's inequality on H^1, i.e. there exist positive constants m and κ such that*

$$A(u,u) + \kappa\|u\|_L^2 \ge m\|u\|_{H^1}^2, \quad \forall\, u \in H^1.$$

If β^k are sufficiently small and $\alpha^k > 0$ $(k=1,2)$, then the bilinear form A is coercive (i.e. $\kappa = 0$). Sufficient conditions are

$$|\beta^1(x,x') + \beta^2(x',x)| \le \frac{2\sqrt{\alpha^1(x)\,\alpha^2(x')}}{d-c}, \quad \forall\, x \in \Gamma_{12}^1,\ \forall\, x' \in \Gamma_{11}^2. \quad (9)$$

Proof. The proof is analogous to the proof of Lemma 3.8 in [7]. Boundedness of A follows from (6) and the trace theorem for the Sobolev spaces

$$\|u^k\|_{L_2(\partial\Omega^k)} \le C\|u^k\|_{H^1(\Omega^k)}.$$

Gårding's inequality follows from (7), (8), multiplicative trace inequality (see, e.g., Proposition 1.6.3 in [3])

$$\|u^k\|_{L_2(\partial\Omega^k)}^2 \le C\|u^k\|_{L_2(\Omega^k)}\|u^k\|_{H^1(\Omega^k)},$$

Cauchy-Schwartz and ϵ-inequalities, for sufficiently small $\epsilon > 0$. ∎

Theorem 1. *Let the assumptions* (6)*,* (7) *and* (9) *hold. Then the BVP* (1)–(5) *has a unique weak solution $u \in H^1(\Omega)$, and it depends continuously on f.*

Proof. The proof is an easy consequence of Lemma 1 and the Lax-Milgram Lemma (see Theorems 17.9 and 17.10 in [15]). ∎

4. Finite difference approximation

Let $\overline{\omega}_{h_k}$ be a uniform mesh in $[a_k,b_k]$, with the step size $h_k = h_{k1} = (b_k - a_k)/n_k$, $k=1,2$. We denote $\omega_{h_k} := \overline{\omega}_{h_k} \cap (a_k,b_k)$. Analogously we define a uniform mesh $\overline{\omega}_{h_3}$ in $[c,d]$, with the step size $h_3 = h_{k2} = (d-c)/n_3$ $(k=1,2)$ and its submesh $\omega_{h_3} = \overline{\omega}_{h_3} \cap (c, d)$. We assume that $h_1 \asymp h_2 \asymp h_3 \asymp h = \max\{h_1,\, h_2,\, h_3\}$. We also define the following meshes: $\overline{\omega}^k = \overline{\omega}_{h_k} \times \overline{\omega}_{h_3}$, $\gamma^k = \bar{\omega}^k \cap \Gamma^k$, $\bar{\gamma}_{ij}^k = \bar{\omega}^k \cap \Gamma_{ij}^k$, $\gamma_{1j}^k = \{x \in \bar{\gamma}_{1j}^k \;:\; c < x_2 < d\}$, $\gamma_{1j}^{k-} = \{x \in \bar{\gamma}_{1j}^k \;:\; c \le x_2 < d\}$, $\gamma_{1j}^{k+} = \{x \in \bar{\gamma}_{1j}^k \;:\; c < x_2 \le d\}$, $\gamma_{1j}^{k\star} = \bar{\gamma}_{1j}^k \setminus \gamma_{1j}^k$, $\gamma_{2j}^k = \{x \in \bar{\gamma}_{2j}^k \;:\; a_k < x_1 < b_k\}$, $\gamma_{2j}^{k-} = \{x \in \bar{\gamma}_{2j}^k \;:\; a_k \le x_1 < b_k\}$, $\gamma_{2j}^{k+} = \{x \in \bar{\gamma}_{2j}^k \;:\; a_k < x_1 \le b_k\}$, $\gamma_{2j}^{k\star} = \bar{\gamma}_{2j}^k \setminus \gamma_{2j}^k$, $\gamma_\star^k = \gamma^k \setminus \left\{\cup_{i,j} \gamma_{ij}^k\right\},\; i, j, k = 1, 2.$

We shall consider vector-functions of the form $v = (v^1, v^2)$ where v^k is a mesh function defined on $\overline{\omega}^k$, $k = 1, 2$.

The finite difference operators are defined in the usual manner [14]:

$$v_{x_i}^k = \frac{(v^k)^{+i} - v^k}{h_{ki}}, \quad v_{\bar{x}_i}^k = \frac{v^k - (v^k)^{-i}}{h_{ki}},$$

where $(v^k)^{\pm i}(x) = v^k(x \pm h_{ki} e_i),\; e_i$ is the unit vector of the axis $x_i,\; i, k = 1, 2.$

We define the following discrete inner products and norms:

$$[v^k, w^k]_k = h_k h_3 \sum_{x \in \omega^k} v^k(x) w^k(x) + \frac{h_k h_3}{2} \sum_{x \in \gamma^k \setminus \gamma_\star^k} v^k(x) w^k(x) + \frac{h_k h_3}{4} \sum_{x \in \gamma_\star^k} v^k(x) w^k(x),$$

$$[v^k, w^k)_{k,i} = h_k h_3 \sum_{x \in \omega^k \cup \gamma_{i1}^k} v^k(x) w^k(x) + \frac{h_k h_3}{2} \sum_{x \in \gamma_{3-i,1}^{k-} \cup \gamma_{3-i,2}^{k-}} v^k(x) w^k(x),$$

$$(v^k, w^k]_{k,i} = h_k h_3 \sum_{x \in \omega^k \cup \gamma_{i2}^k} v^k(x) w^k(x) + \frac{h_k h_3}{2} \sum_{x \in \gamma_{3-i,1}^{k+} \cup \gamma_{3-i,2}^{k+}} v^k(x) w^k(x),$$

$$|[v^k]|_k^2 = [v^k, v^k]_k, \quad |[v^k\|_{k,i}^2 = [v^k, v^k)_{k,i}, \quad \|v^k]|_{k,i}^2 = (v^k, v^k]_{k,i},$$

$$[v^k, w^k)_k = h_k h_3 \sum_{x \in \omega^k \cup \gamma_{11}^{k-} \cup \gamma_{21}^{k-}} v^k(x) w^k(x), \quad |[v^k\|_k^2 = [v^k, v^k)_k,$$

$$(v^k, w^k]_k = h_k h_3 \sum_{x \in \omega^k \cup \gamma_{12}^{k+} \cup \gamma_{22}^{k+}} v^k(x) w^k(x), \quad \|v^k]|_k^2 = (v^k, v^k]_k,$$

$$|[v^k]|_{H^1(\bar{\omega}^k)}^2 = |[v^k]|_k^2 + |[v_{x_1}^k\|_{k,1}^2 + |[v_{x_2}^k\|_{k,2}^2, \quad |[v^k]|_{C(\bar{\omega}^k)} = \max_{x \in \bar{\omega}^k} |v^k(x)|,$$

$$[v^k, w^k]_{\bar{\gamma}_{ij}^k} = h_{k,3-i} \sum_{x \in \gamma_{ij}^k} v^k(x) w^k(x) + \frac{h_{k,3-i}}{2} \sum_{x \in \gamma_{ij}^{k\star}} v^k(x) w^k(x), \quad |[v^k]|_{\bar{\gamma}_{ij}^k}^2 = [v^k, v^k]_{\bar{\gamma}_{ij}^k},$$

$$(v^k, w^k)_{\gamma_{ij}^k} = h_{k,3-i} \sum_{x \in \gamma_{ij}^k} v^k(x) w^k(x), \quad \|v^k\|_{\gamma_{ij}^k}^2 = (v^k, v^k)_{\gamma_{ij}^k},$$

$$[v^k, w^k)_{\gamma_{ij}^{k-}} = h_{k,3-i} \sum_{x \in \gamma_{ij}^{k-}} v^k(x) w^k(x), \quad |[v^k\|_{\gamma_{ij}^{k-}}^2 = [v^k, v^k)_{\gamma_{ij}^{k-}},$$

$$|v^k|_{H^{1/2}(\gamma_{ij}^{k-})}^2 = h_{k,3-i}^2 \sum_{x,\, x' \in \gamma_{ij}^{k-},\; x' \ne x} \left[\frac{v^k(x) - v^k(x')}{x_{3-i} - x'_{3-i}}\right]^2,$$

$$|[v^k\|^2_{H^{1/2}(\gamma_{ij}^{k-})} = |v^k|^2_{H^{1/2}(\gamma_{ij}^{k-})} + |[v^k\|^2_{\gamma_{ij}^{k-}},$$

$$|[v^k\|^2_{\ddot{H}^{1/2}(\gamma_{ij}^{k-})} = |v^k|^2_{H^{1/2}(\gamma_{ij}^{k-})} +$$

$$h_{k,3-i}\sum_{x\in\gamma_{ij}^{k-}}\left(\frac{1}{x_{3-i}+h_{k,3-i}/2}+\frac{1}{l_{ki}-x_{3-i}-h_{k,3-i}/2}\right)|v^k(x)|^2,$$

where $l_{k1} = d - c$ and $l_{k2} = b_k - a_k$.

For $v = (v^1, v^2)$ and $w = (w^1, w^2)$ we denote

$$[v,w] = [v^1,w^1]_1 + [v^2,w^2]_2, \quad |[v]|^2 = [v,v], \quad |[v]|^2_{H^1_h} = |[v^1]|^2_{H^1(\bar\omega^1)} + |[v^2]|^2_{H^1(\bar\omega^2)},$$

We also define the Steklov smoothing operators (see [4]):

$$T^+_{ki}f^k(x) = \int_0^1 f^k(x + h_{ki}x'_ie_i)\,\mathrm{d}x'_i = T^-_{ki}f^k(x + h_{ki}e_i) = T_{ki}f^k(x + 0.5h_{ki}e_i),$$

$$T^{2\pm}_{ki}f^k(x) = 2\int_0^1 (1 - x'_i)f^k(x \pm h_{ki}x'_ie_i)\,\mathrm{d}x'_i, \qquad k, i = 1, 2.$$

These operators commute and transform derivatives into differences, for example:

$$T^+_{ki}\left(\frac{\partial u^k}{\partial x_i}\right) = u^k_{x_i}\,, \quad T^-_{ki}\left(\frac{\partial u^k}{\partial x_i}\right) = u^k_{\bar x_i}\,, \quad T^2_{ki}\left(\frac{\partial^2 u^k}{\partial x_i^2}\right) = u^k_{\bar x_i x_i}.$$

We approximate the boundary-value problem (1)–(5) with the following finite difference scheme:

$$L^k_h v = \tilde f^k, \qquad x\in\bar\omega^k \tag{10}$$

where $L^1_h v$ is equal to:

$$-\frac12\sum_{i,j=1}^2\Big[\big(p^1_{ij}\,v^1_{x_j}\big)_{\bar x_i} + \big(p^1_{ij}\,v^1_{\bar x_j}\big)_{x_i}\Big] + \tilde q^1 v^1, \qquad x\in\omega^1$$

$$\frac{2}{h_1}\Big[-\frac{p^1_{11}+(p^1_{11})^{+1}}{2}\,v^1_{x_1} - p^1_{12}\,\frac{v^1_{x_2}+v^1_{\bar x_2}}{2} + \tilde\alpha^1 v^1\Big] - \big(p^1_{12}\,v^1_{\bar x_2}\big)_{x_1}$$
$$-\big(p^1_{21}\,v^1_{x_1}\big)_{\bar x_2} - \frac12\big(p^1_{22}\,v^1_{x_2}\big)_{\bar x_2} - \frac12\big(p^1_{22}\,v^1_{\bar x_2}\big)_{x_2} + \tilde q^1 v^1, \qquad x\in\gamma^1_{11}$$

$$\frac{2}{h_1}\Big[-\frac{p^1_{11}+(p^1_{11})^{+1}}{2}\,v^1_{x_1} - p^1_{12}v^1_{x_2} + \tilde\alpha^1_1 v^1\Big]$$
$$+\frac{2}{h_3}\Big[-p^1_{21}v^1_{x_1} - \frac{p^1_{22}+(p^1_{22})^{+2}}{2}\,v^1_{x_2} + \tilde\alpha^1_2 v^1\Big] + \tilde q^1 v^1, \qquad x = (a_1, c)$$

$$\frac{2}{h_1}\Big[-\frac{p^1_{11}+(p^1_{11})^{+1}}{2}\,v^1_{x_1} - p^1_{12}v^1_{\bar x_2} + \tilde\alpha^1_1 v^1\Big] - 2\big(p^1_{12}\,v^1_{\bar x_2}\big)_{x_1}$$
$$+\frac{2}{h_3}\Big[p^1_{21}v_{x_1} + \frac{p^1_{22}+(p^1_{22})^{-2}}{2}\,v^1_{\bar x_2} + \tilde\alpha^1_2 v^1\Big] - 2\big(p^1_{21}\,v^1_{x_1}\big)_{\bar x_2} + \tilde q^1 v^1, \quad , x = (a_1, d)$$

$$\frac{2}{h_1}\Big[\frac{p_{11}^1+(p_{11}^1)^{-1}}{2}\,v_{\bar{x}_1}^1+p_{12}^1\,\frac{v_{x_2}^1+v_{\bar{x}_2}^1}{2}+\tilde{\alpha}^1 v^1$$
$$-[\tilde{\beta}^1(x,\cdot),v^2(\cdot)]_{\bar{\gamma}_{11}^2}\Big]-\big(p_{12}^1\,v_{x_2}^1\big)_{\bar{x}_1}$$
$$-\big(p_{21}^1\,v_{\bar{x}_1}^1\big)_{x_2}-\frac{1}{2}\big(p_{22}^1\,v_{x_2}^1\big)_{\bar{x}_2}-\frac{1}{2}\big(p_{22}^1\,v_{\bar{x}_2}^1\big)_{x_2}+\tilde{q}^1 v^1, \qquad x\in\gamma_{12}^1$$
$$\frac{2}{h_1}\Big[\frac{p_{11}^1+(p_{11}^1)^{-1}}{2}\,v_{\bar{x}_1}^1+p_{12}^1 v_{x_2}^1+\tilde{\alpha}_1^1 v^1-[\tilde{\beta}^1(x,\cdot),v^2(\cdot)]_{\bar{\gamma}_{11}^2}\Big]$$
$$+\frac{2}{h_3}\Big[-p_{21}^1 v_{\bar{x}_1}-\frac{p_{22}^1+(p_{22}^1)^{+2}}{2}\,v_{x_2}^1+\tilde{\alpha}_2^1 v^1\Big]$$
$$-2\big(p_{12}^1\,v_{x_2}^1\big)_{\bar{x}_1}-2\big(p_{21}^1\,v_{\bar{x}_1}^1\big)_{x_2}+\tilde{q}^1 v^1, \qquad x=(b_1,c)$$
$$\frac{2}{h_1}\Big[\frac{p_{11}^1+(p_{11}^1)^{-1}}{2}\,v_{\bar{x}_1}^1+p_{12}^1 v_{\bar{x}_2}^1+\tilde{\alpha}_1^1 v^1-[\tilde{\beta}^1(x,\cdot),v^2(\cdot)]_{\bar{\gamma}_{11}^2}\Big]$$
$$+\frac{2}{h_3}\Big[p_{21}^1 v_{\bar{x}_1}^1+\frac{p_{22}^1+(p_{22}^1)^{-2}}{2}\,v_{\bar{x}_2}^1+\tilde{\alpha}_2^1 v^1\Big]+\tilde{q}^1 v^1, \qquad x=(b_1,d)$$

and analogously at the other boundary nodes, $x\in\gamma_{21}^1\cup\gamma_{22}^1$

$L_h^2 v$ is defined in an analogous manner,

$$\tilde{f}^k=\begin{cases} T_{k1}^2T_{k2}^2 f^k, & x\in\omega^k\\ T_{ki}^{2\pm}T_{k,3-i}^2 f^k, & x\in\gamma_{i1}^k\,/\,x\in\gamma_{i2}^k\\ T_{k1}^{2\pm}T_{k2}^{2\pm} f^k, & x\in\gamma_\star^k\end{cases}$$
$$\tilde{q}^k=\begin{cases} T_{k1}^2T_{k2}^2 q^k, & x\in\omega^k\\ T_{ki}^{2\pm}T_{k,3-i}^2 q^k, & x\in\gamma_{i1}^k\,/\,x\in\gamma_{i2}^k\\ T_{k1}^{2\pm}T_{k2}^{2\pm} q^k, & x\in\gamma_\star^k\end{cases}$$
$$\tilde{\alpha}^k=T_{k,3-i}^2\alpha^k, \qquad x\in\gamma_{i1}^k\cup\gamma_{i2}^k, \quad i=1,2,$$
$$\tilde{\alpha}_i^k=T_{k,3-i}^{2\pm}\alpha^k, \qquad x\in\gamma_\star^k, \quad i=1,2$$

and

$$\tilde{\beta}^k=\begin{cases} T_{k2}^2\beta^k, & x\in\gamma_{1,3-k}^k,\\ T_{k2}^{2\pm}\beta^k, & x\in\gamma_{1,3-k}^{k\star}.\end{cases}$$

The FDS (10) may be compactly presented as operator-difference scheme

$$L_h v=\tilde{f}, \tag{11}$$

where $v=(v^1,v^2)$, $\tilde{f}=(\tilde{f}^1,\tilde{f}^2)$ and $L_h v=(L_h^1 v,L_h^2 v)$.

In the sequel we shall assume that the generalized solution of the problem (1)–(5) belongs to the Sobolev space H^s, $2<s\le 3$, while the data satisfy the following smoothness conditions:

$$\begin{gathered} p_{ij}^k\in H^{s-1}(\Omega^k), \quad \alpha^k\in H^{s-3/2}(\Gamma_{ij}^k), \quad \alpha^k\in C(\Gamma^k), \quad \beta^k\in H^{s-1}(\Delta^k),\\ f^k\in H^{s-2}(\Omega^k), \quad q^k\in H^{s-2}(\Omega^k) \quad k,i,j=1,2.\end{gathered} \tag{12}$$

Let us consider the bilinear form $A_h(v,w)$ associated with difference operator $L_h v$:

$$A_h(v,w)=[L_h v,\, w]=\sum_{k=1}^{2}\Big\{\frac{1}{2}\sum_{i=1}^{2}\Big[[p_{ii}^k v_{x_i}^k,\, w_{x_i}^k)_{k,i}+(p_{ii}^k v_{\bar{x}_i}^k,\, w_{\bar{x}_i}^k]_{k,i}$$

$$+[p_{i,3-i}^k v_{x_{3-i}}^k,\, w_{x_i}^k)_k+(p_{i,3-i}^k v_{\bar{x}_{3-i}}^k,\, w_{\bar{x}_i}^k]_k\Big]+[\tilde{q}^k v^k, w^k]_k+\sum_{i,j=1}^{2}\sum_{x\in\gamma_{ij}^k} h_{k,3-i}\tilde{\alpha}^k\, v^k w^k$$

$$+\frac{1}{2}\sum_{x\in\gamma_\star^k}(h_{k2}\tilde{\alpha}_1^k+h_{k1}\tilde{\alpha}_2^k)\, v^k w^k-h_{k2}^2\sum_{x\in\gamma_{1,3-k}^k}\ \sum_{x'\in\gamma_{1,k}^{3-k}}\beta^k(x,x')v^{3-k}(x')w^k(x)$$

$$-\frac{h_{k2}^2}{2}\sum_{x\in\gamma_{1,3-k}^{k\star}}\ \sum_{x'\in\gamma_{1,k}^{(3-k)\star}}\beta^k(x,x')v^{3-k}(x')w^k(x)\Big\}$$

LEMMA 2. *Let p_{ij}^k, $\alpha^k>0$ and β^k satisfy the assumptions* (12)*, q^k satisfy assumption* (7) *and let conditions* (9) *be fulfilled. Then, for a sufficiently small mesh step h, there exist positive constants c_2 and c_3 such that*

$$c_2|[v]|_{H_h^1}^2\le A_h(v,v)=[L_h v,v]\le c_3|[v]|_{H_h^1}^2.$$

The proof is analogous to the proof of Lemma 1.

5. Convergence of the finite difference scheme

Let $u=(u^1,u^2)$ be the solution of the BVP (1)–(5), and let $v=(v^1,v^2)$ denote the solution of the FDS (10). The error $z=(z^1,z^2)=u-v$ satisfies the following conditions

$$L_h^k z=\psi^k,\qquad x\in\bar{\omega}^k, \tag{13}$$

where ψ^1 is equal to:

$$\begin{array}{ll}
\displaystyle\sum_{i,j=1}^{2}\eta_{ij,\,\bar{x}_i}^1+\mu^1, & x\in\omega^1,\\
\displaystyle\frac{2}{h_1}\,\eta_{11}^1+\frac{2}{h_1}\,\eta_{12}^1+\tilde{\eta}_{21,\,\bar{x}_2}^1+\tilde{\eta}_{22,\,\bar{x}_2}^1+\frac{2}{h_1}\,\zeta^1+\tilde{\mu}^1, & x\in\gamma_{11}^1,\\
\displaystyle\frac{2}{h_1}\left(\tilde{\eta}_{11}^1+\tilde{\eta}_{12}^1+\zeta_1^1+\zeta_2^1\right)+\frac{2}{h_3}\left(\tilde{\eta}_{21}^3+\tilde{\eta}_{22}^1+\zeta_2^1\right)+\tilde{\tilde{\mu}}^1, & x=(a_1,c),\\
\displaystyle-\frac{2}{h_1}\,(\eta_{11}^1)^{-1}-\frac{2}{h_1}\,(\eta_{12}^1)^{-1}+\tilde{\eta}_{21,\,\bar{x}_2}^1+\tilde{\eta}_{22,\,\bar{x}_2}^1+\frac{2}{h_1}\,\zeta^1+\frac{2}{h_1}\,\chi^1+\tilde{\mu}^1, & x\in\gamma_{12}^1,\\
\displaystyle\frac{2}{h_1}\left[-(\tilde{\eta}_{11}^1)^{-1}-(\tilde{\eta}_{12}^1)^{-1}+\zeta_1^1+\chi^1\right]+\frac{2}{h_3}\left(\tilde{\eta}_{21}^1+\tilde{\eta}_{22}^1+\zeta_2^1\right)+\tilde{\tilde{\mu}}^1, & x=(b_1,c),
\end{array}$$

and analogously at the other boundary nodes.

ψ^2 is defined in an analogous manner,

$$\eta_{ij}^k = T_{ki}^+ T_{k,3-i}^2 \left(p_{ij}^k \frac{\partial u^k}{\partial x_j} \right) - \frac{1}{2} \left[p_{ij}^k u_{x_j} + (p_{ij}^k)^{+i} (u_{\bar{x}_j}^k)^{+i} \right], \quad x \in \omega^k,$$

$$\tilde{\eta}_{ii}^k = T_{ki}^+ T_{k,3-i}^{2\pm} \left(p_{ii}^k \frac{\partial u^k}{\partial x_i} \right) - \frac{p_{ii}^k + (p_{ii}^k)^{+i}}{2} u_{x_i}^k, \quad x \in \gamma_{3-i,1}^{k-} \ / \ x \in \gamma_{3-i,2}^{k-},$$

$$\tilde{\eta}_{i,3-i}^k = \begin{cases} T_{ki}^+ T_{k,3-i}^{2+} \left(p_{i,3-i}^k \frac{\partial u^k}{\partial x_{3-i}} \right) - p_{i,3-i}^k u_{x_{3-i}}^k, & x \in \gamma_{3-i,1}^{k-}, \\ T_{ki}^+ T_{k,3-i}^{2-} \left(p_{i,3-i}^k \frac{\partial u^k}{\partial x_{3-i}} \right) - (p_{i,3-i}^k)^{+i} (u_{\bar{x}_{3-i}}^k)^{+i}, & x \in \gamma_{3-i,2}^{k-}, \end{cases}$$

$$\begin{aligned}
\zeta^k &= (T_{ki}^2 \alpha^k) u^k - T_{ki}^2 (\alpha^k u^k), \quad x \in \gamma_{3-i,1}^k \cup \gamma_{3-i,2}^k, \\
\zeta_i^k &= (T_{ki}^{2\pm} \alpha^k) u^k - T_{ki}^{2\pm} (\alpha^k u^k), \quad x \in \gamma_\star^k, \\
\mu^k &= (T_{ki}^2 T_{k,3-i}^2 q^k) u^k - T_{ki}^2 T_{k,3-i}^2 (q^k u^k), \quad x \in \omega^k, \\
\tilde{\mu}^k &= (T_{k,3-i}^{2\pm} T_{ki}^2 q^k) u^k - T_{k,3-i}^{2\pm} T_{ki}^2 (q^k u^k), \quad x \in \gamma_{3-i,1}^{k-} \ / \ x \in \gamma_{3-i,2}^{k-}, \\
\tilde{\tilde{\mu}}^k &= (T_{k,3-i}^{2\pm} T_{ki}^{2\pm} q^k) u^k - T_{k,3-i}^{2\pm} T_{ki}^{2\pm} (q^k u^k), \quad x \in \gamma_\star^k, \\
\chi^k &= \int_{\Gamma_{1k}^{3-k}} T_{k2}^2 \beta^k(x,x') u^{3-k}(x') \, \mathrm{d}\Gamma_{1k}^{3-k} - h_3 \sum_{x' \in \bar{\gamma}_{1k}^{3-k}} T_{k2}^2 \beta^k(x,x') u^{3-k}(x') \\
&\quad - \frac{h_3}{2} \sum_{x' \in \bar{\gamma}_{1k}^{3-k}} T_{k2}^2 \beta^k(x,x') u^{3-k}(x'), \quad x \in \gamma_{1,3-k}^k, \\
\chi^k &= \int_{\Gamma_{1k}^{3-k}} T_{k2}^{2\pm} \beta^k(x,x') u^{3-k}(x') \, \mathrm{d}\Gamma_{1k}^{3-k} - h_3 \sum_{x' \in \bar{\gamma}_{1k}^{3-k}} T_{k2}^{2\pm} \beta^k(x,x') u^{3-k}(x') \\
&\quad - \frac{h_3}{2} \sum_{x' \in \bar{\gamma}_{1k}^{3-k}} T_{k2}^{2\pm} \beta^k(x,x') u^{3-k}(x'), \quad x \in \gamma_{1,3-k}^{k\star}.
\end{aligned}$$

We shall prove a suitable a priori estimate for the FDS (13). For this purpose we need the following auxiliary results:

LEMMA 3. [6] *The following inequality holds true:*

$$\left| [v^k, w_{x_{3-i}}^k)_{\gamma_{ij}^{k-}} \right| \le C |[v^k]|_{\ddot{H}^{1/2}(\gamma_{ij}^{k-})} |[w^k]|_{H^1(\bar{\omega}^k)}.$$

LEMMA 4. [6] *Let v^k be a mesh function on $\bar{\omega}^k$, then*

$$|[v^k]|_{C(\bar{\omega}^k)} \le C \sqrt{\log \frac{1}{h}} \, |[v^k]|_{H^1(\bar{\omega}^k)}.$$

Let us rearrange the summands in truncation error ψ in the following manner:

$$\tilde{\eta}_{ij}^k = \eta_{ij}^k + \bar{\eta}_{ij}^k, \qquad \tilde{\mu}^k = \mu^k + \mu^{\star k}, \qquad \tilde{\tilde{\mu}}^k = \mu^k + \mu^{\star\star k}$$

where

$$\bar{\eta}_{ii}^k = \pm \frac{h_{ki}}{3} T_i^+ \left(\frac{\partial}{\partial x_{3-i}} \left(p_{ii}^k \frac{\partial u^k}{\partial x_i} \right) \right), \quad x \in \gamma_{3-i,1}^k \ / \ x \in \gamma_{3-i,2}^k$$

$$\bar{\eta}_{i,3-i}^k = \pm \frac{h_{ki}}{3} T_i^+ \left(\frac{\partial}{\partial x_{3-i}} \left(p_{i,3-i}^k \frac{\partial u^k}{\partial x_{3-i}} \right) \right) \mp \frac{h_{ki}}{2} T_i^+ \left(p_{i,3-i}^k \frac{\partial^2 u^k}{\partial x_{3-i}^2} \right)$$
$$+ \frac{h_{ki}}{2} T_i^+ \left(\frac{\partial}{\partial x_i} \left(p_{i,3-i}^k \frac{\partial u^k}{\partial x_{3-i}} \right) \right), \qquad x \in \gamma_{3-i,1}^k \ / \ x \in \gamma_{3-i,2}^k,$$

$$\mu^{\star k} = \pm \frac{h_{ki}}{3} \left(T_{3-i}^{2\pm} T_i^2 q^k \right) \left(T_i^2 \frac{\partial u^k}{\partial x_{3-i}} \right), \quad x \in \gamma_{3-i,1}^k / x \in \gamma_{3-i,2}^k$$

$$\mu^{\star\star k} = \pm \frac{h_{ki}}{3} \left(T_i^{2\pm} T_{3-i}^{2\pm} q^k \right) \left(T_{3-i}^{2\pm} \frac{\partial u^k}{\partial x_i} \right) \pm \frac{h_{ki}}{3} \left(T_{3-i}^{2\pm} T_i^{2\pm} q^k \right) \left(T_i^{2\pm} \frac{\partial u^k}{\partial x_{3-i}} \right), \ \ x \in \gamma_\star^k,$$

From Lemmas 2–4 one obtains the next assertion.

THEOREM 2. *The finite difference scheme* (13) *is stable in the sense of the a priori estimate*

$$|[z]|_{H_h^1} \le C \sum_{k=1}^{2} \Bigg\{ \sum_{i,j=1}^{2} \left(|[\eta_{ij}^k \|_{k,i} + \|\zeta^k\|_{\gamma_{ij}^k} + h \, \|\mu^{\star k}\|_{\gamma_{ij}^k} \right) + |[\mu^k]|_k + |[\chi^k]|_{\bar{\gamma}_{1,3-k}^k}$$
$$+ h \sum_{i,j,l=1}^{2} |[\bar{\eta}_{ij}^k \|_{\ddot{H}^{1/2}(\gamma_{3-i,l}^{k-})} + h \sqrt{\log \frac{1}{h}} \sum_{x \in \gamma_\star^k} \left(\sum_{i=1}^{2} |\zeta_i^k| + h \, |\mu^{\star\star k}| \right) \Bigg\}. \quad (14)$$

THEOREM 3. *Let the assumptions of Lemma 2 hold. Then the solution of FDS* (10) *converges to the solution of BVP* (1)–(5) *and the convergence rate estimates*

$$|[u - v]|_{H_h^1} \le C h^{s-1} \sqrt{\log \frac{1}{h}} \Big(1 + \max_{i,j,k} \|p_{ij}^k\|_{H^{s-1}(\Omega^k)} + \max_k \|q^k\|_{H^{s-2}(\Omega^k)}$$
$$+ \max_{i,j,k} \|\alpha^k\|_{H^{s-3/2}(\Gamma_{ij}^k)} + \max_k \|\beta^k\|_{H^{s-1}(\Delta^k)} \Big) \|u\|_{H^s}, \qquad 2.5 < s < 3$$

and

$$|[u - v]|_{H_h^1} \le C h^2 \left(\log \frac{1}{h} \right)^{3/2} \Big(1 + \max_{i,j,k} \|p_{ij}^k\|_{H^2(\Omega^k)} + + \max_k \|q^k\|_{H^1(\Omega^k)}$$
$$\max_{i,j,k} \|\alpha^k\|_{H^{3/2}(\Gamma_{ij}^k)} + \max_k \|\beta^k\|_{H^2(\Delta^k)} \Big) \|u\|_{H^3}, \qquad s = 3$$

hold.

Proof. The terms η_{ij}^k and μ^k at the internal nodes of the mesh $\bar{\omega}^k$ can be estimated in the same manner as in the case of the Dirichlet BVP [4, 7]:

$$h_k h_3 \sum_{x\in\omega^k\cup\gamma_{i1}^k} \left(\eta_{ij}^k\right)^2 \le Ch^{2s-2}\,\|p_{ij}^k\|^2_{H^{s-1}(\Omega^k)}\|u^k\|^2_{H^s(\Omega^k)}, \quad 2 < s \le 3.$$

Analogous result at the boundary nodes is obtained in [5]:

$$h_k h_3 \sum_{x\in\gamma_{3-i,1}^{k-}\cup\gamma_{3-i,2}^{k-}} \left(\eta_{ij}^k\right)^2 \le Ch^{2s-2}\,\|p_{ij}^k\|^2_{H^{s-1}(\Omega^k)}\|u^k\|^2_{H^s(\Omega^k)}, \quad 2.5 < s \le 3.$$

From these inequalities, it follows

$$|[\eta_{ij}^k\|_{k,i} \le Ch^{s-1}\,\|p_{ij}^k\|_{H^{s-1}(\Omega^k)}\|u^k\|_{H^s(\Omega^k)}, \quad 2.5 < s \le 3. \tag{15}$$

Analogously,

$$|[\mu^k]|_k \le Ch^{s-1}\,\|q^k\|_{H^{s-2}(\Omega^k)}\|u^k\|_{H^s(\Omega^k)}, \quad 2.5 < s \le 3. \tag{16}$$

Terms $\mu^{\star k}$ and $\mu^{\star\star k}$ can be estimated directly:

$$\|\mu^{\star k}\|_{\gamma_{ij}^k} \le Ch\,\|q^k\|_{H^{s-2}(\Omega^k)}\|u^k\|_{H^s(\Omega^k)}, \qquad s > 2.5, \tag{17}$$

$$|\mu^{\star\star k}(x)| \le C\,\|q^k\|_{H^{s-2}(\Omega^k)}\,\|u\|_{H^s(\Omega^k)}, \qquad x\in\gamma_\star^k, \qquad s > 2. \tag{18}$$

Terms analogous to ζ^k, ζ_i^k, $\bar{\eta}_{ij}^k$, are estimated in [5] whereby it follows that:

$$\|\zeta^k\|_{\gamma_{ij}^k} \le Ch^{s-1}\|\alpha^k\|_{H^{s-3/2}(\Gamma_{ij}^k)}\|u^k\|_{H^s(\Omega^k)}, \qquad 2.5 < s \le 3, \tag{19}$$

$$\|\zeta_i^k\|_{\gamma_\star^k} \le Ch\,\|\alpha^k\|_{H^{s-3/2}(\Gamma_{3-i,j}^k)}\,\|u\|_{H^s(\Omega^k)}, \qquad s > 2, \tag{20}$$

$$\|\bar{\eta}_{ij}^k\|_{\ddot{H}^{1/2}(\gamma_{3-i,l}^{k-})} \le Ch^{s-2}\sqrt{\log\frac{1}{h}}\,\|p_{ij}^k\|_{H^{s-1}(\Omega^k)}\,\|u^k\|_{H^s(\Omega^k)}, \quad 2.5 < s < 3, \tag{21}$$

The term χ^k can be estimated in the following manner: let us denote

$$I_1 = I_1(g) = \int_0^h g(x)\,\mathrm{d}x - \frac{h}{2}\,[g(0)+g(h)].$$

For $r > 0.5$ $I_1(g)$ is a bounded linear functional of $g\in H^r(0,h)$ which vanishes when $g(x)=1$ and $g(x)=x$. Using the Bramble-Hilbert lemma [4] we obtain

$$|I_1| \le Ch^{r+1/2}\,|g|_{H^r(0,h)}, \qquad 0.5 < r \le 2,$$

whereby it follows that

$$\left|\int_0^1 g(x)\,\mathrm{d}x - h\left[\frac{g(0)}{2} + \sum_{i=1}^{n-1} g(ih) + \frac{g(1)}{2}\right]\right| \le Ch^r\,|g|_{H^r(0,1)}, \quad 0.5 < r \le 2.$$

From this inequality, using properties of multipliers in Sobolev spaces [13], we immediately obtain

$$\begin{aligned}|\chi^k(x)| &\le Ch^r \,\big\|T_{k2}^2\beta^k(x,\cdot)\,u^{3-k}(\cdot)\big\|_{H^r(\Gamma_{1k}^{3-k})}\\ &\le Ch^r \,\big\|T_{k2}^2\beta^k(x,\cdot)\big\|_{H^r(\Gamma_{1k}^{3-k})}\,\big\|u^{3-k}(\cdot)\big\|_{H^r(\Gamma_{1k}^{3-k})}, \qquad 1<r\le 2\end{aligned}$$

when $x\in\gamma_{1,3-k}^k$, while for $x\in\gamma_{1,3-k}^{k\star}$ the analogous inequalities hold. After summation over the mesh $\bar\gamma_{1,3-k}^k$ we obtain

$$|[\chi^k]|_{\bar\gamma_{1,3-k}^k}\le Ch^r\,\big\|\beta^k\big\|_{H^r(\Delta^k)}\,\big\|u^{3-k}\big\|_{H^r(\Gamma_{1k}^{3-k})}, \qquad 1<r\le 2.$$

Finally, using the trace theorem for anisotropic Sobolev spaces [12] and denoting $r=s-1$, we get

$$|[\chi^k]|_{\bar\gamma_{1,3-k}^k}\le Ch^{s-1}\,\big\|\beta^k\big\|_{H^{s-1}(\Delta^k)}\,\big\|u^{3-k}\big\|_{H^s(\Omega^{3-k})}, \quad 2<s\le 3. \tag{22}$$

The assertion follows from (13)–(22). ■

Remark 1. Convergence rate estimates of the form

$$\|u-v\|_{H^m(\Omega_h)}\le Ch^{s-m}\|u\|_{H^s(\Omega)}, \qquad m<s\le m+r$$

are often called "compatible with the smoothness of the solution" (se e.g. [4]). Here u is the solution of boundary value problem (defined in the domain Ω), v is the solution of the corresponding finite difference scheme (defined on the mesh $\Omega_h\subset\Omega$), h is discretization parameter (mesh size), r is a given constant (the highest possible order of convergence), $H^s(\Omega)$ is Sobolev space and $H^m(\Omega_h)$ is Sobolev space of mesh-functions. In such a manner, error bounds obtained in Theorem 3 are compatible with the smoothness of the solution, up to a slowly increasing logarithmic factor of the mesh size.

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