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FIXED POINTS OF A PAIR OF LOCALLY CONTRACTIVE MAPPINGS IN ORDERED PARTIAL METRIC SPACES

Abdullah Shoaib, Muhammad Arshad and Akbar Azam

Abstract. Common fixed point results for mappings satisfying locally contractive conditions on a closed ball in a 0-complete ordered partial metric space have been established. The notion of dominated mappings of Economics, Finance, Trade and Industry has also been applied to approximate the unique solution to non-linear functional equations. Our results improve some well-known, primary and conventional results.

1. Introduction

Let $T\colon X \to X$ be a mapping. A point $x \in X$ is called a fixed point of T if $x = Tx$. In 1922, Banach obtained unique fixed point of a mapping $T\colon X \to X$ satisfying:

$$d(Tx, Ty) \le kd(x, y),$$

for all $x, y \in X$, where $0 \le k < 1$ and X is a complete metric space. It is important to note that this theorem has laid down the foundation of modern fixed points theory for contractive type mappings.

Fixed points results for mappings satisfying certain contractive conditions on the entire domain has been at the centre of vigorous research activity, for example see [5–7, 11, 12, 17, 22, 24, 26–28], and it has a wide range of applications in different areas such as nonlinear and adaptive control systems, parameterized estimation problems, fractal image decoding, computing magnetostatic fields in a nonlinear medium, and convergence of recurrent networks, see [19, 21, 31, 32].

From the application point of view the situation is not yet completely satisfactory because it frequently happens that a mapping T is a contraction not on the entire space X but merely on a subset Y of X. However, if Y is closed, then it is complete, so that T has a fixed point x in Y, and $x_n \to x$ as in the case of the whole space X, provided we impose a subtle restriction on the choice of x_0, so that x'_ms remains in Y. Recently, Azam et al. [10] proved a significant result concerning

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the existence of fixed points of a fuzzy mapping satisfying a contractive condition on a closed ball of a complete metric space. Other results on closed balls can be seen in [6, 9, 29].

On the other hand, the notion of a partial metric space was introduced by G. S. Matthews in [20]. In partial metric spaces, the distance of a point from itself may not be zero. After the definition of partial metric space, Matthews proved the partial metric version of Banach fixed point theorem. Then, Oltra et al. [25] and Altun et al. [2] gave some generalizations of the result of Matthews. Altun et al. [3] gave fixed point results for mappings satisfying generalized contractions on partial metric spaces (see also [4]). Further results in this direction under different conditions were proved, e.g., in [14]. Romaguera [30] gave the idea of 0-complete partial metric space. Nashine et al. [23] used this concept and proved some classical results. In this paper, we will exploit this concept for two, three and four mappings on a 0-complete ordered partial metric space X to generalize/improve some classical fixed point results.

Recently, Haghi et al. [15] deduced some partial metric fixed point results from the corresponding results in metric spaces. However, we show that the results proved in this paper cannot be deduced from the corresponding results in metric spaces (see Example 2.3 and Remak 2.4). Our results not only extend some primary theorems to ordered partial metric spaces but also restrict the contractive conditions on a closed ball only. The concept of dominated mapping which comes from real world has been applied to approximate the unique solution of non-linear functional equations. The dominated mapping which satisfies the condition $fx \preceq x$ occurs very naturally in several practical problems. For example, if x denotes the total quantity of food produced over a certain period of time and $f(x)$ gives the quantity of food consumed over the same period in a certain town, then we must have $fx \preceq x$.

Consistent with [1, 4, 8, 15, 20], the following definitions and results will be needed in the sequel.

Definition 1.1. [20] Let X be a nonempty set. If for any $x, y, z \in X$, a mapping $p\colon X \times X \to R^+$ satisfies

(P_1) $p(x,x) = p(y,y) = p(x,y)$ if and only if $x = y$,

(P_2) $p(x,x) \leq p(x,y)$,

(P_3) $p(x,y) = p(y,x)$,

(P_4) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$,

then it is said to be a partial metric on X and the pair (X,p) is called a partial metric space.

Each partial metric p on X induces a T_0 topology τ_p on X which has as a base the family of open balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$. Also, $\overline{B_p(x,r)} = \{y \in X : p(x,y) \leq p(x,x) + r\}$ is a closed ball in (X,p).

It is clear that if $p(x,y) = 0$, then from P_1 and P_2, $x = y$. But if $x = y$, then $p(x,y)$ may not be 0.

EXAMPLE 1.2. [20] If $X = [0, \infty)$ then, $p(x, y) = \max\{x, y\}$ for all $x, y \in X$, defines a partial metric p on X.

DEFINITION 1.3. [20] Let (X, p) be a partial metric space. Then,

(a) A sequence $\{x_n\}$ in (X, p) converges to a point $x \in X$ if and only if $\lim_{n\to\infty} p(x, x_n) = p(x, x)$.

(b) A sequence $\{x_n\}$ in (X, p) is called a Cauchy sequence if the limit $\lim_{n,m\to\infty} p(x_n, x_m)$ exists (and is finite).

(c) [30] A sequence $\{x_n\}$ in (X, p) is called 0-Cauchy if $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. The space (X, p) is called 0-complete if every 0-Cauchy sequence in X converges to a point $x \in X$ such that $p(x, x) = 0$.

If (X, p) is a partial metric space, then $p_s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$, $x, y \in X$, is a metric on X.

LEMMA 1.4. [20] *Let (X, p) be a partial metric space. Then,*

(a) *$\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p_s).*

(b) *(X, p) is complete if and only if the metric space (X, p_s) is complete.*

(c) [30] *Every 0-Cauchy sequence in (X, p) is Cauchy in (X, p_s).*

(d) *If (X, p) is complete, then it is 0-complete.*

Romaguera [30] gave an example which proves that converse assertions of (c) and (d) do not hold. It is easy to see that every closed subset of a 0-complete partial metric space is 0-complete.

DEFINITION 1.5. [4] Let X be a nonempty set. Then $(X, \preceq, p)$ is called an ordered partial metric space if: (i) p is a partial metric on X and (ii) $\preceq$ is a partial order on X.

DEFINITION 1.6. Let $(X, \preceq)$ be a partially ordered set. Then $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.

DEFINITION 1.7. [1] Let $(X, \preceq)$ be a partially ordered set. A self mapping f on X is called dominated if $fx \preceq x$ for each x in X.

EXAMPLE 1.8. [1] Let $X = [0, 1]$ be endowed with the usual ordering and $f : X \to X$ be defined by $fx = x^n$ for some $n \in N$. Since $fx = x^n \leq x$ for all $x \in X$, therefore f is a dominated map.

DEFINITION 1.9. Let X be a nonempty set and $T, f : X \to X$. A point $x \in X$ is called a coincidence point and $y \in X$ is called a point of coincidence of T and f if $y = Tx = fx$. The mappings T, f are said to be weakly compatible if they commute at their coincidence points (i.e., $Tfx = fTx$ whenever $Tx = fx$).

We require the following lemmas for subsequent use:

LEMMA 1.10. [15] *Let X be a nonempty set and $f\colon X \to X$ be a function. Then there exists a subset $E \subset X$ such that $fE = fX$ and $f\colon E \to X$ is one-to-one.*

LEMMA 1.11. [8] *Let X be a nonempty set and the mappings $S, T, f\colon X \to X$ have a unique point of coincidence v in X. If (S, f) and (T, f) are weakly compatible, then S, T, f have a unique common fixed point.*

2. Fixed points of Banach mappings

The following result regarding the existence of a fixed point of a mapping satisfying a contractive condition on the closed ball is given in [18, Theorem 5.1.4]. The result is very useful in the sense that it requires the contraction of the mapping only on the closed ball instead on the whole space.

THEOREM 2.1. [18] *Let (X, d) be a complete metric space, $S\colon X \to X$ be a mapping, $r > 0$ and x_0 be an arbitrary point in X. Suppose there exists $k \in [0, 1)$ with*

$$d(Sx, Sy) \le kd(x, y), \text{ for all } x, y \in Y = \overline{B(x_0, r)}$$

and $d(x_0, Sx_0) < (1 - k)r$. Then there exists a unique point x^* in $\overline{B(x_0, r)}$ such that $x^* = Sx^*$.

In the proof [18], the author considers an iterative sequence $x_n = Sx_{n-1}$, $n \ge 0$ and exploits the contraction condition on the points x_m's to see that

$$d(x_m, x_n) \le \frac{k^m}{1-k} d(x_0, x_1),$$

by using techniques of [18, Theorem 5.1.2] before proving that x_m's lie in the closed ball. The following theorem not only extends the above theorem to ordered partial metric spaces but also rectifies this mistake specially for those researchers who are utilizing the style of the proof of [18, Theorem 5.1.4] to study more general result.

THEOREM 2.2. *Let $(X, \preceq, p)$ be a 0-complete ordered partial metric space, $S, T\colon X \to X$ be dominated maps and x_0 be an arbitrary point in X. Suppose there exists $k \in [0, 1)$ with*

$$p(Sx, Ty) \le kp(x, y), \text{ for all comparable elements } x, y \text{ in } \overline{B(x_0, r)} \tag{2.1}$$

$$\text{and } p(x_0, Sx_0) \le (1 - k)[r + p(x_0, x_0)]. \tag{2.2}$$

Then:

(i) *There exists $x^* \in \overline{B(x_0, r)}$ such that $p(x^*, x^*) = 0$.*

(ii) *If, for a non-increasing sequence $\{x_n\}$ in $\overline{B(x_0, r)}$, $\{x_n\} \to u$ implies that $u \preceq x_n$, then there exists a point x^* in $\overline{B(x_0, r)}$ such that $x^* = Sx^* = Tx^*$.*

(iii) *If for any two points x, y in $\overline{B(x_0, r)}$ there exists a point $z \in \overline{B(x_0, r)}$ such that $z \preceq x$ and $z \preceq y$, that is every pair of elements has a lower bound, then the point x^* is unique.*

Proof. Choose a point x_1 in X such that $x_1 = Sx_0$. As $Sx_0 \preceq x_0$ so $x_1 \preceq x_0$ and let $x_2 = Tx_1$. Now $Tx_1 \preceq x_1$ gives $x_2 \preceq x_1$; continuing this process, we construct a sequence x_n of points in X such that

$$x_{2i+1} = Sx_{2i},\ x_{2i+2} = Tx_{2i+1} \text{ and } x_{2i+1} = Sx_{2i} \preceq x_{2i} \text{ where } i = 0, 1, 2, \dots .$$

First we show that $x_n \in \overline{B(x_0, r)}$ for all $n \in N$. Using inequality (2.2), we have,

$$p(x_0, x_1) \leq (1 - k)\,[r + p(x_0, x_0)] \leq r + p(x_0, x_0).$$

It follows that $x_1 \in \overline{B(x_0, r)}$. Let $x_2, \dots, x_j \in \overline{B(x_0, r)}$ for some $j \in N$. If $j = 2i+1$, then $x_{2i+1} \preceq x_{2i}$, where $i = 0, 1, 2, \dots, \frac{j-1}{2}$ so using inequality (2.1), we obtain

$$\begin{aligned} p(x_{2i+1}, x_{2i+2}) &= p(Sx_{2i}, Tx_{2i+1}) \leq k[p(x_{2i}, x_{2i+1})] \\ &\leq k^2[p(x_{2i-1}, x_{2i})] \leq \dots \leq k^{2i+1} p(x_0, x_1). \end{aligned} \tag{2.3}$$

If $j = 2i + 2$, then as $x_1, x_2, \dots, x_j \in \overline{B(x_0, r)}$ and $x_{2i+2} \preceq x_{2i+1}$, ($i = 0, 1, 2, \dots, \frac{j-2}{2}$), we obtain

$$p(x_{2i+2}, x_{2i+3}) \leq k^{2(i+1)} p(x_0, x_1). \tag{2.4}$$

Thus from inequalities (2.3) and (2.4), we have

$$p(x_j, x_{j+1}) \leq k^j p(x_0, x_1). \tag{2.5}$$

Now,

$$\begin{aligned} p(x_0, x_{j+1}) &\leq p(x_0, x_1) + \dots + p(x_j, x_{j+1}) - [p(x_1, x_1) + \dots + p(x_j, x_j)] \\ &\leq p(x_0, x_1) + \dots + k^j p(x_0, x_1) \qquad \text{(by 2.5)} \\ &\leq p(x_0, x_1)[1 + \dots + k^{j-1} + k^j] \\ &\leq \frac{(1 - k^{j+1})}{1 - k} p(x_0, x_1) \\ &\leq \frac{(1 - k^{j+1})}{1 - k} (1 - k)\,[r + p(x_0, x_0)] \qquad \text{(by 2.2)} \\ &\leq (1 - k^{j+1})[r + p(x_0, x_0)] \\ &\leq r + p(x_0, x_0). \end{aligned}$$

Thus $x_{j+1} \in \overline{B(x_0, r)}$. Hence $x_n \in \overline{B(x_0, r)}$ for all $n \in N$. It implies that

$$p(x_n, x_{n+1}) \leq k^n p(x_0, x_1), \text{ for all } n \in N. \tag{2.6}$$

So we have

$$\begin{aligned} p(x_{n+k}, x_n) &\leq p(x_{n+k}, x_{n+k-1}) + \dots + p(x_{n+1}, x_n) \\ &\leq k^{n+k-1} p(x_0, x_1) + \dots + k^n p(x_0, x_1), \qquad \text{(by 2.6)} \\ p(x_{n+k}, x_n) &\leq k^n p(x_0, x_1)[k^{k-1} + k^{k-2} + \dots + 1] \\ &\leq k^n p(x_0, x_1) \frac{(1 - k^k)}{1 - k} \to 0 \text{ as } n \to \infty. \end{aligned}$$

Notice that the sequence $\{x_n\}$ is a 0-Cauchy sequence in $(\overline{B_p(x_0,r)}, p)$. Therefore there exists a point $x^* \in \overline{B_p(x_0,r)}$ with $\lim_{n\to\infty} x_n = x^*$. Also,

$$\lim_{n\to\infty} p(x_n, x^*) = 0. \tag{2.7}$$

Using the fact that $x^* \preceq x_n$ for all n, we have

$$\begin{aligned} p(x^*, Sx^*) &\le p(x^*, x_{2n+2}) + p(x_{2n+2}, Sx^*) - p(x_{2n+2}, x_{2n+2}) \\ &\le p(x^*, x_{2n+2}) + kp(x_{2n+1}, x^*). \end{aligned}$$

On taking limit as $n \to \infty$, we obtain $p(x^*, Sx^*) \le 0$, and hence $x^* = Sx^*$. Similarly, by using

$$p\,(x^*, Tx^*) \le p(x^*, x_{2n+1}) + p(x_{2n+1}, Tx^*) - p(x_{2n+1}, x_{2n+1}),$$

we can show that $x^* = Tx^*$. Hence S and T have a common fixed point in $\overline{B(x_0,r)}$. Now,

$$\begin{aligned} p(x^*, x^*) = p(Sx^*, Tx^*) &\le kp(x^*, x^*) \\ (1-k)p(x^*, x^*) &\le 0. \end{aligned}$$

This implies that $p(x^*, x^*) = 0$.

For uniqueness, assume that y is another fixed point of T in $\overline{B(x_0,r)}$. If x^* and y are comparable then,

$$p(x^*, y) = p(Sx^*, Ty) \le kp(x^*, y).$$

This shows that $x^* = y$. Now if x^* and y are not comparable then there exists a point $z_0 \in \overline{B(x_0,r)}$ such that $z_0 \preceq x^*$ and $z_0 \preceq y$. Choose a point z_1 in X such that $z_1 = Tz_0$. As $Tz_0 \preceq z_0$, so $z_1 \preceq z_0$ and let $z_2 = Sz_1$. Now $Sz_1 \preceq z_1$ gives $z_2 \preceq z_1$. Continuing this process and choose z_n in X such that

$$z_{2i+1} = Tz_{2i},\ z_{2i+2} = Sz_{2i+1} \text{ and } z_{2i+1} = Tz_{2i} \preceq z_{2i} \text{ where } i = 0, 1, 2, \dots .$$

It follows that $z_{n+1} \preceq z_n \preceq \cdots \preceq z_0 \preceq x^* \preceq x_n$. We will prove that $z_n \in \overline{B(x_0,r)}$ for all $n \in N$ by using mathematical induction. For $n = 1$,

$$\begin{aligned} p(x_0, z_1) &\le p(x_0, x_1) + p(x_1, z_1) - p(x_1, x_1) \\ &\le (1-k)[r + p(x_0, x_0)] + kp(x_0, z) \\ &\le (1-k)r + (1-k)p(x_0, x_0) + k[r + p(x_0, x_0)] \\ &\le r + p(x_0, x_0). \end{aligned}$$

It follows that $z_1 \in \overline{B(x_0,r)}$. Let $z_2, z_3, \dots, z_j \in \overline{B(x_0,r)}$ for some $j \in N$. Note that if j is odd then

$$p(x_{j+1}, z_{j+1}) = p(Tx_j, Sz_j)) \le kp(x_j, z_j) \le \cdots \le k^{j+1} p(x_0, z_0),$$

and if j is even then

$$p(x_{j+1}, z_{j+1}) = p(Sx_j, Tz_j) \le kp(x_j, z_j) \le \cdots \le k^{j+1} p(x_0, z_0).$$

Now,

$$
\begin{aligned}
p(x_0, z_{j+1}) &\le p(x_0, x_1) + p(x_1, x_2) + \cdots + p(x_{j+1}, z_{j+1}) \\
&\qquad - [p(x_1, x_1) + \cdots + p(x_{j+1}, x_{j+1})] \\
&\le p(x_0, x_1) + kp(x_0, x_1) + \cdots + k^{j+1} p(x_0, z_0) \\
&\le p(x_0, x_1)[1 + k + \cdots + k^j] + k^{j+1}[r + p(x_0, x_0)] \\
&\le (1-k)[r + p(x_0, x_0)] \frac{(1-k^{j+1})}{1-k} + k^{j+1} r + k^{j+1} p(x_0, x_0), \\
p(x_0, z_{j+1}) &\le r + p(x_0, x_0).
\end{aligned}
$$

Thus $z_{j+1} \in \overline{B(x_0, r)}$. Hence $z_n \in \overline{B(x_0, r)}$ for all $n \in N$. As $z_0 \preceq x^*$ and $z_0 \preceq y$, it follows that $z_{n+1} \preceq T^n x^*$ and $z_{n+1} \preceq T^n y$ for all $n \in N$ as $T^n x^* = x^*$ and $T^n y = y$ for all $n \in N$. If n is odd then,

$$
\begin{aligned}
p(x^*, y) &= p(T^n x^*, T^n y) \\
&\le p(T^n x^*, S z_n) + p(S z_n, T^n y) - p(S z_n, S z_n) \\
&\le kp(T^{n-1} x^*, z_n) + kp(z_n, T^{n-1} y) \\
&= kp(S^{n-1} x^*, T z_{n-1}) + kp(T z_{n-1}, S^{n-1} y) \\
&\le k^2 p(S^{n-2} x^*, z_{n-1}) + k^2 p(z_{n-1}, S^{n-2} y) \\
&\;\vdots \\
&\le k^{n+1} p(x^*, z_0) + k^{n+1} p(z_0, y) \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

So $x^* = y$. Similarly, we can show that $x^* = y$ if n is even. Hence x^* is a unique common fixed point of T and S in $\overline{B(x_0, r)}$. ∎

EXAMPLE 2.3. Let $X = Q^+ \cup \{0\}$ and $\overline{B(x_0, r)} = [0, 1] \cap X$ be endowed with usual order and let $p \colon X \times X \to X$ be the 0-complete partial metric on X defined by $p(x, y) = \max\{x, y\}$. Let $S, T \colon X \to X$ be defined by

$$
Sx = \begin{cases} \dfrac{x}{7} & \text{if } x \in [0,1] \cap X \\ x - \frac{1}{3} & \text{if } x \in (1, \infty) \cap X \end{cases} \quad \text{and} \quad Tx = \begin{cases} \dfrac{2x}{7} & \text{if } x \in [0,1] \cap X \\ x - \frac{1}{4} & \text{if } x \in (1, \infty) \cap X. \end{cases}
$$

Clearly, S and T are dominated mappings. For all comparable elements with $k = \frac{3}{10} \in [0, \frac{1}{2})$, $x_0 = \frac{1}{2}$, $r = \frac{1}{2}$, $p(x_0, x_0) = \max\{\frac{1}{2}, \frac{1}{2}\} = \frac{1}{2}$,

$$
\begin{aligned}
(1-k)[r + p(x_0, x_0)] &= (1 - \frac{3}{10})[\frac{1}{2} + \frac{1}{2}] = \frac{7}{10}, \\
p(x_0, S x_0) = p(\frac{1}{2}, S\frac{1}{2}) &= p(\frac{1}{2}, \frac{1}{14}) = \max\{\frac{1}{2}, \frac{1}{14}\} = \frac{1}{2} < \frac{7}{10}.
\end{aligned}
$$

Putting $x = y = 2$ we obtain

$$
p(S2, T2) = \max\{\frac{5}{3}, \frac{7}{4}\} = \frac{7}{4} > \frac{3}{5} = \frac{3}{10} \max\{2, 2\}.
$$

So the contractive condition does not hold on X. Now if $x,y \in \overline{B(x_0,r)}$, then

$$p(Sx,Ty) = \max\{\frac{x}{7}, \frac{2y}{7}\} = \frac{1}{7}\max\{x,2y\} \le \frac{3}{10}\max\{x,y\} \le kp(x,y).$$

Therefore, all the conditions of Theorem 2.2 are satisfied and 0 is the common fixed point of S and T. Moreover, note that for any metric d on X,

$$d(S1,T1) = d\left(\frac{1}{7}, \frac{2}{7}\right) > kd(1,1) = 0 \text{ for any } k \in [0,1).$$

Therefore common fixed points of S and T cannot be obtained from a corresponding metric fixed point theorem. Also X is not complete in any metric space.

Remark 2.4. If we impose Banach type contractive condition for a pair $S,T\colon X \to X$ of mappings on a metric space (X,d), that is

$$d(Sx,Ty) \le kd(x,y) \quad \text{for all } x,y \in X,$$

then it follows that $Sx = Tx$, for all $x \in X$ (that is S and T are equal). Therefore the above condition fails to find common fixed points of S and T. However the same condition in a partial metric space does not assert that $S = T$, as is seen in Example 2.3. Hence Theorem 2.2 cannot be obtained from a corresponding metric fixed point theorem.

Corollary 2.5. *Let $(X,\preceq,p)$ be a 0-complete ordered partial metric space, $S,T\colon X \to X$ be dominated maps and x_0 be an arbitrary point in X. Suppose there exists $k \in [0,1)$ with*

$$p(Sx,Ty) \le kp(x,y), \text{ for all comparable elements } x,y \text{ in } \overline{B(x_0,r)}$$
$$\text{and } p(x_0,Sx_0) \le (1-k)[r + p(x_0,x_0)].$$

Then there exists $x^* \in \overline{B(x_0,r)}$ such that $p(x^*,x^*) = 0$. Also if, for a non-increasing sequence $\{x_n\}$ in $\overline{B(x_0,r)}$, $\{x_n\} \to u$ implies that $u \preceq x_n$, then there exists a point x^* in $\overline{B(x_0,r)}$ such that $x^* = Sx^* = Tx^*$.

Corollary 2.6. *Let $(X,\preceq,p)$ be a 0-complete ordered partial metric space, $S,T\colon X \to X$ be the dominated map and x_0 be an arbitrary point in X. Suppose there exists $k \in [0,1)$ with*

$$p(Sx,Ty) \le kp(x,y), \text{ for all comparable elements } x,y \text{ in } X.$$

Then there exists $x^* \in X$ such that $p(x^*,x^*) = 0$. Also if, for a non-increasing sequence $\{x_n\}$ in X, $\{x_n\} \to u$ implies that $u \preceq x_n$ and for any two points x,y in X there exists a point $z \in X$ such that $z \preceq x$ and $z \preceq y$, then there exists a unique point x^* in X such that $x^* = Sx^* = Tx^*$.

Corollary 2.7. *Let (X,p) be a 0-complete partial metric space, $S,T\colon X \to X$ be self-maps and x_0 be an arbitrary point in X. Suppose there exists $k \in [0,1)$ with*

$$p(Sx,Ty) \le kp(x,y), \text{ for all elements } x,y \text{ in } \overline{B(x_0,r)}$$
$$\text{and } p(x_0,Sx_0) \le (1-k)[r + p(x_0,x_0)].$$

Then there exists a unique $x^* \in \overline{B(x_0, r)}$ *such that* $p(x^*, x^*) = 0$ *and* $x^* = Sx^* = Tx^*$. *Further,* S *and* T *have no fixed points other than* x^*.

Proof. By Theorem 2.2, $x^* = Sx^* = Tx^*$. Let y be another point such that $y = Ty$. Then

$$p(x^*, y) = p(Sx^*, Ty) \leq kp(x^*, y)$$

This shows that $x^* = y$. Thus T has no fixed points other than x^*. Similarly, S has no fixed points other than x^*. ■

Now we apply our Theorem 2.2 to obtain unique common fixed point of three mappings on a closed ball in a 0-complete partial ordered metric space.

THEOREM 2.8. *Let* $(X, \preceq, p)$ *be an ordered partial metric space,* S, T *be self mappings and* f *be a dominated mapping on* X *such that* $SX \cup TX \subset fX$ *and* $Tx, Sx \preceq fx$. *Assume that for* $r > 0$ *and an arbitrary point* x_0 *in* X*, the following conditions hold:*

$$p\left(Sx, Ty\right) \leq kp(fx, fy) \tag{2.10}$$

for all comparable elements $fx,\ fy \in \overline{B(fx_0, r)} \subseteq fX$*, and some* $0 \leq k < 1$ *and*

$$p(fx_0, Tx_0) \leq (1 - k)[r + p(fx_0, fx_0)]\ . \tag{2.11}$$

Let for a non-increasing sequence $\{x_n\} \to u$ *implies that* $u \preceq x_n$*; also for any two points* z *and* x *in* $\overline{B(fx_0, r)}$ *there exists a point* $y \in \overline{B(fx_0, r)}$ *such that* $y \preceq z$ *and* $y \preceq x$. *If* fX *is a* 0*-complete subspace of* X *and* (S, f) *and* (T, f) *are weakly compatible, then* S, T *and* f *have a unique common fixed point* fz *in* $\overline{B(fx_0, r)}$. *Also* $p(fz, fz) = 0$.

Proof. By Lemma 1.10, there exists $E \subset X$ such that $fE = fX$ and $f \colon E \to X$ is one-to-one. Now since $SX \cup TX \subset fX$, we define two mappings $g, h \colon fE \to fE$ by $g(fx) = Sx$ and $h(fx) = Tx$, respectively. Since f is one-to-one on E, then g, h are well-defined. As $Sx \preceq fx$ implies that $g(fx) \preceq fx$ and $Tx \preceq fx$ implies that $h(fx) \preceq fx$ therefore g and h are dominated maps. Now $fx_0 \in \overline{B(fx_0, r)} \subseteq fX$. Then $fx_0 \in fX$. Choose a point x_1 in fX such that $x_1 = h(fx_0)$. As $h(fx_0) \preceq fx_0 \preceq x_0$, so $x_1 \preceq x_0$ and let $x_2 = g(fx_1)$. Now $g(fx_1) \preceq fx_1 \preceq x_1$ gives $x_2 \preceq x_1$. Continuing this process and having chosen x_n in fX such that

$$x_{2i+1} = h(fx_{2i}) \text{ and } x_{2i+2} = g(fx_{2i+1}), \text{ where } i = 0, 1, 2, \ldots,$$

then $x_{2i+1} = h(fx_{2i}) \preceq fx_{2i} \preceq x_{2i}$. Following similar arguments as those of Theorem 2.2, $x_n \in \overline{B(fx_0, r)}$. Also by inequality (2.11),

$$p(fx_0, h(fx_0)) \leq (1 - k)[r + p(fx_0, fx_0)].$$

Note that for $fx, fy \in \overline{B(fx_0, r)}$, where $fx,\ fy$ are comparable. Then by using inequality (2.10), we have

$$p\left(g(fx), h(fy)\right) \leq kp(fx, fy).$$

As fE is a 0-complete space, all conditions of Theorem 2.2 are satisfied, we deduce that there exists a unique common fixed point $fz \in \overline{B(fx_0, r)}$ of g and h. Also $p(fz, fz) = 0$. Now $fz = g(fz) = h(fz)$ or $fz = Sz = Tz = fz$. Thus fz is the point of coincidence of S, T and f. Let $v \in \overline{B(fx_0, r)}$ be another point of coincidence of f, S and T. Then there exists $u \in \overline{B(fx_0, r)}$ such that is $v = fu = Su = Tu$, which implies that $fu = g(fu) = h(fu)$. A contradiction as $fz \in \overline{B(fx_0, r)}$ is a unique common fixed point of g and h. Hence $v = fz$. Thus S, T and f have a unique point of coincidence $fz \in \overline{B(fx_0, r)}$. Now since (S, f) and (T, f) are weakly compatible, by Lemma 1.11 fz is a unique common fixed point of S, T and f. ■

In the following theorem we use Corollary 2.7 to establish the existence of a unique common fixed point of four mappings on closed ball in 0-complete partial metric space.

THEOREM 2.9. *Let (X, p) be a partial metric space and S, T, g and f be self mappings on X such that SX, $TX \subset fX = gX$. Assume that for some $r > 0$ and an arbitrary point x_0 in X, the following conditions hold:*

$$p\,(Sx, Ty) \leq kp(fx, gy) \tag{2.12}$$

for all elements fx, $gy \in \overline{B(fx_0, r)} \subseteq fX$ and some $0 \leq k < 1$, and

$$p(fx_0, Sx_0) \leq (1 - k)[r + p(fx_0, fx_0)]. \tag{2.13}$$

If fX is a 0-complete subspace of X then there exists $fz \in X$ such that $p(fz, fz) = 0$. Also if (S, f) and (T, g) are weakly compatible, then S, T, f and g have a unique common fixed point fz in $\overline{B(fx_0, r)}$. Further, S and T have no fixed points other than x^*.

Proof. By Lemma 1.10, there exists $E_1, E_2 \subset X$ such that $fE_1 = fX = gX = gE_2$, $f\colon E_1 \to X$, $g\colon E_2 \to X$ are one-to-one. Now define the mappings $A, B\colon fE_1 \to fE_1$ by $A(fx) = Sx$ and $B(gx) = Tx$ respectively. Since f, g are one to one on E_1, and E_2 respectively, then the mappings A, B are well-defined. As $fx_0 \in \overline{B(fx_0, r)} \subseteq fX$, then $fx_0 \in fX$. Choose a point x_1 in fX such that $x_1 = A(fx_0)$ and let $x_2 = B(gx_1)$. Continuing this process choose x_n in fX such that

$$x_{2i+1} = A(fx_{2i}) \text{ and } x_{2i+2} = B(gx_{2i+1}), \text{ where } i = 0, 1, 2, \ldots$$

following similar arguments of Theorem 2.2, and we have $x_n \in \overline{B(fx_0, r)}$. Also by inequality (2.13)

$$p(fx_0, A(fx_0)) \leq (1 - k)[r + p(fx_0, fx_0)].$$

By using inequality (2.12), for $fx, gy \in \overline{B(fx_0, r)}$, we have

$$p\,(A(fx), B(gy)) \leq kp(fx, gy).$$

As fX is a 0-complete space, all conditions of Corollary 2.7 are satisfied, and we deduce that there exists a unique common fixed point $fz \in \overline{B(fx_0, r)}$ of A and B.

Further A and B have no fixed points other than fz. Also $p(fz, fz) = 0$. Now $fz = A(fz) = B(fz)$ or $fz = Sz = fz$. Thus fz is a point of coincidence of f and S. Let $w \in \overline{B(fx_0, r)}$ be another point of coincidence of S and f. Then there exists $u \in \overline{B(fx_0, r)}$ such that is $w = fu = Su$, which implies that $fu = A(fu)$. A contradiction as $fz \in \overline{B(fx_0, r)}$ is a unique fixed point of A. Hence $w = fz$. Thus S and f have a unique point of coincidence $fz \in \overline{B(fx_0, r)}$. Since (S, f) are weakly compatible, by Lemma 1.11 fz is a unique common fixed point of S and f. As $fX = gX$, there exist $v \in X$ such that $fz = gv$. Now as $A(fz) = B(fz) = fz \Rightarrow A(gv) = B(gv) = gv \Rightarrow Tv = gv$, thus gv is a point of coincidence of T and g. Now if $Tx = gx \Rightarrow B(gx) = gx$. A contradiction. This implies that $gv = gx$. As (T, g) are weakly compatible, we obtain gv, a unique common fixed point for T and g. But $gv = fz$. Thus S, T, g and f have a unique common fixed point $fz \in \overline{B(fx_0, r)}$. ■

COROLLARY 2.10. *Let $(X, \preceq, p)$ be an ordered partial metric space and S, T be self-mappings and f be a dominated mapping on X such that $SX \cup TX \subset fX$ and $Tx, Sx \preceq fx$. Assume that for $r > 0$ and an arbitrary point x_0 in X, the following conditions hold:*

$$p\,(Sx, Ty) \leq kp(fx, fy)$$

for all comparable elements $fx,\ fy \in \overline{B(fx_0, r)} \subseteq fX$ and some $0 \leq k < 1$, and

$$p(fx_0, Sx_0) \leq (1 - k)[r + p(fx_0, fx_0)].$$

If for a non-increasing sequence $\{x_n\}$ in $\overline{B(fx_0, r)}$, $\{x_n\} \to u$ implies that $u \preceq x_n$, and for any two points z and x in $\overline{B(fx_0, r)}$ there exists a point $y \in \overline{B(fx_0, r)}$ such that $y \preceq z$ and $y \preceq x$, and if fX is a 0-complete subspace of X, then S, T and f have a unique point of coincidence $fz \in \overline{B(fx_0, r)}$. Also $p(fz, fz) = 0$.

COROLLARY 2.11. *Let (X, p) be a partial metric space and S, T, g and f be self mappings on X such that $SX,\ TX \subset fX = gX$. Assume that for $r > 0$ and an arbitrary point x_0 in X, the following conditions hold:*

$$p\,(Sx, Ty) \leq kp(fx, gy)$$

for all elements $fx,\ gy \in \overline{B(fx_0, r)} \subseteq fX$ and some $0 \leq k < 1$, and

$$p(fx_0, Sx_0) \leq (1 - k)[r + p(fx_0, fx_0)].$$

If fX is 0-complete subspace of X and (S, f) and (T, g) are weakly compatible, then S, T, f and g have a unique point of coincidence fz in $\overline{B(fx_0, r)}$. Also $p(fz, fz) = 0$.

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A. Shoaib, Department of Mathematics, International Islamic University, H-10, Islamabad - 44000, Pakistan.

E-mail: abdullahshoaib15@yahoo.com

M. Arshad, Department of Mathematics, International Islamic University, H-10, Islamabad - 44000, Pakistan.

E-mail: marshad_zia@yahoo.com

A. Azam Department of Mathematics COMSATS Institute of InformationTechnology, Chack Shahzad, Islamabad - 44000, Pakistan.

E-mail: akbarazam@yahoo.com