МАТЕМАТИЧКИ ВЕСНИК
67, 1 (2015), 52–55
March 2015

оригинални научни рад
research paper

A UNIQUENESS RESULT FOR THE FOURIER TRANSFORM OF MEASURES ON THE PARABOLOID

Francisco Javier González Vieli

Abstract. A finite measure supported by a paraboloid of revolution Σ in $\mathbb{R}^3$ and absolutely continuous with respect to the natural measure on Σ is entirely determined by the restriction of its Fourier transform to a plane if and only if this plane is normal to the axis of Σ.

1. Introduction

Hedenmalm and Montes-Rodríguez asked in [4] the following: given Γ a smooth curve in $\mathbb{R}^2$ and Λ a subset of $\mathbb{R}^2$, when is it possible to recover uniquely a finite measure ν supported by Γ and absolutely continuous with respect to the arc length measure on Γ from the restriction to Λ of its Fourier transform $\mathcal{F}\nu$ on $\mathbb{R}^2$? Equivalently, when does $\mathcal{F}\nu(\lambda) = 0$ for all $\lambda \in \Lambda$ imply $\nu = 0$? If this is the case, they call (Γ, Λ) a *Heisenberg uniqueness pair*.

This initiated a series of papers in the subject [1–3, 5–8]. For example, in [8]. Sjölin established that if Γ is the parabola $y = x^2$ and Λ is a straight line, (Γ, Λ) is a Heisenberg uniqueness pair if and only if this straight line is parallel to the x-axis.

The definition of Heisenberg uniqueness pairs can easily be extended to all $\mathbb{R}^n$ $(n \geq 2)$:

Definition. Let Σ be a C^1 submanifold of $\mathbb{R}^n$ $(n \geq 2)$, μ_Σ the natural measure on Σ and Λ a subset of $\mathbb{R}^n$. The pair (Σ, Λ) is a *Heisenberg uniqueness pair* if, for every finite measure ν on Σ which is absolutely continuous with respect to μ_Σ, $\mathcal{F}\nu(\lambda) = 0$ for all $\lambda \in \Lambda$ implies $\nu = 0$, where $\mathcal{F}\nu$ is the Fourier transform of ν on $\mathbb{R}^n$:

$$\mathcal{F}\nu(x) = \int_\Sigma \mathrm{e}^{-2\pi i x \cdot \eta} d\nu(\eta)$$

for all $x \in \mathbb{R}^n$.

2010 Mathematics Subject Classification: 42B10, 46F12
Keywords and phrases: Heisenberg uniqueness; Fourier transform; measure; paraboloid

We have obtained the following generalization to paraboloids of Sjölin's result.

THEOREM. *Let* Σ *be the paraboloid* $x_n = x_1^2 + \cdots + x_{n-1}^2$ *in* $\mathbb{R}^n$ *and* Λ *an affine hyperplane in* $\mathbb{R}^n$ *of dimension* $n-1$. *The pair* (Σ, Λ) *is a Heisenberg uniqueness pair if and only if* Λ *is parallel to the hyperplane* $x_n = 0$.

2. Preliminaries

If (Σ, Λ) is a Heisenberg uniqueness pair in $\mathbb{R}^n$, it follows from elementary properties of the Fourier transform that $(\Sigma, \Lambda + b)$ is also a Heisenberg uniqueness pair for any $b \in \mathbb{R}^n$. By the theorem of Radon-Nykodým, a measure ν is absolutely continuous with respect to a measure μ if and only if ν has a density function f with respect to μ, that is, $\nu = f \cdot \mu$. Moreover, if ν is finite, then f is integrable with respect to μ.

Let Σ be the paraboloid $x_n = x_1^2 + \cdots + x_{n-1}^2$ in $\mathbb{R}^n$. It is the graph of the function h on $\mathbb{R}^{n-1}$ given by $h(u) := \|u\|^2$. The natural measure μ_Σ on Σ is defined by

$$\begin{aligned}\mu_\Sigma(\varphi) &:= \int_{\mathbb{R}^{n-1}} \varphi(u, h(u)) \sqrt{1 + \|\operatorname{grad} h(u)\|^2}\, du \\ &= \int_{\mathbb{R}^{n-1}} \varphi(u, \|u\|^2) \sqrt{1 + 4\|u\|^2}\, du.\end{aligned}$$

By ν we will always designate a finite measure on Σ which is absolutely continuous with respect to μ_Σ, i.e. of the form

$$\nu(\varphi) := \int_{\mathbb{R}^{n-1}} \varphi(u, \|u\|^2) f(u) \sqrt{1 + 4\|u\|^2}\, du,$$

where $f \in L^1(\mathbb{R}^{n-1}, \sqrt{1 + 4\|u\|^2}\, du)$.

We will need two auxiliary functions: let $\psi \in C^\infty(\mathbb{R})$ be odd with compact support and $\psi(1) = 1$; let $\chi \in C^\infty(\mathbb{R}^{n-2})$ with compact support and $\chi(0) = 1$.

Let now Λ be an affine hyperplane in $\mathbb{R}^n$ of dimension $n-1$. By the first remark above, we may assume that $0 \in \Lambda$, which means that Λ is a linear subspace of $\mathbb{R}^n$. Since Σ is invariant with respect to any rotation in the first $n-1$ variables $x_1, \ldots, x_{n-1}$, we may further assume that Λ is either of the type $x_n = \lambda x_1$ ($\lambda \in \mathbb{R}$) or the hyperplane $x_1 = 0$.

3. Proof

First, we take Λ of the type $x_n = \lambda x_1$ with $\lambda = 0$, i.e. $x_n = 0$. Let us suppose the measure ν has its Fourier transform null on Λ:

$$\mathcal{F}\nu(x_1, \ldots, x_{n-1}, 0) = 0$$

for all $(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. This can be written as

$$\int_{\mathbb{R}^{n-1}} \mathrm{e}^{-2\pi i (\xi, 0) \cdot (u, \|u\|^2)} f(u) \sqrt{1 + 4\|u\|^2}\, du = 0$$

for all $\xi \in \mathbb{R}^{n-1}$, or

$$\int_{\mathbb{R}^{n-1}} \mathrm{e}^{-2\pi i \xi \cdot u} f(u) \sqrt{1+4\|u\|^2}\, du = 0,$$

that is, the Fourier transform of the integrable function $f(u)\sqrt{1+4\|u\|^2}$ is 0 on all $\mathbb{R}^{n-1}$. Therefore $f = 0$ a.e. and $\nu = 0$. This shows that when Λ is the hyperplane $x_n = 0$, (Σ, Λ) is a Heisenberg uniqueness pair.

Next, we take Λ to be the hyperplane $x_1 = 0$. We choose the measure ν with

$$f(u_1, \ldots, u_{n-1}) := \psi(u_1) \cdot \chi(u_2, \ldots, u_{n-1}) \Big/ \sqrt{1+4\|u\|^2}.$$

For all $x \in \Lambda$, we have

$$\begin{aligned}
\mathcal{F}\nu(x) &= \mathcal{F}\nu(0, x_2, \ldots, x_n) \\
&= \int_{\mathbb{R}^{n-1}} \mathrm{e}^{-2\pi i (0, x_2, \ldots, x_n)\cdot(u_1, \ldots, u_{n-1}, \|u\|^2)} f(u) \sqrt{1+4\|u\|^2}\, du \\
&= \int_{\mathbb{R}^{n-1}} \mathrm{e}^{-2\pi i [x_2 u_2 + \cdots + x_{n-1}u_{n-1} + x_n \|u\|^2]} \psi(u_1) \cdot \chi(u_2, \ldots, u_{n-1})\, du \\
&= \int_{\mathbb{R}^{n-2}} \mathrm{e}^{-2\pi i [x_2 u_2 + \cdots + x_{n-1}u_{n-1} + x_n u_2^2 + \cdots + x_n u_{n-1}^2]} \times \\
&\qquad \times \left(\int_{\mathbb{R}} \mathrm{e}^{-2\pi i x_n u_1^2} \psi(u_1)\, du_1 \right) \chi(u_2, \ldots, u_{n-1})\, du_2 \ldots du_{n-1}.
\end{aligned}$$

Since the function $u_1 \mapsto \mathrm{e}^{-2\pi i x_n u_1^2}\psi(u_1)$ is odd (with compact support), its integral over $\mathbb{R}$ is equal to 0, for any value of x_n. So we get $\mathcal{F}\nu(0, x_2, \ldots, x_n) = 0$ for all $(x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$, i.e. $\mathcal{F}\nu(x) = 0$ for all $x \in \Lambda$. This shows that when Λ is the hyperplane $x_1 = 0$, (Σ, Λ) is not a Heisenberg uniqueness pair.

Finally, we take Λ of the type $x_n = \lambda x_1$ with $\lambda \neq 0$. We choose the measure ν with

$$f(u_1, \ldots, u_{n-1}) := \psi(u_1 + 1/2\lambda) \cdot \chi(u_2, \ldots, u_{n-1}) \Big/ \sqrt{1+4\|u\|^2}.$$

For all $x \in \Lambda$, we have

$$\begin{aligned}
&\mathcal{F}\nu(x) \\
&= \mathcal{F}\nu(x_1, \ldots, x_{n-1}, \lambda x_1) \\
&= \int_{\mathbb{R}^{n-1}} \mathrm{e}^{-2\pi i (x_1, \ldots, x_{n-1}, \lambda x_1)\cdot(u_1, \ldots, u_{n-1}, \|u\|^2)} f(u) \sqrt{1+4\|u\|^2}\, du \\
&= \int_{\mathbb{R}^{n-1}} \mathrm{e}^{-2\pi i [x_1 u_1 + \cdots + x_{n-1}u_{n-1} + \lambda x_1 \|u\|^2]} \psi(u_1 + 1/2\lambda) \cdot \chi(u_2, \ldots, u_{n-1})\, du \\
&= \int_{\mathbb{R}^{n-2}} \mathrm{e}^{-2\pi i [x_2 u_2 + \cdots + x_{n-1}u_{n-1} + \lambda x_1 u_2^2 + \cdots + \lambda x_1 u_{n-1}^2]} \times \\
&\qquad \times \left(\int_{\mathbb{R}} \mathrm{e}^{-2\pi i [x_1 u_1 + \lambda x_1 u_1^2]} \psi(u_1 + 1/2\lambda) du_1 \right) \chi(u_2, \ldots, u_{n-1})\, du_2 \ldots du_{n-1}.
\end{aligned}$$

The integral over u_1 can be written as

$$\begin{aligned}
&\int_{\mathbb{R}} \mathrm{e}^{-2\pi i\lambda x_1[u_1^2+u_1/\lambda]}\psi(u_1+1/2\lambda)\,du_1\\
&\quad= \int_{\mathbb{R}} \mathrm{e}^{-2\pi i\lambda x_1[(u_1+1/2\lambda)^2-1/4\lambda^2]}\psi(u_1+1/2\lambda)\,du_1\\
&\quad= \mathrm{e}^{\pi i x_1/2\lambda}\int_{\mathbb{R}} \mathrm{e}^{-2\pi i\lambda x_1(u_1+1/2\lambda)^2}\psi(u_1+1/2\lambda)\,du_1\\
&\quad= \mathrm{e}^{\pi i x_1/2\lambda}\int_{\mathbb{R}} \mathrm{e}^{-2\pi i\lambda x_1 t^2}\psi(t)\,dt,
\end{aligned}$$

where $t := u_1 + 1/2\lambda$. Since the function $t \mapsto \mathrm{e}^{-2\pi i\lambda x_1 t^2}\psi(t)$ is odd (with compact support), its integral over $\mathbb{R}$ is equal to 0, for any value of x_1. We get in this way $\mathcal{F}\nu(x_1,\ldots,x_{n-1},\lambda x_1) = 0$ for all $(x_1,\ldots,x_{n-1}) \in \mathbb{R}^{n-1}$, i.e. $\mathcal{F}\nu(x) = 0$ for all $x \in \Lambda$. This shows that when Λ is a hyperplane of the type $x_n = \lambda x_1$ with $\lambda \neq 0$, (Σ,Λ) is not a Heisenberg uniqueness pair. ■

Acknowledgement. We wood like to thank the referees for the suggested improvements.

REFERENCES

[**1**] D. Blasi Babot, *Heisenberg uniqueness pairs in the plane. Three parallel lines*, Proc. Amer. Math. Soc. **141** (2013), 3899–3904.

[**2**] F. Canto-Martín, H. Hedenmalm, A. Montes-Rodríguez, *Perron-Frobenius operators and the Klein-Gordon equation*, J. Eur. Math. Soc. **16** (2014), 31–66.

[**3**] F. J. González Vieli, *A uniqueness result for the Fourier transform of measures on the sphere*, Bull. Aust. Math. Soc. **86** (2012), 78–82.

[**4**] H. Hedenmalm, A. Montes-Rodríguez, *Heisenberg uniqueness pairs and the Klein-Gordon equation*, Ann. of Math. **173** (2011), 1507–1527.

[**5**] H. Hedenmalm, A. Montes-Rodríguez, *The transfer operator intertwines the Hilbert transform: the Klein-Gordon equation*, Preprint.

[**6**] N. Lev, *Uniqueness theorems for Fourier transforms*, Bull. Sci. Math. **135** (2011), 135–140.

[**7**] P. Sjölin, *Heisenberg uniqueness pairs and a theorem of Beurling and Malliavin*, Bull. Sci. Math. **135** (2011), 125–133.

[**8**] P. Sjölin, *Heisenberg uniqueness pairs for the parabola*, J. Fourier Anal. Appl. **19** (2013), 410–416.

(received 14.08.2013; in revised form 06.01.2014; available online 03.03.2014)

Montoie 45, 1007 Lausanne, Switzerland

E-mail: Francisco-Javier.Gonzalez@gmx.ch