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GENERALIZED DERIVATIONS AS A GENERALIZATION OF JORDAN HOMOMORPHISMS ACTING ON LIE IDEALS

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Abstract. Let R be a prime ring with extended centroid C, L a non-central Lie ideal of R and $n \geq 1$ a fixed integer. If R admits the generalized derivations H and G such that $H(u^2)^n = G(u)^{2n}$ for all $u \in L$, then one of the following holds:

(1) $H(x) = ax$ and $G(x) = bx$ for all $x \in R$, with $a, b \in C$ and $a^n = b^{2n}$;

(2) $\mathrm{char}(R) \neq 2$, R satisfies s_4, $H(x) = ax + [p, x]$ and $G(x) = bx$ for all $x \in R$, with $b \in C$ and $a^n = b^{2n}$;

(3) $\mathrm{char}(R) = 2$ and R satisfies s_4.

As an application we also obtain some range inclusion results of continuous generalized derivations on Banach algebras.

1. Introduction

Let R be an associative prime ring with center $Z(R)$ and U the Utumi quotient ring of R. The center of U, denoted by C, is called the extended centroid of R (we refer the reader to [2] for these objects). For given $x, y \in R$, the Lie commutator of x, y is denoted by $[x, y] = xy - yx$. A linear mapping $d : R \to R$ is called a derivation, if it satisfies the Leibnitz rule $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In particular, d is said to be an inner derivation induced by an element $a \in R$, if $d(x) = [a, x]$ for all $x \in R$. In [5], Bresar introduced the definition of generalized derivation: An additive mapping $F : R \to R$ is called generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$, and d is called the associated derivation of F. Hence, the concept of generalized derivations covers the concept of derivations. In [20], Lee extended the definition of generalized derivation as follows: by a generalized derivation we mean an additive mapping $F : I \to U$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in I$, where I is a dense left ideal of R and d is a derivation from I into U. Moreover, Lee also proved that every generalized derivation can be uniquely extended to a generalized derivation of U, and thus all generalized derivations of

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R will be implicitly assumed to be defined on the whole of U. Lee obtained the following: every generalized derivation F on a dense left ideal of R can be uniquely extended to U and assumes the form $F(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U. Let S be a nonempty subset of R and $F : R \to R$ be an additive mapping. Then we say that F acts as homomorphism or anti-homomorphism on S if $F(xy) = F(x)F(y)$ or $F(xy) = F(y)F(x)$ holds for all $x, y \in S$ respectively. The additive mapping F acts as a Jordan homomorphism on S if $F(x^2) = F(x)^2$ holds for all $x \in S$.

Let us introduce the background of our investigation. In [25], Singer and Wermer obtained a fundamental result which stated investigation into the ranges of derivations on Banach algebras. They proved that any continuous derivation on a commutative Banach algebra has the range in the Jacobson radical of the algebra. Very interesting question is how to obtain non-commutative version of Singer-Wermer theorem. In [24] Sinclair obtained a fundamental result which stated investigation into the ranges of derivations on a non-commutative Banach algebra. He proved that every continuous derivation of a Banach algebra leaves primitive ideals of the algebra invariant. In the meanwhile many authors obtained more information about derivations satisfying certain suitable conditions in Banach algebra. For example, in [23] Park proved that if d is a linear continuous derivation of a non-commutative Banach algebra A such that $[[d(x), x], d(x)] \in rad(A)$ for all $x \in A$ then $d(A) \subseteq rad(A)$. In [9], De Filippis extended the Park's result to generalized derivations.

Many results in literature indicate that global structure of a prime ring R is often tightly connected to the behavior of additive mappings defined on R. A. Ali, S. Ali and N. Ur Rehman in [1] proved that if d is a derivation of a 2-torsion free prime ring R which acts as a homomorphism or anti-homomorphism on a non-central Lie ideal of R such that $u^2 \in L$, for all $u \in L$, then $d = 0$. At this point the natural question is what happens in case the derivation is replaced by generalized derivation. In [14], Golbasi and Kaya respond this question. More precisely, they proved the following: Let R be a prime ring of characteristic different from 2, H a generalized derivation of R, L a Lie ideal of R such that $u^2 \in L$ for all $u \in L$. If H acts as a homomorphism or anti-homomorphism on L, then either $d = 0$ or L is central in R. More recently in [8], Filippis studied the situation when generalized derivation H acts as a Jordan homomorphism on a non-central Lie ideal L.

In [10], we generalize these results when conditions are more widespread. More precisely we prove that if H is a non-zero generalized derivation of prime ring R such that $H(u^2)^n = H(u)^{2n}$ for all $u \in L$, a non-central Lie ideal of R, where $n \geq 1$ is a fixed integer, then one of the following holds:

(1) $\mathrm{char}(R) = 2$ and R satisfies s_4;

(2) $H(x) = bx$ for all $x \in R$, for some $b \in C$ and $b^n = 1$.

The present article is motivated by the previous results. The main results of this paper are as follows:

THEOREM 1.1. *Let R be a prime ring with extended centroid C, L a non-central*

Lie ideal of R and $n \geq 1$ a fixed integer. If R admits the generalized derivations H and G such that $H(u^2)^n = G(u)^{2n}$ for all $u \in L$, then one of the following holds:

(1) *$H(x) = ax$ and $G(x) = bx$ for all $x \in R$, with $a, b \in C$ and $a^n = b^{2n}$;*

(2) *$char(R) \neq 2$, R satisfies s_4, $H(x) = ax + [p, x]$ and $G(x) = bx$ for all $x \in R$, with $b \in C$ and $a^n = b^{2n}$;*

(3) *$char(R) = 2$ and R satisfies s_4.*

We prove the following result regarding the non-commutative Banach algebra.

THEOREM 1.2. *Let A be a non-commutative Banach algebra, $\zeta = L_a + d$, $\eta = L_b + \delta$ continuous generalized derivations of A and n a fixed positive integer. If $\zeta([x, y]^2)^n - \eta([x, y])^{2n} \in rad(A)$, for all $x, y \in A$, then $d(A) \subseteq rad(A), \delta(A) \subseteq rad(A)$, $[a, A] \subseteq rad(A), [b, A] \subseteq rad(A)$ and $a^n - b^{2n} \subseteq rad(A)$ or $s_4(a_1, a_2, a_3, a_4) \in rad(A)$ for all $a_1, a_2, a_3, a_4 \in A$.*

The following remarks are useful tools for the proof of main results.

REMARK 1.3. Let R be a prime ring and L a noncentral Lie ideal of R. If $\mathrm{char}(R) \neq 2$, by [4, Lemma 1] there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$. If $\mathrm{char}(R) = 2$ and $\dim_C RC > 4$, i.e., $\mathrm{char}(R) = 2$ and R does not satisfy s_4, then by [19, Theorem 13] there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$. Thus if either $\mathrm{char}(R) \neq 2$ or R does not satisfy s_4, then we may conclude that there exists a nonzero ideal I of R such that $[I, I] \subseteq L$.

REMARK 1.4. We denote by $\mathrm{Der}(U)$ the set of all derivations on U. By a derivation word Δ of R we mean $\Delta = d_1 d_2 d_3 \ldots d_m$ for some derivations $d_i \in \mathrm{Der}(U)$.

For $x \in R$, we denote by x^Δ the image of x under Δ, that is $x^\Delta = (\cdots(x^{d_1})^{d_2}\cdots)^{d_m}$. By a differential polynomial, we mean a generalized polynomial, with coefficients in U, of the form $\Phi(x_i^{\Delta_j})$ involving noncommutative indeterminates x_i on which the derivations words Δ_j act as unary operations. $\Phi(x_i^{\Delta_j}) = 0$ is said to be a differential identity on a subset T of U if it vanishes for any assignment of values from T to its indeterminates x_i.

Let D_{int} be the C-subspace of $\mathrm{Der}(U)$ consisting of all inner derivations on U and let d be a non-zero derivation on R. By [17, Theorem 2] we have the following result:

If $\Phi(x_1, x_2, \cdots, x_n, d(x_1), d(x_2) \cdots d(x_n)$ is a differential identity on R, then one of the following holds:

(1) $d \in D_{int}$;

(2) R satisfies the generalized polynomial identity $\Phi(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n)$.

2. Proof of the main results

Now we begin with the following lemmas.

Lemma 2.1. *Let $R = M_k(F)$ be the ring of all $k \times k$ matrices over the field F with $k \geq 2$ and $a, b, p, q \in R$. Suppose that*

$$(a[x,y]^2 + [x,y]^2 b)^n = (p[x,y] + [x,y]q)^{2n}$$

for all $x, y \in R$, where $n \geq 1$ a fixed integer. Then one of the following holds:

(1) *$k = 2$, $p, q \in F.I_2$ and $(a+b)^n - (p+q)^{2n} = 0$;*

(2) *$k \geq 3$, $a, b, p, q \in F.I_k$ and $(a+b)^n - (p+q)^{2n} = 0$.*

Proof. Let $a = (a_{ij})_{k\times k}$, $b = (b_{ij})_{k\times k}$, $p = (p_{ij})_{k\times k}$ and $q = (q_{ij})_{k\times k}$, where a_{ij}, b_{ij}, p_{ij} and $q_{ij} \in F$. Denote e_{ij} the usual matrix unit with 1 in (i,j)-entry and zero elsewhere. By choosing $x = e_{ii}$, $y = e_{ij}$ for any $i \neq j$, we have

$$0 = (pe_{ij} + e_{ij}q)^{2n}. \tag{1}$$

Multiplying this equality from right by e_{ij}, we arrive at

$$0 = (pe_{ij} + e_{ij}q)^{2n} e_{ij} = (q_{ji})^{2n} e_{ij}.$$

This implies $q_{ji} = 0$. Thus for any $i \neq j$, we have $q_{ji} = 0$, which implies that q is diagonal matrix. Let $q = \sum_{i=1}^{k} q_{ii}e_{ii}$. For any F-automorphism θ of R, we have

$$(a^\theta[x,y]^2 + [x,y]^2 b^\theta)^n = (p^\theta[x,y] + [x,y]q^\theta)^{2n}$$

for every $x, y \in R$. Hence q^θ must also be diagonal. We have

$$(1 + e_{ij})q(1 - e_{ij}) = \sum_{i=1}^{k} q_{ii}e_{ii} + (q_{jj} - q_{ii})e_{ij}$$

diagonal. Therefore, $q_{jj} = q_{ii}$ and so $q \in F.I_k$.

Now left multiplying (1) by e_{ij}, we have $p_{ji} = 0$ for any $i \neq j$, that is p is diagonal. Then by same manner as above, we have $p \in F.I_k$.

Case-I: Let $k = 2$. We know the fact that for any $x, y \in M_2(F)$, $[x,y]^2 \in F.I_2$. Thus our assumption reduces to

$$((a+b)^n - (p+q)^{2n})[x,y]^{2n} = 0$$

for all $x, y \in R$. We choose $[x,y] = [e_{12}, e_{21}] = e_{11} - e_{22}$ and so $[x,y]^2 = I_2$. Thus from above relation, we have that $(a+b)^n - (p+q)^{2n} = 0$.

Case-II: Let $k \geq 3$. Choose $x = e_{it} - e_{tj}$ and $y = e_{tt}$, where i, j, t are any three distinct indices. Then $[x,y] = e_{it} + e_{tj}$ and so $[x,y]^2 = e_{ij}$. Thus by assumption, we have

$$(ae_{ij} + e_{ij}b)^n = 0$$

for all $x, y \in R$. Left multiplying by e_{ij}, above relation yields $a_{ji}^n = 0$ that is $a_{ji} = 0$ for any $i \neq j$. This gives that a is diagonal, and hence by above argument a is central. By the same manner, right multiplying above relation by e_{ij}, we have b diagonal and hence central. Then our identity reduces to

$$((a+b)^n - (p+q)^{2n})[x,y]^{2n} = 0$$

for all $x, y \in R$. This implies that $(a+b)^n - (p+q)^{2n} = 0$. ∎

Lemma 2.2. *Let R be a non-commutative prime ring with extended centroid C and $a, b, p, q \in R$. Suppose that*

$$(a[x,y]^2 + [x,y]^2 b)^n = (p[x,y] + [x,y]q)^{2n}$$

for all $x, y \in R$, where $n \geq 1$ a fixed integer. Then one of the following holds:

(1) *R satisfies s_4, $p, q \in C$ and $(a+b)^n - (p+q)^{2n} = 0$;*

(2) *R does not satisfy s_4, $a, b, p, q \in C$ and $(a+b)^n - (p+q)^{2n} = 0$.*

Proof. By assumption, R satisfies the generalized polynomial identity (GPI)

$$f(x,y) = (a[x,y]^2 + [x,y]^2 b)^n - (p[x,y] + [x,y]q)^{2n}.$$

By Chuang [7, Theorem 2], this generalized polynomial identity (GPI) is also satisfied by U. Now we consider the following two cases:

Case-I. U does not satisfy any nontrivial GPI

Let $T = U *_C C\{x,y\}$, the free product of U and $C\{x,y\}$, the free C-algebra in noncommuting indeterminates x and y. Thus

$$(a[x,y]^2 + [x,y]^2 b)^n - (p[x,y] + [x,y]q)^{2n}$$

is zero element in $T = U *_C C\{x,y\}$. Let $q \notin C$. Then $\{1, q\}$ is C-independent. If $b \notin Span_C\{1, q\}$, then expanding above expression, we see that $([x,y]q)^{2n}$ appears nontrivially, a contradiction. Let $b = \alpha + \beta q$ for some $\alpha, \beta \in C$. Then we have

$$(a[x,y]^2 + \alpha[x,y]^2 + \beta[x,y]^2 q)^n - (p[x,y] + [x,y]q)^{2n}$$

is zero in T. Since $q \notin C$, we have from above

$$(a[x,y]^2 + \alpha[x,y]^2 + \beta[x,y]^2 q)^{n-1}\beta[x,y]^2 q - (p[x,y] + [x,y]q)^{2n-1}[x,y]q,$$

that is,

$$\{(a[x,y]^2 + \alpha[x,y]^2 + \beta[x,y]^2 q)^{n-1}\beta[x,y] - (p[x,y] + [x,y]q)^{2n-1}\}[x,y]q$$

is zero in T. In the above expression, $([x,y]q)^{2n-1}[x,y]q$ appears nontrivially, a contradiction. Thus we conclude that $q \in C$. Then the identity reduces to

$$(a[x,y]^2 + [x,y]^2 b)^n - ((p+q)[x,y])^{2n}$$

which is zero element in T. Again, if $b \notin C$, then $([x,y]^2 b)^n$ becomes a nontrivial element in the above expansion, a contradiction. Hence $b \in C$. Thus we have

$$((a+b)[x,y]^2)^n - ((p+q)[x,y])^{2n},$$

that is,

$$\{((a+b)[x,y]^2)^{n-1}(a+b)[x,y] - ((p+q)[x,y])^{2n-1}(p+q)\}[x,y]$$

is zero element in T. If $p+q \notin C$, then $((p+q)[x,y])^{2n-1}(p+q)[x,y]$ is not cancelled in the above expansion, leading again contradiction. Hence $p+q \in C$ and so

$$((a+b)[x,y]^2)^n - [x,y]^{2n}(p+q)^{2n} = 0$$

in T. If $a+b\notin C$, then from above, $((a+b)[x,y]^2)^n$ appears nontrivially, a contradiction. Hence, $a+b\in C$. Therefore, we have

$$\{(a+b)^n-(p+q)^{2n}\}[x,y]^{2n}=0$$

in T, implying $(a+b)^n-(p+q)^{2n}=0$. This is our conclusion (2).

Case-II. U satisfies a nontrivial GPI

Thus we assume that

$$(a[x,y]^2+[x,y]^2b)^n-(p[x,y]+[x,y]q)^{2n}=0$$

is a nontrivial GPI for U. In case C is infinite, we have $f(x,y)=0$ for all $x,y\in U\otimes_C\overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both U and $U\otimes_C\overline{C}$ are prime and centrally closed [11], we may replace R by U or $U\otimes_C\overline{C}$ according to C finite or infinite. Thus we may assume that R centrally closed over C which either finite or algebraically closed and $f(x,y)=0$ for all $x,y\in R$. By Martindale's Theorem [22], R is then primitive ring having non-zero socle $soc(R)$ with C as the associated division ring. Hence by Jacobson's Theorem [15], R is isomorphic to a dense ring of linear transformations of a vector space V over C. If $dim_CV<\infty$, then $R\simeq M_k(C)$ for some $k\geq 2$. In this case by Lemma 2.1, we obtain our conclusions.

Now we assume that $\dim_CV=\infty$. Let e be an idempotent element of $soc(R)$. Then replacing x with e and y with $er(1-e)$, we have

$$(per(1-e)+er(1-e)q)^{2n}=0. \tag{2}$$

Left multiplying by $(1-e)$ we get $(1-e)(per(1-e))^{2n}=0$. This implies that $((1-e)per)^{2n+1}=0$ for all $r\in R$. By [12], it follows that $(1-e)pe=0$. Similarly replacing x with e and y with $(1-e)re$, we shall get $ep(1-e)=0$. Thus for any idempotent $e\in soc(R)$, we have $(1-e)pe=0=ep(1-e)$ that is $[p,e]=0$. Therefore, $[p,E]=0$, where E is the additive subgroup generated by all idempotents of $soc(R)$. Since E is non-central Lie ideal of $soc(R)$, this implies $p\in C$ (see [4, Lemma 2]). Now by similar argument we can prove that $q\in C$.

Then our identity reduces to

$$(a[x,y]^2+[x,y]^2b)^n-\alpha^{2n}[x,y]^{2n}=0$$

for all $x,y\in R$, where $\alpha=p+q\in C$. Let for some $v\in V$, v and bv are linearly independent over C. Since $\dim_CV=\infty$, there exists $w\in V$ such that v,bv,w are linearly independent over C. By density there exist $x,y\in R$ such that

$$xv=v,\quad xbv=-bv,\quad xw=0;$$
$$yv=0,\quad ybv=w,\quad yw=v.$$

Then $[x,y]v=0$, $[x,y]bv=w$, $[x,y]w=v$ and hence $0=\{(a[x,y]^2+[x,y]^2b)^n-\alpha^{2n}[x,y]^{2n}\}v=v$, a contradiction. Thus v and bv are linearly C-dependent for all $v\in V$. By standard argument, it follows that $b\in C$. Then again our identity reduces to

$$(a'[x,y]^2)^n-\alpha^{2n}[x,y]^{2n}=0$$

for all $x,y\in R$, where $a'=a+b$.

Let for some $v \in V$, v and $a'v$ are linearly independent over C. Since $\dim_C V = \infty$, there exists $w \in V$ such that $v, a'v, w, u$ are linearly independent over C. By density there exist $x, y \in R$ such that

$$xv = v, \quad xa'v = -bv, \quad xw = 0, \quad xu = v + u;$$
$$yv = u, \quad ya'v = w, \quad yw = v, \quad yu = 0.$$

Then $[x,y]v = v$, $[x,y]a'v = w$, $[x,y]w = v$ and hence $0 = \{(a'[x,y]^2)^n - \alpha^{2n}[x,y]^{2n}\}v = a'v - \alpha^{2n}v$, a contradiction. Thus v and $a'v$ are linearly C-dependent for all $v \in V$. Then again by standard argument, we have that $a' \in C$. Thus our identity reduces to

$$({a'}^n - \alpha^{2n})[x,y]^{2n} = 0$$

for all $x, y \in R$. This gives ${a'}^n - \alpha^{2n} = 0$ i.e., $(a+b)^n = (p+q)^{2n}$ or $[x,y]^{2n} = 0$ for all $x, y \in R$. The last case implies R to be commutative, a contradiction. ∎

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. If $\mathrm{char}(R) = 2$ and R satisfies s_4, then we have our conclusion (3). So we assume that either $\mathrm{char} R \neq 2$ or R does not satisfy s_4. Since L is non central by Remark 1.3, there exists a nonzero ideal I of R such that $[I, I] \subseteq L$. Thus by assumption I satisfies the differential identity

$$H([x,y]^2)^n = G([x,y])^{2n}.$$

Now since R is a prime ring and H, G are generalized derivations of R, by Lee [20, Theorem 3], $H(x) = ax + d(x)$ and $G(x) = bx + \delta(x)$ for some $a, b \in U$ and derivations d, δ on U. Since I, R and U satisfy the same differential identity [21], without loss of generality,

$$H([x,y]^2)^n = G([x,y])^{2n}$$

for all $x, y \in U$. Hence U satisfies

$$(a[x,y]^2 + d([x,y]^2))^n = (b[x,y] + \delta([x,y]))^{2n}. \tag{3}$$

Here we divide the proof into three cases:

Case 1. Let d and δ be both inner derivations induced by elements $p, q \in U$ respectively; that is, $d(x) = [p, x]$ and $\delta(x) = [q, x]$ for all $x \in U$. It follows that

$$(a[x,y]^2 + [p, [x,y]^2])^n - (b[x,y] + [q, [x,y]])^{2n} = 0$$

that is

$$((a+p)[x,y]^2 - [x,y]^2 p)^n - ((b+q)[x,y] - [x,y]q)^{2n} = 0$$

for all $x, y \in U$. Now by Lemma 2.2, one of the following holds:

(1) R satisfies s_4, $b+q, q \in C$ and $a^n - b^{2n} = 0$. Thus $H(x) = ax + [p, x]$ and $G(x) = (b+q)x - xq = bx$ for all $x \in R$, with $b \in C$ and $a^n = b^{2n}$. In this case by assumption, char $(R) \neq 2$.

(2) R does not satisfy s_4, $a+p, p, b+q, q \in C$ and $a^n - b^{2n} = 0$. Thus $H(x) = ax + [p, x] = ax$ and $G(x) = bx + [q, x] = bx$ for all $x \in R$, with $a, b \in C$ and $a^n = b^{2n}$.

Case 2. Assume that d and δ are not both inner derivations of U. Suppose that d and δ be C-linearly dependent modulo D_{int}. Let $\delta = \beta d + ad(p)$, for some $\beta \in C$ and $ad(p)$ the inner derivation induced by element $p \in U$. Notice that if d is inner or $\beta = 0$, then δ is also inner, a contradiction.

Therefore consider the case when d is not inner and $\beta \neq 0$. Then by (3), we have that U satisfies

$$(a[x,y]^2 + d([x,y]^2))^n = (b[x,y] + \beta d([x,y]) + [p,[x,y]])^{2n}$$

that is

$$\begin{aligned}(a[x,y]^2 + ([d(x),y] + [x,d(y)])[x,y] + [x,y]([d(x),y] + [x,d(y)]))^n \\ = (b[x,y] + \beta([d(x),y] + [x,d(y)]) + [p,[x,y]])^{2n}.\end{aligned}$$

Then by Kharchenko's Theorem [17],

$$\begin{aligned}(a[x,y]^2 + ([z,y] + [x,w])[x,y] + [x,y]([z,y] + [x,w]))^n \\ = (b[x,y] + \beta([z,y] + [x,w]) + [p,[x,y]])^{2n}. \quad (4)\end{aligned}$$

Setting $z = w = 0$, we obtain

$$(a[x,y]^2)^n = ((b+p)[x,y] - [x,y]p)^{2n}$$

for all $x, y \in U$. Then by Lemma 2.2, we have $b + p, p \in C$, that gives $b, p \in C$. Therefore, in particular for $x = 0$, (4) becomes $0 = \beta^{2n}[z,y]^{2n}$. Since $\beta \neq 0$, we have $0 = [z,y]^{2n}$ for all $z, y \in U$. This implies that U and so R is commutative. This contradicts with the fact that L is noncentral Lie ideal of R.

The situation when $d = \lambda\delta + ad(q)$, for some $\lambda \in C$ and $ad(q)$ the inner derivation induced by element $q \in U$, is similar.

Case 3. Assume now that d and δ be C-linearly independent modulo D_{int}. In this case from (3), we have that U satisfies

$$\begin{aligned}(a[x,y]^2 + ([d(x),y] + [x,d(y)])[x,y] + [x,y]([d(x),y] + [x,d(y)]))^n \\ = (b[x,y] + [\delta(x),y] + [x,\delta(y)])^{2n}. \quad (5)\end{aligned}$$

By Kharchenko's Theorem [17], U satisfies

$$(a[x,y]^2 + ([z,y] + [x,w])[x,y] + [x,y]([z,y] + [x,w]))^n = (b[x,y] + [s,y] + [x,t])^{2n}.$$

In particular, for $x = 0$ we have $[s,y]^{2n} = 0$ for all $s, y \in U$. As above this leads that U and so R is commutative, a contradiction. ■

In particular, the proof of Theorem 1.1 yields:

Corollary 2.3. *Let R be a prime ring and $n \geq 1$ a fixed integer. If R admits the generalized derivations H and G such that $H(x^2)^n = G(x)^{2n}$ for all $x \in [R,R]$, then one of the following holds: (1) $H(x) = ax$ and $G(x) = bx$ for all $x \in R$, with $a, b \in C$ and $a^n = b^{2n}$; (2) R satisfies s_4.*

Here A will denote a complex non-commutative Banach algebras. Our final result in this paper is about continuous generalized derivations on non-commutative Banach algebras.

The following results are useful tools needed in the proof of Theorem 1.2.

Remark 2.4. (see [24]). Any continuous derivation of Banach algebra leaves the primitive ideals invariant.

Remark 2.5. (see [25]). Any continuous linear derivation on a commutative Banach algebra maps the algebra into its radical.

Remark 2.6. (see [16]). Any linear derivation on semisimple Banach algebra is continuous.

Proof of Theorem 1.2. By the hypothesis, ζ, η are continuous. Again, since L_a, L_b, the left multiplication by some element $a, b \in A$, are continuous, we have that the derivations d, δ are also continuous. By Remark 2.4, for any primitive ideal P of A, we have $\zeta(P) \subseteq aP + d(P) \subseteq P$ and $\eta(P) \subseteq aP + d(P) \subseteq P$. It means that the continuous generalized derivations ζ, η leaves the primitive ideal invariant. Denote $\bar{A} = A/P$ for any primitive ideals P. Thus we can define the generalized derivations $\zeta_P : \bar{A} \to \bar{A}$ by $\zeta_P(\bar{x}) = \zeta_P(x + P) = \zeta(x) + P$ and $\eta_P : \bar{A} \to \bar{A}$ by $\eta_P(\bar{x}) = \eta_P(x+P) = \eta(x)+P$ for all $\bar{x} \in \bar{A}$, where $A/P = \bar{A}$. Since P is primitive ideal, $\bar{A}$ is primitive and so it is prime. The hypothesis $\zeta([x, y]^2)^n - \eta([x, y])^{2n} \in rad(A)$ yields that $\zeta_P([\bar{x}, \bar{y}]^2)^n - \eta_P([\bar{x}, \bar{y}])^{2n} = \bar{0}$ for all $\bar{x}, \bar{y} \in \bar{A}$. Now from Corollary 2.3, it is immediate that either (1) $d = \bar{0}, \delta = \bar{0}$, $\bar{a} \in Z(\bar{A}), \bar{b} \in Z(\bar{A})$ and $(a + P)^n = (b + P)^{2n}$, that is, $d(A) \subseteq P, \delta(A) \subseteq P$, $[a, A] \subseteq P, [b, A] \subseteq P$ and $a^n - b^{2n} \in P$; or (2) $\bar{A}$ satisfies s_4, that is $s_4(a_1, a_2, a_3, a_4) \in P$ for all $a_1, a_2, a_3, a_4 \in A$. Since the radical of A is the intersection of all primitive ideals, we arrive the required conclusions. ■

Corollary 2.7. *Let A be a non-commutative semisimple Banach algebra $\zeta = L_a + d$, $\eta = L_b + \delta$ continuous generalized derivations of A and n a fixed positive integer. If $\zeta([x, y]^2)^n - (\eta[x, y])^{2n} = 0$, for all $x, y \in A$, then $\zeta(x) = \alpha x, \eta(x) = \beta x$ for some $\alpha, \beta \in Z(A)$ and $\alpha^n = \beta^{2n}$ or A satisfies s_4.*

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