МАТЕМАТИЧКИ ВЕСНИК
67, 2 (2015), 143–154
June 2015

оригинални научни рад
research paper

GENERALIZED RELATIVE LOWER ORDER OF ENTIRE FUNCTIONS

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Abstract. The basic properties of the generalized relative lower order of entire functions are discussed in this paper. In fact, we improve here some results of Datta, Biswas and Biswas [Casp. J. Appl. Math. Ecol. Econ., **1**, 2 (2013), 3–18].

1. Introduction, definitions and notations

Let f and g be any two entire functions defined in the complex plane $\mathbb{C}$ and $M_f(r) = \max\{|f(z)| : |z| = r\}$, $M_g(r) = \max\{|g(z)| : |z| = r\}$. Sato [9] defined the generalized order $\rho_f^{[l]}$ and generalized lower order $\lambda_f^{[l]}$ of an entire function f for any integer $l \geq 2$, in the following way:

$$\rho_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log r} \text{ and } \lambda_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log r},$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$, $k = 1, 2, 3, \dots$ and $\log^{[0]} x = x$.

When $l = 2$, the above definition coincides with the classical definition of order and lower order, which are as follows:

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r}.$$

If f is non-constant then $M_f(r)$ is strictly increasing and continuous, and its inverse $M_f{}^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that $\lim_{s\to\infty} M_f{}^{-1}(s) = \infty$.

Bernal ([1], see also [2]) introduced the definition of relative order of g with respect to f, denoted by $\rho_f(g)$ as follows :

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0\}\\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.\end{aligned}$$

The definition coincides with the classical one [10] if $g(z) = \exp z$.

2010 Mathematics Subject Classification: 30D20, 30D30, 30D35
Keywords and phrases: Entire function; generalized relative lower order; Property (A).

Similarly, one can define the relative lower order of g with respect to f, denoted by $\lambda_f(g)$ as follows

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.$$

Extending this notion, Lahiri and Banerjee [7] gave a more generalized concept of relative order which may be given in the following way.

Definition 1. [7] If $l \geq 1$ is a positive integer, then the l-th generalized relative order of f with respect to g, denoted by $\rho_f^{[l]}(g)$, is defined by

$$\begin{aligned}\rho_g^{[l]}(f) &= \inf\{\mu > 0 : M_f(r) < M_g(\exp^{[l-1]} r^\mu) \text{ for all } r > r_0(\mu) > 0\}\\ &= \limsup_{r\to\infty} \frac{\log^{[l]} M_g^{-1} M_f(r)}{\log r},\end{aligned}$$

If $l = 1$ then $\rho_g^{[l]}(f) = \rho_g(f)$. If $l = 1$, $g(z) = \exp z$ then $\rho_g^{[l]}(f) = \rho_f$, the classical order of f (cf. [10]).

Analogously, one can define the l-th generalized relative lower order of g with respect to f, denoted by $\lambda_f^{[l]}(g)$ as follows

$$\lambda_f^{[l]}(g) = \liminf_{r\to\infty} \frac{\log^{[l]} M_f^{-1} M_g(r)}{\log r}.$$

During the past decades, several authors (see [4–7]) made close investigations on the properties of relative order of entire functions. In this connection the following definition is relevant.

Definition 2. [2] A non-constant entire function f is said have the Property (A) if for any $\sigma > 1$ and for all sufficiently large r, $[M_f(r)]^2 \leq M_f(r^\sigma)$ holds.

For examples of functions with or without the Property (A), one may see [2].

It is well known that the order of the products and of the sums of two entire functions is not greater than the maximal order of the two functions. Bernal [2] and Lahiri and Banerjee [7] extended these results for relative order and generalized relative order. Our aim in this paper is to study some parallel basic properties of generalized relative lower order of entire functions. We do not explain the standard definitions and notations in the theory of entire functions as those are available in [11].

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. [2] *Suppose f is a nonconstant entire function, $\alpha > 1$, $0 < \beta < \alpha$, $s > 1$, $0 < \mu < \lambda$ and n is a positive integer. Then*

(a) $M_f(\alpha r) > \beta M_f(r)$.

(*b*) *There exists* $K = K(s, f) > 0$ *such that* $(M_f(r))^s \le K M_f(r^s)$ *for* $r > 0$.

(*c*) $\lim_{r\to\infty} \frac{M_f(r^s)}{M_f(r)} = \infty = \lim_{r\to\infty} \frac{M_f(r^\lambda)}{M_f(r^\mu)}$.

(*d*) *If* f *is transcendental then*

$$\lim_{r\to\infty} \frac{M_f(r^s)}{r^n M_f(r)} = \infty = \lim_{r\to\infty} \frac{M_f(r^\lambda)}{r^n M_f(r^\mu)}.$$

Lemma 2. [2] *Let* f *be an entire function satisfying the Property (A), and let* $\delta > 1$ *and* n *be a given positive integer. Then the inequality* $[M_f(r)]^n \le M_f(r^\delta)$ *holds for* r *large enough.*

Lemma 3. *Let* f, g *and* h *are any three entire functions. If* $M_g(r) \le M_h(r)$ *for all sufficiently large values of* r, *then* $\lambda_h^{[l]}(f) \le \lambda_g^{[l]}(f)$, *where* $l \ge 1$.

Proof. As $M_g(r) \le M_h(r)$ and $M_f(r)$ is an increasing function of r we get for all sufficiently large values of r that

$$M_h^{-1}(r) \le M_g^{-1}(r)$$
$$\text{i.e., } M_h^{-1} M_f(r) \le M_g^{-1} M_f(r)$$
$$\text{i.e., } \liminf_{r\to\infty} \frac{\log^{[l]} M_h^{-1} M_f(r)}{\log r} \le \liminf_{r\to\infty} \frac{\log^{[l]} M_g^{-1} M_f(r)}{\log r}$$
$$\text{i.e., } \lambda_h^{[l]}(f) \le \lambda_g^{[l]}(f).$$

This proves the lemma. ∎

Lemma 4. [7] *Every entire function* f *satisfying the Property (A) is transcendental.*

Lemma 5. [8, p. 21] *Let* $f(z)$ *be holomorphic in the circle* $|z| = 2eR$ $(R > 0)$ *with* $f(0) = 1$ *and* η *be an arbitrary positive number not exceeding* $\frac{3e}{2}$. *Then inside the circle* $|z| = R$, *but outside of a family of excluded circles the sum of whose radii is not greater than* $4\eta R$, *we have*

$$\log |f(z)| > -T(\eta) \log M_f(2eR),$$

for $T(\eta) = 2 + \log \frac{3e}{2\eta}$.

3. Results

In this section we present the main results of the paper.

Theorem 1. *If* $f_1, f_2, \ldots, f_n$ $(n \ge 2)$ *and* g *are entire functions, then*

$$\lambda_f^{[l]}(g) \ge \lambda_{f_i}^{[l]}(g),$$

where $l \geq 1$, $f = f_1 \pm \sum_{k=2}^{n} f_k$ *and* $\lambda_{f_i}^{[l]}(g) = \min\{\lambda_{f_k}^{[l]}(g) \mid k = 1, 2, \ldots, n\}$. *The equality holds when* $\lambda_{f_i}^{[l]}(g) \neq \lambda_{f_k}^{[l]}(g)$ $(k = 1, 2, \ldots, n$ *and* $k \neq i)$.

Proof. If $\lambda_f^{[l]}(g) = \infty$ then the result is obvious. So we suppose that $\lambda_f^{[l]}(g) < \infty$. We can clearly assume that $\lambda_{f_i}^{[l]}(g)$ is finite. By hypothesis, $\lambda_{f_i}^{[l]}(g) \leq \lambda_{f_k}^{[l]}(g)$ for all $k = 1, 2, \ldots, i, \ldots, n$. We can suppose $\lambda_{f_i}^{[l]}(g) > 0$ (the proof for the case $\lambda_{f_i}^{[l]}(g) = 0$ is easier and left to the interested reader).

Now for any arbitrary $\varepsilon > 0$, we get for all sufficiently large values of r that

$$M_{f_k}[\exp^{[l-1]} r^{(\lambda_{f_k}^{[l]}(g)-\varepsilon)}] < M_g(r) \quad \text{where } k = 1, 2, \ldots, n$$

$$\text{i.e., } M_{f_k}(r) < M_g[(\log^{[l-1]} r)^{\frac{1}{(\lambda_{f_k}^{[l]}(g)-\varepsilon)}}] \quad \text{where } k = 1, 2, \ldots, n \text{ , so}$$

$$M_{f_k}(r) \leq M_g[(\log^{[l-1]} r)^{\frac{1}{(\lambda_{f_k}^{[l]}(g)-\varepsilon)}}] \quad \text{where } k = 1, 2, \ldots, n. \tag{1}$$

Now for all sufficiently large values of r,

$$M_f(r) < \sum_{k=1}^{n} M_{f_k}(r)$$

$$\text{i.e., } M_f(r) < \sum_{k=1}^{n} M_g[(\log^{[l-1]} r)^{\frac{1}{(\lambda_{f_k}^{[l]}(g)-\varepsilon)}}]$$

$$\text{i.e., } M_f(r) < nM_g[(\log^{[l-1]} r)^{\frac{1}{(\lambda_{f_i}^{[l]}(g)-\varepsilon)}}]. \tag{2}$$

Now in view of the first part of Lemma 1, we obtain from (2) for all sufficiently large values of r that

$$M_f(r) < M_g[(n+1)(\log^{[l-1]} r)^{\frac{1}{(\lambda_{f_i}^{[l]}(g)-\varepsilon)}}]$$

$$\text{i.e., } M_f[\exp^{[l-1]}(\frac{r}{n+1})^{(\lambda_{f_i}^{[l]}(g)-\varepsilon)}] < M_g(r)$$

$$\text{i.e., } \exp^{[l-1]}(\frac{r}{n+1})^{(\lambda_{f_i}^{[l]}(g)-\varepsilon)} < M_f^{-1} M_g(r)$$

$$\text{i.e., } (\lambda_{f_i}^{[l]}(g) - \varepsilon) \log(\frac{r}{n+1}) < \log^{[l]} M_f^{-1} M_g(r)$$

$$\text{i.e., } \lambda_{f_i}^{[l]}(g) - \varepsilon < \frac{\log^{[l]} M_f^{-1} M_g(r)}{\log r + O(1)}$$

$$\text{i.e., } \frac{\log^{[l]} M_f^{-1} M_g(r)}{\log r + O(1)} > \lambda_{f_i}^{[l]}(g) - \varepsilon.$$

So

$$\lambda_f^{[l]}(g) = \liminf_{r \to \infty} \frac{\log^{[l]} M_f^{-1} M_g(r)}{\log r + O(1)} \geq \lambda_{f_i}^{[l]}(g) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$\lambda_f^{[l]}(g) \geq \lambda_{f_i}^{[l]}(g). \tag{3}$$

Next let $\lambda_{f_i}^{[l]}(g) < \lambda_{f_k}^{[l]}(g)$ where $k = 1, 2, \ldots, n$ and $k \neq i$. As $\varepsilon(> 0)$ is arbitrary, from the definition of generalized lower order it follows for a sequence of values of r tending to infinity that

$$M_g(r) < M_{f_i}[\exp^{[l-1]} r^{(\lambda_{f_i}^{[l]}(g)+\varepsilon)}]$$
$$\text{i.e., } M_g[(\log^{[l-1]} r)^{\frac{1}{(\lambda_{f_i}^{[l]}(g)+\varepsilon)}}] < M_{f_i}(r). \tag{4}$$

Since $\lambda_{f_i}^{[l]}(g) < \lambda_{f_k}^{[l]}(g)$ where $k = 1, 2, \ldots, n$ and $k \neq i$, then in view of the third part of Lemma 1 we obtain that

$$\lim_{r\to\infty} \frac{M_g[(\log^{[l-1]} r)^{\frac{1}{(\lambda_{f_i}^{[l]}(g)+\varepsilon)}}]}{M_g[(\log^{[l-1]} r)^{\frac{1}{(\lambda_{f_k}^{[l]}(g)-\varepsilon)}}]} = \infty \text{ where } k = 1, 2, \ldots, n \text{ and } k \neq i. \tag{5}$$

Therefore from (5) we obtain for all sufficiently large values of r that

$$M_g[(\log^{[l-1]} r)^{\frac{1}{(\lambda_{f_i}^{[l]}(g)+\varepsilon)}}] > nM_g[(\log^{[l-1]} r)^{\frac{1}{(\lambda_{f_k}^{[l]}(g)-\varepsilon)}}], \tag{6}$$

for all $k \in \{1, 2, \ldots, n\}\backslash\{i\}$.

Thus from (1), (4) and (6) we get for a sequence of values of r tending to infinity that

$$M_{f_i}(r) > M_g[(\log^{[l-1]} r)^{\frac{1}{(\lambda_{f_i}^{[l]}(g)+\varepsilon)}}]$$
$$\text{i.e., } M_{f_i}(r) > nM_g[(\log^{[l-1]} r)^{\frac{1}{(\lambda_{f_k}^{[l]}(g)-\varepsilon)}}]$$
$$\text{i.e., } M_{f_i}(r) > nM_{f_k}(r) \text{ for all } k = 1, 2, \ldots, n \text{ with } k \neq i. \tag{7}$$

So from (4) and (7) and in view of the first part of Lemma 1 it follows for a sequence of values of r tending to infinity that

$$M_f(r) \geq M_{f_i}(r) - \sum_{\substack{k=1\\k\neq i}}^{n} M_{f_k}(r)$$
$$\text{i.e., } M_f(r) \geq M_{f_i}(r) - \frac{1}{n}\sum_{\substack{k=1\\k\neq i}}^{n} M_{f_i}(r)$$
$$\text{i.e., } M_f(r) > M_{f_i}(r) - (\frac{n-1}{n})M_{f_i}(r)$$
$$\text{i.e., } M_f(r) > (\frac{1}{n})M_{f_i}(r)$$
$$\text{i.e., } M_f(r) > (\frac{1}{n})M_g[(\log^{[l-1]} r)^{\frac{1}{(\lambda_{f_i}^{[l]}(g)+\varepsilon)}}]$$

$$\text{i.e., } M_f(r) > M_g[\frac{(\log^{[l-1]} r)^{\frac{1}{(\lambda_{f_i}^{[l]}(g)+\varepsilon)}}}{n+1}] \ .$$

This gives for a sequence of values of r tending to infinity that

$$M_f[\exp^{[l-1]}\{(n+1)r\}^{(\lambda_{f_i}^{[l]}(g)+\varepsilon)}] > M_g(r)$$

$$\text{i.e., } \{(n+1)r\}^{(\lambda_{f_i}^{[l]}(g)+\varepsilon)} > \log^{[l-1]} M_f^{-1} M_g(r)$$

$$\text{i.e., } \lambda_{f_i}^{[l]}(g)+\varepsilon > \frac{\log^{[l]} M_f^{-1} M_g(r)}{\log((n+1)r)}$$

$$\text{i.e., } \lambda_{f_i}^{[l]}(g)+\varepsilon > \frac{\log^{[l]} M_f^{-1} M_g(r)}{\log r + O(1)}$$

$$\text{i.e., } \lambda_{f_i}^{[l]}(g) \geq \liminf_{r\to\infty} \frac{\log^{[l]} M_f^{-1} M_g(r)}{\log r + O(1)}$$

$$\text{i.e., } \lambda_{f}^{[l]}(g) = \liminf_{r\to\infty} \frac{\log^{[l]} M_f^{-1} M_g(r)}{\log r} \leq \lambda_{f_i}(g). \tag{8}$$

So from (3) and (8), we finally obtain that

$$\lambda_f^{[l]}(g) = \lambda_{f_i}^{[l]}(g),$$

whenever $\lambda_{f_i}^{[l]}(g) \neq \lambda_{f_k}^{[l]}(g)$ for all $k \in \{1, 2, \ldots, n\} \backslash \{i\}$. ■

THEOREM 2. *Let n, l be two positive integers with $n, l \geq 2$. Then*

$$\frac{1}{n}\lambda_f^{[l]}(g) \leq \lambda_{f^n}^{[l]}(g) \leq \lambda_f^{[l]}(g).$$

Proof. From the first and second parts of Lemma 1, we obtain that

$$\{M_f(r)\}^n \leq K M_f(r^n) < M_f((K+1)r^n), \ n > 1 \text{ and } r > 0 \tag{9}$$

where $K = K(n, f) > 0$. Therefore from (9) we obtain that

$$M_f^{-1}(r^n) < (K+1)\{M_f^{-1}(r)\}^n$$

So

$$\lambda_{f^n}^{[l]}(g) \geq \frac{\log^{[l]} \frac{1}{(K+1)} M_f^{-1} M_g(r^n)}{\log r^n}$$

$$\text{i.e., } \lambda_{f^n}^{[l]}(g) \geq \frac{1}{n}\lambda_f^{[l]}(g). \tag{10}$$

On the other hand since $\{M_f(r)\}^n > M_f(r)$ for all sufficiently large values of r, we have by Lemma 3

$$\lambda_{f^n}^{[l]}(g) \leq \lambda_f^{[l]}(g). \tag{11}$$

Thus the theorem follows from (10) and (11). ■

Proposition 1. *Let n, l be two positive integers with $n, l \geq 2$. Then*

$$\frac{1}{n}\rho_f^{[l]}(g) \leq \rho_{f^n}^{[l]}(g) \leq \rho_f^{[l]}(g).$$

The proof is omitted as it can be carried out under the lines of Theorem 2.

Theorem 3. *Let P be a polynomial. If f is transcendental then $\lambda_{Pf}^{[l]}(g) = \lambda_f^{[l]}(g)$, and if g is transcendental, then $\lambda_f^{[l]}(Pg) = \lambda_f^{[l]}(g)$. If f and g are both transcendental then $\lambda_{Pf}^{[l]}(g) = \lambda_f^{[l]}(Pg) = \lambda_f^{[l]}(g) = \lambda_{Pf}^{[l]}(Pg)$. Here Pf and Pg denote the ordinary product of P with f and g respectively and $l \geq 1$.*

Proof. Let m be the degree of $P(z)$. Then there exists α such that $0 < \alpha < 1$ and a positive integer $n(> m)$ for which

$$2\alpha \leq |P(z)| \leq r^n$$

holds on $|z| = r$ for all sufficiently large values of r. Now by the first part of Lemma 1 we obtain that $M_g(\frac{1}{\alpha} \cdot \alpha r) > \frac{1}{2\alpha} M_g(\alpha r)$, i.e.,

$$M_g(\alpha r) < 2\alpha M_g(r). \tag{12}$$

Now let us consider $h(z) = P(z) \cdot f(z)$. Then from (2) and in view of the fourth part of Lemma 1 we get for any $s(> 1)$ and for all sufficiently large values of r that

$$M_g(\alpha r) < 2\alpha M_g(r) \leq M_h(r) \leq r^n M_g(r) < M_g(r^s).$$

So

$$\liminf_{r\to\infty} \frac{\log^{[l]} M_f^{-1} M_g(\alpha r)}{\log r} \leq \liminf_{r\to\infty} \frac{\log^{[l]} M_f^{-1} M_h(r)}{\log r} \leq \liminf_{r\to\infty} \frac{\log^{[l]} M_f^{-1} M_g(r^s)}{\log r}$$

i.e.

$$\liminf_{r\to\infty} \frac{\log^{[l]} M_f^{-1} M_g(\alpha r)}{\log(\alpha r) + O(1)} \leq \liminf_{r\to\infty} \frac{\log^{[l]} M_f^{-1} M_h(r)}{\log r} \leq \liminf_{r\to\infty} \frac{\log^{[l]} M_f^{-1} M_g(r^s)}{\log r^s} \cdot s$$

$$\text{i.e., } \lambda_f^{[l]}(g) \leq \lambda_h^{[l]}(g) \leq s \cdot \lambda_f^{[l]}(g),$$

and letting $s \to 1+$ we get

$$\lambda_{Pf}^{[l]}(g) = \lambda_f^{[l]}(g). \tag{13}$$

Similarly, when g is transcendental one can easily prove that

$$\lambda_f^{[l]}(Pg) = \lambda_f^{[l]}(g). \tag{14}$$

If f and g are both transcendental then the conclusion of the theorem can easily be obtained by combining (13) and (14), and the theorem follows. ∎

Theorem 4. *If $f_1, f_2, \dots, f_n$ $(n \geq 2)$, g are entire functions and g has the Property (A), then*

$$\lambda_f^{[l]}(g) \geq \lambda_{f_i}^{[l]}(g)$$

where $f = \prod_{k=1}^{n} f_k$ and $\lambda_{f_i}^{[l]}(g) = \min\{\lambda_{f_k}^{[l]}(g) \mid k = 1, 2, \dots, n\}$. The equality holds when $\lambda_{f_i}^{[l]}(g) \neq \lambda_{f_k}^{[l]}(g)$ $(k = 1, 2, \dots, n$ and $k \neq i)$. Finally, assume that F_1 and F_2 are entire functions such that $f := \frac{F_1}{F_2}$ is also an entire function. Then $\lambda_f^{[l]}(g) = \min\{\lambda_{F_1}^{[l]}(g), \lambda_{F_2}^{[l]}(g)\}$.

Proof. By Lemma 4, g is transcendental. Suppose that $\lambda_f^{[l]}(g) < \infty$. Otherwise if $\lambda_f^{[l]}(g) = \infty$ then the result is obvious. We can clearly assume that $\lambda_{f_i}^{[l]}(g)$ is finite. Also suppose that $\lambda_{f_i}^{[l]}(g) \leq \lambda_{f_k}^{[l]}(g)$ where $k = 1, 2, \dots, n$. We can suppose $\lambda_{f_i}^{[l]}(g) > 0$ (the proof for the case $\lambda_{f_i}^{[l]}(g) = 0$ is easier and left to the interested reader).

Now for any arbitrary $\varepsilon > 0$, with $\varepsilon < \lambda_{f_i}^{[l]}(g)$, we have for all sufficiently large values of r that

$$M_{f_k}[\exp^{[l-1]} r^{(\lambda_{f_k}^{[l]}(g)-\frac{\varepsilon}{2})}] < M_g(r) \quad \text{where } k = 1, 2, \dots, n$$

$$\text{i.e., } M_{f_k}(r) < M_g[(\log^{[l-1]} r)^{\frac{1}{(\lambda_{f_k}^{[l]}(g)-\frac{\varepsilon}{2})}}] \quad \text{where } k = 1, 2, \dots, n, \text{ so}$$

$$M_{f_k}(r) \leq M_g[(\log^{[l-1]} r)^{\frac{1}{(\lambda_{f_i}^{[l]}(g)-\frac{\varepsilon}{2})}}] \quad \text{for } k = 1, 2, \dots, n. \tag{15}$$

From (15) we have for all sufficiently large values of r that

$$M_f(r) < \prod_{k=1}^{n} M_{f_k}(r),$$

$$\text{i.e., } M_f(r) < \prod_{k=1}^{n} M_g[(\log^{[l-1]} r)^{\frac{1}{(\lambda_{f_k}^{[l]}(g)-\frac{\varepsilon}{2})}}]$$

$$\text{i.e., } M_f(r) < [M_g[(\log^{[l-1]} r)^{\frac{1}{(\lambda_{f_i}^{[l]}(g)-\frac{\varepsilon}{2})}}]]^n. \tag{16}$$

Observe that

$$\delta := \frac{\lambda_{f_i}^{[l]}(g) - \frac{\varepsilon}{2}}{\lambda_{f_i}^{[l]}(g) - \varepsilon} > 1. \tag{17}$$

Since g has the Property (A), in view of Lemma 2 and (17) we obtain from (16) for all sufficiently large values of r that

$$M_f(r) < M_g(\log^{[l-1]} r)^{\frac{\delta}{(\lambda_{f_i}^{[l]}(g)-\frac{\varepsilon}{2})}} = M_g[(\log^{[l-1]} r)^{\frac{1}{(\lambda_{f_i}^{[l]}(g)-\varepsilon)}}]$$

$$\text{i.e., } M_f[\exp^{[l-1]} r^{(\lambda_{f_i}^{[l]}(g)-\varepsilon)}] < M_g(r)$$

$$\text{i.e., } r^{(\lambda_{f_i}^{[l]}(g)-\varepsilon)} < \log^{[l-1]} M_f^{-1} M_g(r)$$

$$\text{i.e., } (\lambda_{f_i}^{[l]}(g) - \varepsilon) \log r < \log^{[l]} M_f^{-1} M_g(r)$$

$$\text{i.e., } \lambda_{f_i}^{[l]}(g) - \varepsilon < \frac{\log^{[l]} M_f^{-1} M_g(r)}{\log r}$$

So

$$\lambda_f^{[l]}(g) = \liminf_{r\to\infty} \frac{\log^{[l]} M_f^{-1} M_g(r)}{\log r} \geq \lambda_{f_i}^{[l]}(g) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$\lambda_f^{[l]}(g) \geq \lambda_{f_i}^{[l]}(g). \tag{18}$$

Next, let $\lambda_{f_i}^{[l]}(g) < \lambda_{f_k}^{[l]}(g)$ where $k = 1, 2, \ldots, n$ and $k \neq i$. Fix $\varepsilon > 0$ with $\varepsilon < \frac{1}{4} \min\{\lambda_{f_k}^{[l]}(g) - \lambda_{f_i}^{[l]}(g) : k \in \{1, \ldots, n\} \backslash \{i\}\}$. Without loss of any generality, we may assume that $f_k(0) = 1$ where $k = 1, 2, \ldots, n$ and $k \neq i$.

Now from the definition of relative lower order we obtain for a sequence of values of R tending to infinity that

$$M_g(R) < M_{f_i}[\exp^{[l-1]} R^{(\lambda_{f_i}^{[l]}(g)+\varepsilon)}]$$

$$\text{i.e., } M_{f_i}(R) > M_g[(\log^{[l-1]} R)^{\frac{1}{(\lambda_{f_i}^{[l]}(g)+\varepsilon)}}]\ . \tag{19}$$

Also for all sufficiently large values of r we get that

$$M_{f_k}[\exp^{[l-1]} r^{(\lambda_{f_k}^{[l]}(g)-\varepsilon)}] < M_g(r) \quad \text{where } k = 1, 2, \ldots, n \text{ and } k \neq i$$

$$\text{i.e., } M_{f_k}(r) < M_g[(\log^{[l-1]} r)^{\frac{1}{(\lambda_{f_k}^{[l]}(g)-\varepsilon)}}] \quad \text{where } k = 1, 2, \ldots, n \text{ and } k \neq i.$$

Since $\lambda_{f_i}^{[l]}(g) < \lambda_{f_k}^{[l]}(g)$, we get from above that

$$M_{f_k}(r) < M_g[(\log^{[l-1]} r)^{\frac{1}{(\lambda_{f_i}^{[l]}(g)-\varepsilon)}}] \tag{20}$$

where $k = 1, 2, \ldots, n$ and $k \neq i$.

Now in view of Lemma 5, taking $f_k(z)$ for $f(z)$, $\eta = \frac{1}{16}$ and $2R$ for R, it follows for the values of z specified in the lemma that

$$\log |f_k(z)| > -T(\eta) \log M_{f_k}(2e \cdot 2R),$$

where

$$T(\eta) = 2 + \log(\frac{3e}{2 \cdot \frac{1}{16}}) = 2 + \log(24e).$$

Therefore

$$\log |f_k(z)| > -(2 + \log(24e)) \log M_{f_k}(4eR)$$

holds within and on $|z| = 2R$ but outside a family of excluded circles the sum of whose radii is not greater than $4 \cdot \frac{1}{16} \cdot 2R = \frac{R}{2}$. If $r \in (R, 2R)$ then on $|z| = r$

$$\log |f_k(z)| > -7 \log M_{f_k}(4e \cdot R). \tag{21}$$

Since $r > R$, we have from above and (19) for a sequence of values of r tending to infinity that

$$M_{f_i}(r) > M_{f_i}(R) > M_g[(\log^{[l-1]} R)^{\frac{1}{(\lambda^{[l]}_{f_i}(g)+\varepsilon)}}] > M_g[(\log^{[l-1]} \frac{r}{2})^{\frac{1}{(\lambda^{[l]}_{f_i}(g)+\varepsilon)}}]. \quad (22)$$

Let z_r be a point on $|z| = r$ such that $M_{f_i}(r) = |f_i(z_r)|$. Therefore as $r > R$, from (20), (21) and (22) it follows for a sequence of values of r tending to infinity that

$$M_f(r) = \max\{|f(z)| : |z| = r\} = \max\{\prod_{k=1}^{n} |f_k(z)| : |z| = r\}, \text{ so}$$

$$M_f(r) \geq \prod_{\substack{k=1\\k\neq i}}^{n} |f_k(z_r)||f_i(z_r)|$$

$$\text{i.e., } M_f(r) \geq \prod_{\substack{k=1\\k\neq i}}^{n} |f_k(z_r)| M_{f_i}(r)$$

$$\text{i.e., } M_f(r) \geq \prod_{\substack{k=1\\k\neq i}}^{n} [M_{f_k}(4eR)]^{-7} M_g[(\log^{[l-1]}(\tfrac{r}{2}))^{\frac{1}{(\lambda^{[l]}_{f_i}(g)+\varepsilon)}}]$$

$$\geq \prod_{\substack{k=1\\k\neq i}}^{n} [M_g[(\log^{[l-1]}(4eR))^{\frac{1}{(\lambda^{[l]}_{f_k}(g)-\varepsilon)}}]]^{-7} M_g[(\log^{[l-1]}(\tfrac{r}{2}))^{\frac{1}{(\lambda^{[l]}_{f_i}(g)+\varepsilon)}}]$$

$$= \prod_{\substack{k=1\\k\neq i}}^{n} [M_g[(\log^{[l-1]}(4eR))^{\frac{1}{(\lambda^{[l]}_{f_k}(g)-\varepsilon)}}]]^{-7} M_g[(\log^{[l-1]}(\tfrac{4er}{8e}))^{\frac{1}{(\lambda^{[l]}_{f_i}(g)+\varepsilon)}}],$$

hence

$$M_f(r) \geq \prod_{\substack{k=1\\k\neq i}}^{n} [M_g[(\log^{[l-1]} 4er)^{\frac{1}{(\lambda^{[l]}_{f_k}(g)-\varepsilon)}}]]^{-7} M_g[(\log^{[l-1]} \tfrac{4er}{8e})^{\frac{1}{(\lambda^{[l]}_{f_i}(g)+\varepsilon)}}]. \quad (23)$$

On the other hand, we have $(\log^{[l-1]}(\frac{4er}{8e}))^{\frac{1}{(\lambda^{[l]}_{f_i}(g)+\varepsilon)}} \geq (\log^{[l-1]}(4er))^{\frac{1}{(\lambda^{[l]}_{f_i}(g)+2\varepsilon)}}$ asymptotically. By using this fact together with Lemma 2 (with $n = 2$ and $\delta := \frac{\lambda^{[l]}_{f_i}(g)+3\varepsilon}{\lambda^{[l]}_{f_i}(g)+2\varepsilon} > 1$) we get for r large enough that

$$M_g[(\log^{[l-1]}(\frac{4er}{8e}))^{\frac{1}{(\lambda^{[l]}_{f_i}(g)+\varepsilon)}}] \geq M_g[\{(\log^{[l-1]}(4er))^{\frac{1}{(\lambda^{[l]}_{f_i}(g)+3\varepsilon)}}\}^2]. \quad (24)$$

Let $L := \min\{\lambda^{[l]}_{f_k}(g) : k \neq i\}$. Now, by choosing this time $\delta := \frac{L-\varepsilon}{\lambda^{[l]}_{f_i}(g)+3\varepsilon}$ (which is > 1 due to our selection of ε), a further application of Lemma 2 yields

$$M_g[(\log^{[l-1]}(4er))^{\frac{1}{\lambda^{[l]}_{f_i}(g)+3\varepsilon}}] \geq M_g[(\log^{[l-1]}(4er))^{\frac{1}{L-\varepsilon}}]^{7n}$$

$$\geq \prod_{\substack{k=1\\k\neq i}}^{n} M_g[[(\log^{[l-1]}(4er))^{\frac{1}{\lambda^{[l]}_{f_k}(g)-\varepsilon}}]]^7 \quad (25)$$

for r large enough. Now from (23), (24) and (25), it follows for a sequence of values of r tending to infinity that

$$M_f(r) \geq M_g[(\log^{[l-1]}(4er))^{\frac{1}{(\lambda_{f_i}^{[l]}(g)+3\varepsilon)}}]$$
$$\text{i.e., } M_f[\exp^{[l-1]} r^{(\lambda_{f_i}^{[l]}(g)+3\varepsilon)}] \geq M_g(4er)$$
$$\text{i.e., } r^{(\lambda_{f_i}^{[l]}(g)+3\varepsilon)} \geq \log^{[l-1]} M_f^{-1} M_g(4er)$$
$$\text{i.e., } (\lambda_{f_i}^{[l]}(g)+3\varepsilon)\log r \geq \log^{[l]} M_f^{-1} M_g(4er)$$
$$\text{i.e., } \lambda_{f_i}^{[l]}(g)+3\varepsilon \geq \frac{\log^{[l]} M_f^{-1} M_g(4er)}{\log(4er)+O(1)}$$

If we let $\varepsilon \to 0^+$ then we get

$$\lambda_{f_i}^{[l]}(g) \geq \liminf_{r\to\infty} \frac{\log^{[l]} M_f^{-1} M_g(4er)}{\log(4er)+O(1)} .$$

Therefore

$$\lambda_f^{[l]}(g) = \liminf_{r\to\infty} \frac{\log^{[l]} M_f^{-1} M_g(r)}{\log r} \leq \lambda_{f_i}^{[l]}(g). \tag{26}$$

So from (18) and (26) we finally obtain that

$$\lambda_f^{[l]}(g) = \lambda_{f_i}^{[l]}(g),$$

if one assumes that $\lambda_{f_i}^{[l]}(g) \neq \lambda_{f_k}^{[l]}(g)$ for all $k \in \{1,2,\ldots,n\}\backslash\{i\}$.

Let now $f = \frac{F_1}{F_2}$ with F_1, F_2, f entire, and suppose $\lambda_{F_1}^{[l]}(g) \geq \lambda_{F_2}^{[l]}(g)$. We have $F_1 = f.F_2$. Thus $\lambda_{F_1}^{[l]}(g) = \lambda_f^{[l]}(g)$ if $\lambda_f^{[l]}(g) < \lambda_{F_2}^{[l]}(g)$. So it follows that $\lambda_{F_1}^{[l]}(g) < \lambda_{F_2}^{[l]}(g)$, which contradicts the hypothesis "$\lambda_{F_1}^{[l]}(g) \geq \lambda_{F_2}^{[l]}(g)$". Hence $\lambda_f^{[l]}(g) = \lambda_{\frac{F_1}{F_2}}^{[l]}(g) \geq \lambda_{F_2}^{[l]}(g) = \min\{\lambda_{F_1}^{[l]}(g), \lambda_{F_2}^{[l]}(g)\}$. Also suppose that $\lambda_{F_1}^{[l]}(g) > \lambda_{F_2}^{[l]}(g)$. Then $\lambda_{F_1}^{[l]}(g) = \min\{\lambda_f^{[l]}(g), \lambda_{F_2}^{[l]}(g)\} = \lambda_{F_2}^{[l]}(g)$, if $\lambda_f^{[l]}(g) > \lambda_{F_2}^{[l]}(g)$, which also a contradiction. Thus $\lambda_f^{[l]}(g) = \lambda_{\frac{F_1}{F_2}}^{[l]}(g) = \min\{\lambda_{F_1}^{[l]}(g), \lambda_{F_2}^{[l]}(g)\}$. This proves the theorem. ■

Acknowledgement. The authors are thankful to the referee for his/her valuable suggestions towards the improvement of the paper.

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(received 19.11.2013; in revised form 01.03.2014; available online 15.04.2014)

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