

FUZZY REPRESENTABLE MODULES AND FUZZY ATTACHED PRIMES

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Abstract. Let M be a non-zero unitary module over a non-zero commutative ring R . A kind of uniqueness theorem for a non-zero fuzzy representable submodule μ of M will be proved, and then the set of fuzzy attached primes of μ will be defined. Then among other things, it will be shown that, whenever R is Noetherian, a fuzzy prime ideal ξ is attached to μ if and only if ξ is the annihilator of a fuzzy quotient of μ . The behavior of fuzzy attached primes with fuzzy quotient and fuzzy localization techniques will be studied.

1. Introduction

Let R be a commutative ring with identity and let M be a unitary R -module. In the last two decades a considerable amount of works has been done in fuzzifying various concepts and results of classical ring and module theory. Among them, the fuzzy primary decomposition of ideals/submodules and fuzzy associated prime ideals are of particular significance (see, for example, [4–6]). Recently, in [12], the concept of fuzzy coprimary submodules of M , which is in fact a dual of fuzzy primary submodules, has been defined and some of its properties studied. A fuzzy submodule μ of M is said to be *fuzzy coprimary* if for each $r \in R$, $1_r \cdot \mu = \mu$ or else $1_r \in \mathcal{R}(1_\theta : \mu)$. We continue our study on the fuzzy coprimary and fuzzy representable submodules. It should be noted here that this concepts in crisp case has been introduced in [1,2] and has led to extensive applications in the commutative algebra area (see, for example, [3,7,10,11]). Let μ be a fuzzy submodule of M . A *fuzzy coprimary representation* of μ is an expression

$$\mu = \mu_1 + \cdots + \mu_n \tag{1}$$

of μ as a sum of finitely many fuzzy coprimary submodules μ_i , $1 \leq i \leq n$ of M . This representation is said to be *minimal* if the fuzzy prime ideals $\xi_i = \mathcal{R}(1_\theta : \mu_i)$, $1 \leq i \leq n$, are all distinct and non of the summands μ_i is redundant (i.e., for all $j = 1, \dots, n$, $\mu_j \not\subseteq \sum_{i \neq j}^n \mu_i$).

2010 Mathematics Subject Classification: 08A72

Keywords and phrases: Fuzzy coprimary submodules; fuzzy coprimary representation; fuzzy attached primes.

In this paper, it will be shown that the prime ideals ξ_i depend only on μ and not on the minimal coprimary representation of μ . Using this fact the fuzzy attached prime ideals of μ will be defined. Then, among other things, we show that whenever R is Noetherian, a fuzzy prime ideal ξ is attached to μ if and only if ξ is the annihilator of a fuzzy quotient of μ . These will be done in section 2. The behavior of fuzzy coprimary representation and fuzzy attached primes with fuzzy localization technique will be studied in Section 3.

2. Fuzzy coprimary representations

Throughout the paper, the set of all fuzzy ideals of R will be denoted by $FI(R)$ and the set of all fuzzy R -submodules of M will be denoted by $FS(M)$. The zero fuzzy ideal of R (resp. the zero fuzzy submodule of M) will be denoted by 1_0 (resp. by 1_θ), where 0 and θ are the zero elements of R and M respectively. For any unexplained notation and terminology we refer to the textbook [8].

Let $n \in \mathbb{N}$, $\zeta \in FI(R)$ and $\lambda, \lambda_i \in FS(M), i = 1, \dots, n$. As usual,

$$\lambda^* = \{x \in M \mid \lambda(x) > 0\}, \quad \lambda_* = \{x \in M \mid \lambda(x) = 1\},$$

$$(\zeta \cdot \lambda)(x) = \bigvee_{r \in R, y \in M, x=ry} \{\zeta(r) \wedge \lambda(t)\}$$

for all $x \in M$, and

$$(\lambda_1 + \dots + \lambda_n)(x) = \bigvee \left\{ \bigwedge_{i=1}^n \lambda_i(x_i) \mid x_i \in M, \sum_{i=1}^n x_i = x \right\},$$

for all $x \in M$. It is well known that λ^*, λ_* are submodules of M and $\zeta \cdot \lambda, \lambda_1 + \dots + \lambda_n \in FS(M)$.

REMARK 2.1. (A). Let $\nu, \lambda \in FS(M)$. Like in the ordinary commutative algebra, by $\frac{\lambda}{\nu}$, we mean the fuzzy quotient $\frac{\lambda + \nu}{\nu} \in FS(\frac{\lambda^* + \nu^*}{\nu^*})$, (note that $(\lambda + \nu)^* = \lambda^* + \nu^*$).

(B). Let $\nu, \eta, \lambda \in FS(M)$ and let $\nu \subseteq \eta, \nu \subseteq \lambda$. Then $\frac{\eta}{\nu} \in FS(\frac{\eta^*}{\nu^*})$, $\frac{\lambda}{\nu} \in FS(\frac{\lambda^*}{\nu^*})$ and $\frac{\eta + \lambda}{\nu} \in FS(\frac{\eta^* + \lambda^*}{\nu^*})$, so that we can consider each of this as a fuzzy submodule of $\frac{M}{\nu^*}$ by (for example)

$$\left(\frac{\eta}{\nu}\right)(x + \nu^*) = \bigvee_{e \in \nu^*} \{\eta(x + e)\}, \quad \forall x \in M,$$

etc. In this situation it is not hard to see that given $x \in M$,

$$\begin{aligned} \left(\frac{\eta}{\nu} + \frac{\lambda}{\nu}\right)(x + \nu^*) &= \bigvee_{y, z \in M, y+z+\nu^*=x+\nu^*} \left\{ \left(\frac{\eta}{\nu}\right)(y + \nu^*) \wedge \left(\frac{\lambda}{\nu}\right)(z + \nu^*) \right\} \\ &= \bigvee_{e \in \nu^*} \{(\eta + \lambda)(x + e)\} = \left(\frac{\eta + \lambda}{\nu}\right)(x + \nu^*). \end{aligned}$$

Thus, as for fuzzy submodules of $\frac{M}{\nu^*}$ (and so as for fuzzy submodules of $(\frac{\eta^* + \lambda^*}{\nu^*})$) we have

$$\frac{\eta}{\nu} + \frac{\lambda}{\nu} = \frac{\eta + \lambda}{\nu}.$$

(C). Let $\nu, \lambda \in FS(M)$ such that $\nu \subset \lambda$ and $\nu^* \subset \lambda^*$. Let $\eta \in FS(\frac{\lambda^*}{\nu^*})$ and $\eta \subseteq \frac{\lambda}{\nu}$. Assume that $\pi : \lambda^* \rightarrow \frac{\lambda^*}{\nu^*}$ is the natural epimorphism. Then $\frac{\pi^{-1}(\eta)}{\nu} \in FS(\frac{\pi^{-1}(\eta)^*}{\nu^*})$, so that we can consider $\frac{\pi^{-1}(\eta)}{\nu} \in FS(\frac{\lambda^*}{\nu^*})$. It is easy to see that $\eta = \frac{\pi^{-1}(\eta)}{\nu}$ as fuzzy submodules of $\frac{\lambda^*}{\nu^*}$.

The next proposition which is the fuzzy version of the Prime Avoidance Theorem is needed in the sequel. It may be well-known and exists in the fuzzy literature, but we were not able to find a suitable reference. Because of its importance, we supply the statement without the proof.

PROPOSITION 2.2. *Let $\zeta \in FI(R)$ and let $\xi, \xi_i, 1 \leq i \leq n$, be fuzzy prime ideals of R .*

- (1) *If $\zeta \subseteq \cup_i^n \xi_i$. Then $\zeta \subseteq \xi_i$ for some i .*
- (2) *If $\xi = \cap_i^n \xi_i$, then $\xi = \xi_i$ for some i .*

DEFINITION 2.3. (1) Let $\lambda, \eta \in FS(M)$. The fuzzy residual of λ by η , denoted by $(\lambda : \eta)$, is defined by

$$(\lambda : \eta)(r) = \bigvee \{ \zeta(r) \mid \zeta \cdot \eta \subseteq \lambda \} \quad \forall r \in R.$$

By [8, Theorem 4.5.6], $(\lambda : \eta) \in FI(R)$.

- (2) For $\zeta \in FI(R)$, the \mathcal{R} -radical of ζ , denoted as $\mathcal{R}(\zeta)$, is defined by

$$\mathcal{R}(\zeta)(r) = \bigvee_{n \in \mathbb{N}} \zeta(r^n) \quad \forall r \in R,$$

which is a fuzzy ideal of R containing ζ . It is well known that the \mathcal{R} -radical commutes with finite intersection of fuzzy ideals.

- (3) Let $\mu \in FS(M)$ and let the situation be as in (1). By [12, Theorem 3.7(1)], and omitting the redundant summands, any fuzzy coprimary representation of μ can be refined to a minimal one. If μ has a fuzzy coprimary representation, we shall say that μ is *fuzzy representable*. Note that as the sum of empty family of fuzzy submodules is 1_θ , we shall regard 1_θ as fuzzy representable.

REMARK 2.4. As it was stated in Introduction, the fuzzy coprimary submodules were studied in [12]. Let μ be a fuzzy coprimary submodule of M . Following [12, Theorem 2.4], if μ is a fuzzy coprimary submodule of M , then $(1_\theta : \mu)$ is a fuzzy primary ideal of R so that $\xi = \mathcal{R}(1_\theta : \mu)$ is a fuzzy prime ideal. In this case we will say that μ is ξ -fuzzy coprimary.

It is easy to see that $1_\theta \neq \lambda \in FS(M)$ is ξ -fuzzy coprimary if and only if $\lambda \in FS(\lambda^*)$ is ξ -fuzzy coprimary.

Now, we can state and prove one of the main theorems of this section.

THEOREM 2.5. *Assume that μ is a fuzzy representable submodule of M and $\mu = \mu_1 + \dots + \mu_n$ is a minimal fuzzy coprimary representation of μ with $\xi_i = \mathcal{R}(1_\theta : \mu_i)$. Then the set of fuzzy prime ideals $\{\xi_1, \dots, \xi_n\}$ depends only on μ and not on the minimal fuzzy coprimary representation of μ . In fact for a fuzzy prime ideal ξ , the following conditions are equivalent:*

- (1) $\xi = \xi_i$ for some $i \in \{1, \dots, n\}$.
- (2) μ has a ξ -fuzzy coprimary quotient module.
- (3) There exists a fuzzy submodule $\nu \subset \mu$ such that $\nu^* \subset \mu^*$ and that $\mathcal{R}(1_\theta : \frac{\mu}{\nu}) = \xi$.

Proof. (1) \Rightarrow (2). Let $\nu_i = \sum_{j \neq i} \mu_j$. By the minimality of the representation, we have $\nu_i \subset \mu$ and $\frac{\mu_i^*}{\nu_i^*} \neq \theta$. Now, by [8, Theorem 4.2.5] we have

$$\frac{\mu_i}{\nu_i \cap \mu_i} \approx \frac{\nu_i + \mu_i}{\nu_i} = \frac{\mu}{\nu_i}.$$

Thus, using [8, 4.2.4] for $\frac{\mu_i}{\nu_i \cap \mu_i} \in FS(M)$ and $\frac{\mu}{\nu_i} \in FS(M)$ (note that $\frac{\mu_i}{\nu_i \cap \mu_i} \in FS(\frac{\mu_i^*}{(\nu_i \cap \mu_i)^*})$ which we can consider as a fuzzy submodule of M by a similar argument as in Remark 2.1(A)), there exists $\lambda \in FS(\frac{\mu_i^*}{(\nu_i \cap \mu_i)^*})$ (and so in $FS(M)$) such that $\lambda \subseteq \frac{\mu_i}{\nu_i \cap \mu_i}$ and that

$$\frac{\mu_i / (\nu_i \cap \mu_i)}{\lambda} \cong \frac{\mu}{\nu_i |_{(\frac{\mu_i}{\nu_i})^*}} = \frac{\mu}{\nu_i},$$

note that $(\frac{\mu}{\nu_i})^* = \frac{\mu^*}{\nu_i^*}$. But the right-hand side is non-zero ($\neq 1_\theta$), which in turn gives that $\lambda^* \subset (\frac{\mu_i}{\nu_i \cap \mu_i})^*$ and so $\lambda \subset \frac{\mu_i}{\nu_i \cap \mu_i}$. Now by [12, Theorem 3.7(2)], $\frac{\mu_i / (\nu_i \cap \mu_i)}{\lambda}$ is ξ -fuzzy coprimary. Therefore $\frac{\mu}{\nu_i}$ is ξ -fuzzy coprimary.

(2) \Rightarrow (3). It is clear.

(3) \Rightarrow (1). We enumerate μ_i in such a way that $\mu_i \not\subseteq \nu$ for $1 \leq i \leq r$ and $\mu_i \subseteq \nu$ for $r + 1 \leq i \leq n$. Then, using Remark 2.1, as fuzzy submodules of $\frac{\mu^*}{\nu^*}$ we have

$$\begin{aligned} \frac{\mu}{\nu} &= \frac{\mu + \nu}{\nu} = \frac{\mu_1 + \nu + \dots + \mu_n + \nu}{\nu} \\ &= \frac{\mu_1 + \nu}{\nu} + \dots + \frac{\mu_n + \nu}{\nu} \\ &= \frac{\mu_1 + \nu}{\nu} + \dots + \frac{\mu_r + \nu}{\nu}. \end{aligned}$$

Now for each $i = 1 \dots, r$, $\exists \lambda_i \in F(\frac{\mu_i^*}{(\mu_i \cap \nu)^*})$ such that $\lambda_i \subseteq \frac{\mu_i}{\mu_i \cap \nu}$ and that

$$\frac{\mu_i / (\mu_i \cap \nu)}{\lambda_i} \cong \frac{\mu_i + \nu}{\nu |_{(\frac{\mu_i + \nu}{\nu})^*}} = \frac{\mu_i + \nu}{\nu}.$$

Then after deleting the 1_θ components (if any), in the family

$$\frac{\mu_i/(\mu_i \cap \nu)}{\lambda_i}, \quad i = 1, \dots, r$$

and renumbering, we have the family

$$\frac{\mu_i/(\mu_i \cap \nu)}{\lambda_i}, \quad i = 1, \dots, k$$

(note that since $\frac{\mu}{\nu} \neq 1_\theta$, $k \geq 1$), with suitable related domains, each of which is ξ_i -coprimary by [12, Theorem 3.7(2)], for $i = 1, \dots, k$.

Therefore we have

$$\begin{aligned} \xi &= \mathcal{R}(1_\theta : \frac{\mu}{\nu}) = \mathcal{R}(1_\theta : \frac{\mu_1 + \nu}{\nu} + \dots + \frac{\mu_r + \nu}{\nu}) \\ &= \mathcal{R}(\bigcap_{i=1}^r (1_\theta : \frac{\mu_i + \nu}{\nu})) = \bigcap_{i=1}^k \mathcal{R}(1_\theta : \frac{(\mu_i/\mu_i \cap \nu)}{\lambda_i}) = \bigcap_{i=1}^k \xi_i. \end{aligned}$$

So by Proposition 2.2(2), $\xi = \xi_i$ for some $i \in \{1, \dots, k\}$ and the proof is complete. ■

Let $\zeta \in FI(R)$ and ξ be a fuzzy prime ideal of R . Following [8, Definition 6.2.19], ξ is called *fuzzy prime divisor* of ζ if $\zeta \subseteq \xi$ and $\zeta_* \subseteq \xi_*$. Then ξ is called an *isolated* (or *minimal*) *prime divisor* of ζ , if there does not exist a fuzzy prime divisor ξ' of ζ such that $\xi' \subset \xi$.

DEFINITION 2.6. Let M be an R -module and let μ be a fuzzy representable submodule of M . Let

$$\mu = \mu_1 + \dots + \mu_n, \tag{*}$$

be a minimal fuzzy coprimary representation of μ , where μ_i is ξ_i -fuzzy coprimary, $1 \leq i \leq n$. Then the set $\{\xi_1, \dots, \xi_n\}$ which, by Theorem 2.5 is independent of the choice of minimal coprimary representation of μ , is called the set of *fuzzy attached primes* of μ . We will denote this set by $\text{Att}(\mu)$. We note that for each $i = 1, \dots, n$, $(1_\theta : \mu) \subseteq \xi_i$, and that $(1_\theta : \mu)_* \subseteq (1_\theta : \mu_i)_* \subseteq (\xi_i)_*$. The elements of $\text{Att}(\mu)$ which are isolated prime divisor of $(1_\theta : \mu)$ are said to be *fuzzy isolated attached primes* of μ and the others are said *fuzzy embedded attached primes* of μ . We call a subset \mathcal{A} of $\text{Att}(\mu)$ an isolated subset of $\text{Att}(\mu)$ if for each fuzzy prime ideal $\xi' \in \text{Att}(\mu)$, $\xi' \subseteq \xi$ for some $\xi \in \mathcal{A}$, gives that $\xi' \in \mathcal{A}$.

To state the next lemma, we need the following concept of (fuzzy) primary decomposition of a fuzzy ideal of R .

Let ζ be a proper fuzzy ideal of R . A *fuzzy primary decomposition* of ζ is an expression for ζ as an intersection of finitely many fuzzy primary ideals of R . Such a fuzzy primary decomposition

$$\zeta = \pi_1 \cap \dots \cap \pi_n, \quad \text{with } \xi_i = \mathcal{R}(\pi_i) \quad \text{for } i = 1, \dots, n$$

of ζ is said to be *reduced* (or *irredundant*) if ξ_i 's are different fuzzy primes and $\bigcap_{j \neq i} \pi_j \not\subseteq \pi_i$, for all $i = 1, \dots, n$.

We say that ζ is a *fuzzy decomposable* ideal precisely when it has a primary decomposition. It is well-known that any fuzzy decomposable ideal has a reduced fuzzy primary decomposition and the set of fuzzy prime ideals $\{\xi_1, \dots, \xi_n\}$ (in any reduced fuzzy primary decomposition), which is uniquely determined by ζ , is called the set of *associated primes* of ζ , (see for example [8, Chapter,6]). We denote this set by $\text{ass}(\zeta)$.

PROPOSITION 2.7. *Let μ be a fuzzy representable submodule of M with a minimal fuzzy coprimary representation such as $(*)$. Then the fuzzy ideal $(1_\theta : \mu)$ is a fuzzy decomposable ideal and $\text{ass}(1_\theta : \mu) \subseteq \text{Att}(\mu)$.*

Proof. By [12, Theorem 2.4], $(1_\theta : \mu_i)$ is ξ_i -fuzzy primary for $i = 1, \dots, n$. Now $(1_\theta : \mu) = (1_\theta : \mu_1 + \dots + \mu_n) = \bigcap_i^n (1_\theta : \mu_i)$ and the result follows by the above observation. ■

The next theorem examines the fuzzy attached primes of the fuzzy quotient modules.

THEOREM 2.8. *Let μ be a fuzzy representable submodule of M with a minimal fuzzy coprimary representation such as $(*)$. Let $\nu \in FS(M)$ be such that $\nu \subset \mu$. Then μ/ν is fuzzy representable and $\text{Att}(\frac{\mu}{\nu}) \subseteq \text{Att}(\mu)$.*

Proof. We enumerate μ_i in such a way that $\mu_i \not\subseteq \nu$, for $i = 1, \dots, r$ and $\mu_i \subseteq \nu$ for $i = r + 1, \dots, n$. Then

$$\frac{\mu}{\nu} = \frac{\mu_1 + \nu}{\nu} + \dots + \frac{\mu_n + \nu}{\nu} = \frac{\mu_1 + \nu}{\nu} + \dots + \frac{\mu_r + \nu}{\nu}.$$

Now just as in the proof of Theorem 2.5 (3) \Rightarrow (1), for each $i = 1, \dots, r$, there exist $\lambda_i \in F(\frac{\mu_i^*}{(\mu_i \cap \nu)^*})$ such that $\lambda_i \subseteq \frac{\mu_i}{\mu_i \cap \nu}$ and that

$$\frac{\mu_i / (\mu_i \cap \nu)}{\lambda_i} \cong \frac{\mu_i + \nu}{\nu} \Big|_{(\frac{\mu_i + \nu}{\nu})^*} = \frac{\mu_i + \nu}{\nu}.$$

Since each

$$\frac{\mu_i / (\mu_i \cap \nu)}{\lambda_i}, \quad i = 1, \dots, r$$

is ξ_i -coprimary, we see that $\text{Att}(\frac{\mu}{\nu}) \subseteq \{\xi_1, \dots, \xi_r\}$, and the result follows. ■

We note that for $\nu, \lambda \in FS(M)$ with $\nu \subset \lambda$, $(1_\theta : \frac{\lambda}{\nu})(0) = 1$. So if ζ is a fuzzy ideal of R such that $(1_\theta : \frac{\lambda}{\nu}) \subseteq \zeta$, then $(1_\theta : \frac{\lambda}{\nu})_* \subseteq \zeta_*$ automatically.

PROPOSITION 2.9. *Let μ be a fuzzy representable submodule of M . Then $\xi \in \text{Att}(\mu)$ if and only if there exists $\nu \subset \mu$ such that ξ is an isolated prime divisor of $(1_\theta : \frac{\mu}{\nu})$.*

Proof. Let $\xi \in \text{Att}(\mu)$. By Theorem 2.5, there exists a fuzzy submodule $\nu \subset \mu$ such that $\nu^* \subset \mu^*$ and $\mathcal{R}(1_\theta : \frac{\mu}{\nu}) = \xi$. Now, by the observation just before the statement of the proposition, we see that ξ is the only isolated prime divisor of $(1_\theta : \frac{\mu}{\nu})$.

Conversely, let $\nu \subset \mu$ be such that ξ is an isolated prime divisor of $(1_\theta : \frac{\mu}{\nu})$. By Proposition 2.8, $\frac{\mu}{\nu}$ is fuzzy representable, and by Proposition 2.7, $\zeta = (1_\theta : \frac{\mu}{\nu})$ is a fuzzy decomposable ideal. So

$$\xi \in \text{ass}(1_\theta : \frac{\mu}{\nu}) \subseteq \text{Att}(\frac{\mu}{\nu}) \subseteq \text{Att}(\mu),$$

by Proposition 2.7 and Theorem 2.8. ■

When our ring R is Noetherian, we can improve Proposition 2.9 as follows.

THEOREM 2.10. *Suppose that R is a Noetherian ring and that μ is a fuzzy representable submodule of M . Let ξ be a fuzzy prime ideal of R . Then $\xi \in \text{Att}(\mu)$ if and only if $\exists \lambda \in FS(M)$ such that $\lambda \subset \mu$ and that $\xi = (1_\theta : \frac{\mu}{\lambda})$.*

Proof. The direction (\Leftarrow) follows by Theorem 2.5 (3) \Rightarrow (1).

To prove (\Rightarrow) , let $\xi \in \text{Att}(\mu)$. By Theorem 2.5, there exists $\nu \subset \mu$ such that $\frac{\mu}{\nu}$ is ξ -fuzzy coprimary and so $\mathcal{R}(1_\theta : \frac{\mu}{\nu}) = \xi$. Since R is Noetherian there exists $m \in \mathbb{N}$ such that $\xi^m \subseteq (1_\theta : \frac{\mu}{\nu})$ (see [9, Lemma 5.1]). This, by [12, Remark 2.8(B)], gives that $\xi^m \cdot \mu \subseteq 1_{\nu^*}$ so that $\xi^{*m} \cdot \mu^* \subseteq \nu^*$. We claim that $(1_\theta : \frac{\mu}{\xi \cdot \mu + \nu}) = \xi$. To do this, first we note that $\mu^* \neq \xi^* \mu^* + \nu^*$. Otherwise, we must have

$$\mu^* = \xi^* \mu^* + \nu^* = \xi^*(\xi^* \mu^* + \nu^*) + \nu^* = \dots = \xi^{*m} \mu^* + \nu^* \subseteq \nu^*,$$

which is not the case. So $\frac{\mu}{\xi \cdot \mu + \nu} \neq 1_\theta$ (note that $(\frac{\mu}{\xi \cdot \mu + \nu})^* = \frac{\mu^*}{\xi^* \mu^* + \nu^*}$). Thus using [12, Theorem 3.7], $\frac{\mu}{\xi \cdot \mu + \nu}$ is ξ -fuzzy coprimary and so $\mathcal{R}(1_\theta : \frac{\mu}{\xi \cdot \mu + \nu}) = \xi$. On the other hand, since $\xi \cdot \mu \subseteq \xi \cdot \mu + \nu$, using [12, Remark 2.8(A)], we see that $\xi \cdot \frac{\mu}{\xi \cdot \mu + \nu} \subseteq 1_\theta$. Thus $\xi \subseteq (1_\theta : \frac{\mu}{\xi \cdot \mu + \nu})$ and we must have $\xi = (1_\theta : \frac{\mu}{\xi \cdot \mu + \nu})$ as desired. ■

THEOREM 2.11. *Let μ be a fuzzy representable submodule of M and let $\nu \in FS(M)$ such that $\nu \subset \mu$. Then*

$$\text{Att}(\frac{\mu}{\nu}) \subseteq \text{Att}(\mu) \subseteq \text{Att}(\nu) \cup \text{Att}(\frac{\mu}{\nu}).$$

Proof. The first inclusion was proved in Theorem 2.8.

For the second one. let $\xi \in \text{Att}(\mu)$. By Theorem 2.5, there exists a fuzzy submodule λ of M such that $\lambda \subset \mu$ and $\frac{\mu}{\lambda}$ is ξ -fuzzy coprimary. If $\mu = \lambda + \nu$, then

$$\frac{\nu}{\lambda \cap \nu} \approx \frac{\lambda + \nu}{\lambda} = \frac{\mu}{\lambda},$$

and so there exists $\eta \in FS(\frac{\nu^*}{(\lambda \cap \nu)^*})$ such that $\frac{\nu/(\lambda \cap \nu)}{\eta} \cong \frac{\mu}{\nu}|_{(\mu/\nu)^*}$. Thus, using Remark 2.1(C) we have

$$\frac{\nu}{\pi^{-1}(\eta)} \cong \frac{\nu/(\lambda \cap \nu)}{\pi^{-1}(\eta)/(\lambda \cap \nu)} \cong (\frac{\mu}{\nu})|_{(\frac{\mu}{\nu})^*} = \frac{\mu}{\nu}.$$

Now, $\frac{\mu}{\nu}$ is ξ -fuzzy coprimary which gives that $\frac{\nu}{\pi^{-1}(\eta)}$ is ξ -fuzzy coprimary. Therefore by Theorem 2.5 we must have $\xi \in \text{Att}(\nu)$.

If $\lambda + \nu \subset \mu$, then, $\frac{\mu}{\lambda + \nu} \cong \frac{(\mu/\lambda)}{(\lambda + \nu)/\lambda}$ (see [8, p. 137]) and so $\frac{\mu}{\lambda + \nu}$ as a quotient of $\frac{\mu}{\lambda}$ is ξ -fuzzy coprimary. Since $\frac{\mu}{\lambda + \nu} \cong \frac{(\mu/\nu)}{(\lambda + \nu)/\nu}$, using Theorem 2.5, we must have $\xi \in \text{Att}(\frac{\mu}{\nu})$. ■

PROPOSITION 2.12. *Let μ be a fuzzy representable submodule of M and let $r \in R$. Then*

- (1) $1_r \cdot \mu = \mu$ if and only if $1_r \notin \cup_{i=1}^n \xi_i$.
- (2) $1_{r^n} \cdot \mu \subseteq 1_\theta$ if and only if $1_r \in \cap_{i=1}^n \xi_i$.

Proof. (1) Let $1_r \notin \cup_{i=1}^n \xi_i$. Then $1_r \notin \xi_i, \forall 1 \leq i \leq n$, and since each μ_i is ξ_i -fuzzy coprimary, this gives that $1_r \cdot \mu_i = \mu_i$ for $i = 1, \dots, n$. So, $1_r \cdot \mu = \mu$.

Conversely, if $1_r \in \cup_{i=1}^n \xi_i$, then $1_r \in \xi_i$ for some i . This gives that $1_{r^m} \cdot \mu_i \subseteq 1_\theta$ for some $m \in \mathbb{N}$. Thus

$$1_{r^m} \cdot \mu = 1_{r^m} \cdot \left(\sum_i^n \mu_i \right) \subseteq \sum_{j \neq i} \mu_j \subset \mu,$$

and hence $1_r \cdot \mu \neq \mu$.

- (2) can be proved by a similar argument using Proposition 2.2(2). ■

3. Local examinations

In this section, our aim is to examine the behavior of the fuzzy coprimary representation with multiplicatively closed subsets of R and fuzzy localizations.

Let S be a multiplicatively closed subset of R . Then the set

$$\mathfrak{D}(S) := \{x \in M \mid \exists s \in S \text{ such that } sx = \theta\},$$

is a submodule of M , so that we can consider $M/\mathfrak{D}(S)$ as a submodule of $S^{-1}M$. Let $\pi : M \rightarrow M/\mathfrak{D}(S) =: \bar{M}$ be the natural epimorphism $x \rightarrow \bar{x} = x + \mathfrak{D}(S)$ for all $x \in M$.

Let $I = [0, 1]$. For a fuzzy submodule μ of M we define the fuzzy subset $S^{-1}\mu$ of $S^{-1}M$ by

$$(S^{-1}\mu)(y/s) = \vee \{t \in I \mid \bar{y}/s \in S^{-1}(\pi(\mu)_t)\},$$

for $y \in M$ and $s \in S$, where $\pi(\mu)_t = \{x + \mathfrak{D}(S) \mid (\pi(\mu))(x + \mathfrak{D}(S)) \geq t\}$, and put

$$S(\mu) = \bigcap_{s \in S} (1_s \cdot \mu).$$

The following Proposition is, in a sense, the general case of [12, Theorem 3.7(2)].

PROPOSITION 3.1. *Let S be a multiplicatively closed subset of R . Suppose that μ is a fuzzy coprimary representable submodule of M with $\text{Att}(M) = \{\xi_1, \dots, \xi_n\}$ enumerated in such a way that $(\xi_i)_* \cap S = \emptyset$ for $1 \leq i \leq r$ and $(\xi_i)_* \cap S \neq \emptyset$ for $r + 1 \leq i \leq n$. Then*

$$S(\mu) = \mu_1 + \dots + \mu_r = \sum_{\xi \in \Xi} \lambda,$$

where Ξ is the set of all fuzzy prime ideals ξ with $\xi_* \cap S = \emptyset$, and $\lambda \subseteq \mu$ runs over all ξ -fuzzy coprimary submodules of M .

Proof. Let $s_i \in (\xi_i)_* \cap S$, for $i = r + 1, \dots, n$. Then $\exists l_i \in \mathbb{N}$ such that $1_{s_i^{l_i}} \cdot \mu_i \subseteq 1_\theta$, for $i = r + 1, \dots, n$. Hence $\exists l \in \mathbb{N}$ such that $1_{s_i^l} \cdot \mu_i \subseteq 1_\theta$. Put $u = \prod_{i=r+1}^n s_i^l \in S$. Then

$$\begin{aligned} S(\mu) &= \bigcap_{s \in S} (1_s \cdot \mu) \subseteq 1_u \cdot \mu \\ &= (1_u \cdot (\mu_1 + \dots + \mu_r)) \subseteq \mu_1 + \dots + \mu_r. \end{aligned}$$

Next, it is clear that

$$\mu_1 + \dots + \mu_r \subseteq \sum_{\xi \in \Xi} \lambda.$$

Now let ξ be a fuzzy prime ideal of R such that $\xi_* \cap S = \emptyset$ and that λ be ξ -fuzzy coprimary. Then, by [12, Theorem 3.7(2)], $\lambda = S(\lambda) \subseteq S(\mu)$ and the proof is complete. ■

THEOREM 3.2. *Let μ be a fuzzy representable submodule of M and let $\Psi = \{\xi_1, \dots, \xi_r\}$, ($0 \leq r \leq n$), be an enumerated isolated subset of $\text{Att}(\mu)$. Then the fuzzy submodule $\mu_1 + \dots + \mu_r$ is independent of the choice of minimal fuzzy coprimary representation of μ .*

Proof. By [12, Corollary 2.3], $\text{Im}(\xi_i) = \{0, 1\}$ for $i = 1, \dots, n$. Set $S = R \setminus \cup_{i=1}^r (\xi_i)_*$. Then $S \cap (\xi_i)_* = \emptyset$ for $i = 1, \dots, r$. Also by our assumption on Ψ , $S \cap (\xi_i)_* \neq \emptyset$ for $i = r + 1, \dots, n$. So, by Proposition 3.1, we have $\mu_1 + \dots + \mu_r = S(\mu)$ which is independent of the choice of minimal fuzzy coprimary representation of μ . ■

In the light of Theorems 2.5 and 3.1, we deduce

COROLLARY 3.3. *In any minimal fuzzy coprimary representation of μ , each coprimary term corresponding to a fuzzy isolated prime of μ is uniquely determined by μ and is independent of the choice of minimal fuzzy coprimary representation.*

THEOREM 3.4. *Let S be a multiplicatively closed subset of R and let μ be a fuzzy representable submodule of M with a minimal fuzzy coprimary representation as in (*). Then $S(\mu) = (1_s \cdot \mu)$ for some $s \in S$.*

Proof. Let $S \cap (\xi_i)_* = \emptyset$ for $i = 1, \dots, r$ and $S \cap (\xi_i)_* \neq \emptyset$ for $i = r + 1, \dots, n$. Then, by Theorem 3.2, $S(\mu) = \mu_1 + \dots + \mu_r$. For each $i = r + 1, \dots, n$, let $s_i \in S \cap (\xi_i)_*$. Then we can find $l \in \mathbb{N}$ such that $1_{s_i^l} \cdot \mu \subseteq 1_\theta$. Let $s = \prod_{i=r+1}^n s_i^l$. Then

$$\begin{aligned} (1_s \cdot \mu) &= 1_s \cdot (\mu_1 + \dots + \mu_n) \\ &= 1_s \cdot (\mu_1 + \dots + \mu_r) = S(\mu) \cup 1_\theta = S(\mu), \end{aligned}$$

as desired. ■

PROPOSITION 3.5. *Let S be a multiplicatively closed subset of R and let $\eta, \lambda \in FS(M)$. Then $S^{-1}(\eta + \lambda) = S^{-1}\eta + S^{-1}\lambda$ as fuzzy submodules of $S^{-1}R$ -module $S^{-1}M$.*

Proof. Clearly $S^{-1}\eta + S^{-1}\lambda \subseteq S^{-1}(\eta + \lambda)$. To prove the other inclusion, consider

$$S' = \frac{S}{1} = \{s/1 \mid s \in S\}$$

which is a multiplicatively closed subset of $S^{-1}R$, and

$$\mathfrak{D}(S)' = \{x/s \in S^{-1}M \mid \exists s' \in S', \text{ such that } s'x/s = \theta\},$$

a submodule of $S^{-1}M$. Let $\bar{\cdot}: S^{-1}M \rightarrow S^{-1}M/\mathfrak{D}(S)'$ be the natural epimorphism and $\phi \in FS(S^{-1}M)$. Then it is straightforward to see that for each $x \in M$ and $s, u \in S$,

$$S'^{-1}(\phi)\left(\frac{x/s}{u/1}\right) = \phi(x/1). \quad (\#)$$

Also, it is clear, for example, that $\eta(x) \leq (S^{-1}\eta)(x/1)$ for all $x \in M$. Thus, for $x \in M$,

$$\begin{aligned} S^{-1}(\eta+\lambda)(x/1) &= \bigvee \{t \in I \mid \exists s, u \in S, z \in M, us\bar{x} = u\bar{z}, \bigvee_{e \in \mathfrak{D}(S)} (\eta+\lambda)(z+e) \geq t\} \\ &\leq \bigvee \{t \in I \mid \exists s, u \in S, z \in M, us\bar{x} = u\bar{z}, \bigvee_{e \in \mathfrak{D}(S)} (S^{-1}\eta + S^{-1}\lambda)\left(\frac{z+e}{1}\right) \geq t\} \\ &= \bigvee \{t \in I \mid \exists s', u' \in S', z \in M, u's'\overline{x/1} = u'\overline{z/1}, \bigvee_{e \in \mathfrak{D}(S)'} (S^{-1}\eta + S^{-1}\lambda)(z/1+e/1) \geq t\} \\ &\leq S'^{-1}(S^{-1}\eta + S^{-1}\lambda)\left(\frac{x/1}{1/1}\right) = (S^{-1}\eta + S^{-1}\lambda)(x/1), \end{aligned}$$

where in the last equality we use (#). Hence $S^{-1}(\eta + \lambda) \subseteq S^{-1}\eta + S^{-1}\lambda$. ■

THEOREM 3.6. *Let S be a multiplicatively closed subset of R . Assume that μ is a fuzzy representable submodule of M as in (*) such that each $\mu_i, i = 1, \dots, n$ has the sup property. Then $S^{-1}\mu$ is a fuzzy representable submodule of $S^{-1}M$, with $\text{Att}(S^{-1}M) \subseteq \{S^{-1}\xi_1, \dots, S^{-1}\xi_n\}$.*

Proof. This follows by Proposition 3.5 and [12, Theorem 3.7]. ■

We close the paper with the following theorem which is a kind of connection between the fuzzy attached primes of μ and the (crisp) attached primes of μ^* (cf. [2]).

THEOREM 3.7. *Let μ be a fuzzy coprimary representable submodule of M with the minimal fuzzy coprimary representation as in (*). Then $\mu^* = \mu_1^* + \dots + \mu_n^*$ is a minimal coprimary representation of μ^* with $\text{Att}(\mu^*) = \{\xi_1^*, \dots, \xi_n^*\}$.*

Proof. Using the fact that $\text{Im } \xi_i = \{0, 1\}$ for $i = 1, \dots, n$ and that μ is absorbing, the proof is rather straightforward. ■

The results of this paper raise various dual questions on the concept of fuzzy coprimary modules and fuzzy coprimary representation. However some dual statements which one would hope, are topics for the further research.

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(received 08.05.2014; in revised form 09.03.2015; available online 01.04.2015)

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