

## GENERALIZED CONE b-METRIC SPACES AND CONTRACTION PRINCIPLES

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**Abstract.** The concept of generalized cone b-metric space is introduced as a generalization of cone metric space, cone b-metric space and cone rectangular metric space. An analogue of Banach contraction principle and Kannan's fixed point theorem is proved in this space. Our result generalizes many known results in fixed point theory.

### 1. Introduction

Due to the wide applications of fixed point theorems in different fields, the study of existence and uniqueness of fixed points and common fixed points has become a subject of great interest. The main idea is to extend or generalize the famous Banach Contraction Principle in different directions. In the recent past many authors generalized the Banach contraction Principle by generalizing the concept of a metric space. Rectangular metric [7], b-metric space [3], cone metric space [11], cone rectangular metric space [2] and cone b-metric space [13] are some of the generalized metric spaces introduced by different authors in the recent past. Analogue of Banach contraction principle, Kannan contraction principle, Ćirić contraction principle etc. and many fixed point theorems for various generalized contractions were proved in these generalized spaces by different authors (see, e.g., [1,2,4–6,8–10,12,14,16,17,19,20,22,26]).

It can be seen that many of the generalized metric spaces are not necessarily Hausdorff (see [15,21,24,25]). Proper examples of non Hausdorff rectangular metric space and rectangular b-metric space can be found in [21,23,24]. Note that spaces with non Hausdorff topology play an important role in Tarskian approach to programming language semantics used in computer science.

In this paper we have introduced the concept of generalized cone b-metric space which is not necessarily Hausdorff, and which generalizes the concepts of metric space, rectangular metric space, b-metric space, cone metric space, cone rectangular metric space and cone b-metric space. We have also proved an analogue of the

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Banach Contraction Principle as well as a fixed point theorem for a generalized contraction from which analogue of the Kannan contraction principle is deduced as a special case.

## 2. Preliminaries

Let  $E$  be a real Banach space and  $P$  a subset of  $E$ . Then  $P$  is called a *cone* if

- (i)  $P$  is closed, nonempty, and satisfies  $P \neq \{\theta\}$ ,
- (ii)  $ax + by \in P$  for all  $x, y \in P$  and non-negative real numbers  $a, b$ ,
- (iii)  $x \in P$  and  $-x \in P \implies x = \theta$ , i.e.,  $P \cap (-P) = \theta$ .

Given a cone  $P \subset E$ , we define a partial ordering  $\preceq$  with respect to  $P$  by  $x \preceq y$  if and only if  $y - x \in P$ . We shall write  $x < y$  if  $x \preceq y$  and  $x \neq y$ , and  $x \ll y$  if  $y - x \in \text{int}P$ , where  $\text{int}P$  denote the interior of  $P$ . A cone  $P$  is a solid cone if  $\text{int}P \neq \phi$ .

DEFINITION 2.1. [11] Let  $X$  be a nonempty set. Suppose that the mapping  $d : X \times X \rightarrow E$  satisfies:

- (CM1)  $0 \preceq d(x, y)$  for all  $x, y \in X$  and  $d(x, y) = 0$  if and only if  $x = y$ ,
- (CM2)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ,
- (CM3)  $d(x, y) \preceq d(x, z) + d(z, y)$  for all  $x, y, z \in X$ .

Then  $d$  is called a *cone metric* on  $X$  and  $(X, d)$  is called a *cone metric space*.

DEFINITION 2.2. [13] Let  $X$  be a nonempty set and  $s \geq 1$  be a real number. Suppose that the mapping  $d : X \times X \rightarrow E$  satisfies:

- (CbM1)  $0 \preceq d(x, y)$  for all  $x, y \in X$  and  $d(x, y) = 0$  if and only if  $x = y$ ,
- (CbM2)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ,
- (CbM3)  $d(x, y) \preceq s[d(x, z) + d(z, y)]$  for all  $x, y, z \in X$ .

Then  $d$  is called a *cone b-metric* on  $X$  and  $(X, d)$  is called a *cone b-metric space* (in short *CbMS*).

DEFINITION 2.3. [2] Let  $X$  be a nonempty set. Suppose that the mapping  $d : X \times X \rightarrow E$  satisfies:

- (CRM1)  $0 \preceq d(x, y)$  for all  $x, y \in X$  and  $d(x, y) = 0$  if and only if  $x = y$ ,
- (CRM2)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ,
- (CRM3)  $d(x, y) \preceq d(x, u) + d(u, v) + d(v, y)$  for all  $x, y, \in X$  and for all distinct points  $u, v \in X \setminus \{x, y\}$  (rectangular property).

Then  $d$  is called a *cone rectangular metric* on  $X$  and  $(X, d)$  is called a *cone rectangular metric space* (in short *CRMS*).

Note that any cone metric space is *CbMS* and *CRMS* but the converse is not true in general (see [2,3]).

DEFINITION 2.4. Let  $X$  be a nonempty set. Suppose that the mapping  $d : X \times X \rightarrow E$  satisfies:

(GCbM1)  $0 \preceq d(x, y)$  for all  $x, y \in X$  and  $d(x, y) = 0$  if and only if  $x = y$ ,

(GCbM2)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ,

(GCbM3) there exists a real number  $s \geq 1$  such that  $d(x, y) \preceq s[d(x, u) + d(u, v) + d(v, y)]$  for all  $x, y, \in X$  and for all distinct points  $u, v \in X \setminus \{x, y\}$ .

Then  $d$  is called a *generalized cone b-metric* on  $X$  and  $(X, d)$  is called a *generalized cone b-metric space* (in short *GCbMS*) with coefficient  $s$ .

Note that every *CbMS* with coefficient  $s$  is a *GCbMS* with coefficient  $s^2$  and every *CRMS* is *GCbMS* but the converses are not true in general. As in [21,23,24] we furnish the following examples in support of our claim.

EXAMPLE 2.5. Let  $E = \mathbb{R}^2$ ,  $P = \{(x, y) \in E \mid x, y \geq 0\}$ ,  $X = A \cup B$ , where  $A = \{\frac{1}{n} : n \in \mathbb{N}\}$  and  $B$  is the set of all positive integers. Define  $d: X \times X \rightarrow E$  such that  $d(x, y) = d(y, x)$  for all  $x, y \in X$  and

$$d(x, y) = \begin{cases} (0, 0) & \text{if } x = y; \\ (2, 2) & \text{if } x, y \in A; \\ (\frac{1}{2n}, \frac{1}{2n}) & \text{if } x = \frac{1}{n} \in A \text{ and } y \in \{2, 3\}; \\ (1, 1) & \text{otherwise.} \end{cases}$$

Then  $(X, d)$  is a generalized cone b-metric space with coefficient  $s = 2 > 1$ . However there does not exist  $s > 0$  satisfying  $d(x, y) \leq s[d(x, z) + d(z, y)]$  for all  $x, y, z \in X$ , and so  $(X, d)$  is not a cone b-metric space.

EXAMPLE 2.6 Let  $E = \mathbb{R}^2$ ,  $P = \{(x, y) \in E \mid x, y \geq 0\}$ ,  $X = \mathbb{N}$ ,  $d: X \times X \rightarrow E$  such that

$$d(x, y) = \begin{cases} (0, 0), & \text{for all } x, y \in X \text{ and } x = y; \\ d(y, x), & \text{for all } x, y \in X; \\ (10, 10), & \text{if } x = 1 \text{ and } y = 2; \\ (1, 1), & \text{if } x \in \{1, 2\} \text{ and } y = 3; \\ (2, 2), & \text{if } x \in \{1, 2, 3\} \text{ and } y = 4; \\ (3, 3), & \text{if } x \text{ or } y \notin \{1, 2, 3, 4\} \text{ and } x \neq y. \end{cases}$$

Then  $(X, d)$  is a *GCbMS* but it is not a *CRMS* as we have  $d(1, 2) = (10, 10) \succ d(1, 3) + d(3, 4) + d(4, 2) = (5, 5)$ .

For any  $x \in X$  we define the open ball with center  $x$  and radius  $r > 0$  by

$$B_r(x) = \{y \in X : d(x, y) < r\}$$

The open balls in *GCbMS* are not necessarily open (see Remark 2.8 below). Let  $\mathcal{U}$  be the collection of all subsets  $A$  of  $X$  satisfying the condition that for each  $x \in A$  there exist  $r > 0$  such that  $B_r(x) \subseteq A$ . Then  $\mathcal{U}$  defines a topology for the *GCbMS*  $(X, d)$  which is not necessarily Hausdorff (see Remark 2.8 below).

Now we define convergent and Cauchy sequences in *GCbMS* and completeness of *GCbMS*.

DEFINITION 2.7. [3] Let  $(X, d)$  be a *GCbMS*. The sequence  $\{x_n\}$  in  $X$  is said to be:

(a) a *convergent sequence* if for every  $c \in E$  with  $0 \ll c$ , there is  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$ ,  $d(x_n, x) \ll c$  for some  $x \in X$ . We say that the sequence  $\{x_n\}$  converges to  $x$  and we denote this by  $\lim_{n \rightarrow \infty} x_n = x$ ;

(b) a *Cauchy sequence* if for all  $c \in E$  with  $0 \ll c$ , there is  $n_0 \in \mathbb{N}$  such that  $d(x_m, x_n) \ll c$ , for all  $m, n \geq n_0$ .

(c) The *GCBMS*  $(X, d)$  is said to be *complete* if every Cauchy sequence in  $X$  is convergent in  $X$ .

REMARK 2.8. In Example 2.5 above we note the following:

(i)  $B_{\frac{1}{2}}(\frac{1}{2}) = \{2, 3, \frac{1}{2}\}$  and there does not exist any open ball with center 2 and contained in  $B_{\frac{1}{2}}(\frac{1}{2})$ . So  $B_{\frac{1}{2}}(\frac{1}{2})$  is not an open set.

(ii) The sequence  $\{\frac{1}{n}\}$  converges to 2 and 3 in *GCBMS* and so the limit is not unique. Also,  $d(\frac{1}{n}, \frac{1}{n+p}) = (2, 2) \not\rightarrow (0, 0)$  as  $n \rightarrow \infty$ ; therefore  $\{\frac{1}{n}\}$  is not a Cauchy sequence in *GCBMS*. Thus in a *GCBMS* not every convergent sequence is necessarily a Cauchy sequence.

(iii) There does not exist  $r_1, r_2 > 0$  such that  $B_{r_1}(2) \cap B_{r_2}(3) = \emptyset$  and so  $(X, d)$  is not Hausdorff.

### 3. Main results

For  $x_1, x_2, \dots, x_k \in E$ , we define

$$\min\{x_1, x_2, \dots, x_k\} = \begin{cases} x_r & (1 \leq r \leq k), \text{ if } x_r \preceq x_i \text{ for each } i = 1, 2, \dots, k; \\ 0 \in E, & \text{otherwise.} \end{cases}$$

THEOREM 3.1. Let  $(X, d)$  be a complete generalized cone b-metric space with coefficient  $s > 1$ ,  $P$  be a solid cone and  $T: X \rightarrow X$  be a mapping satisfying

$$d(Tx, Ty) \preceq \lambda d(x, y) \tag{3.1}$$

for all  $x, y \in X$ , where  $\lambda \in [0, \frac{1}{s})$ . Then  $T$  has a unique fixed point.

*Proof.* Let  $x_0 \in X$  be arbitrary. We define a sequence  $\{x_n\}$  by  $x_{n+1} = Tx_n$  for all  $n \geq 0$ . We shall show that  $\{x_n\}$  is a Cauchy sequence. If  $x_n = x_{n+1}$  then  $x_n$  is a fixed point of  $T$ . So, suppose that  $x_n \neq x_{n+1}$  for all  $n \geq 0$ . Setting  $d(x_n, x_{n+1}) = d_n$ , it follows from (3.1) that

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \preceq \lambda d(x_{n-1}, x_n) \\ d_n &\preceq \lambda d_{n-1}. \end{aligned}$$

Repeating this process, we obtain

$$d_n \preceq \lambda^n d_0. \tag{3.2}$$

Also, we can assume that  $x_0$  is not a periodic point of  $T$ . Indeed, if  $x_0 = x_n$  then using (3.2), for any  $n \geq 2$ , we have

$$d_0 = d(x_0, x_1) = d(x_0, Tx_0) = d(x_n, Tx_n) = d(x_n, x_{n+1}) = d_n \preceq \lambda^n d_0.$$

Since  $\lambda \in [0, 1)$  we obtain  $-d_0 \in P$ . Therefore, we must have  $d_0 = 0$ , i.e.,  $x_0 = x_1$ , and so  $x_0$  is a fixed point of  $T$ . Thus, we assume that  $x_n \neq x_m$  for all distinct  $n, m \in \mathbb{N}$ . Again setting  $d(x_n, x_{n+2}) = d_n^*$  and using (3.1) for any  $n \in \mathbb{N}$ , we obtain

$$d_n^* = d(x_n, x_{n+2}) = d(Tx_{n-1}, Tx_{n+1}) \preceq \lambda d(x_{n-1}, x_{n+1}) \preceq \lambda d_{n-1}^*.$$

Repeating this process we obtain

$$d(x_n, x_{n+2}) \preceq \lambda^n d_0^*. \quad (3.3)$$

For the obtained sequence  $\{x_n\}$ , we consider two possible cases for  $d(x_n, x_{n+p})$ .

If  $p$  is odd, say  $p = 2m + 1$ , then using (3.2) we obtain

$$\begin{aligned} d(x_n, x_{n+2m+1}) &\preceq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m+1})] \\ &\preceq s[d_n + d_{n+1}] + s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) \\ &\quad + d(x_{n+4}, x_{n+2m+1})] \\ &\preceq s[d_n + d_{n+1}] + s^2[d_{n+2} + d_{n+3}] + s^3[d_{n+4} + d_{n+5}] \\ &\quad + \cdots + s^m d_{n+2m} \\ &\preceq s[\lambda^n d_0 + \lambda^{n+1} d_0] + s^2[\lambda^{n+2} d_0 + \lambda^{n+3} d_0] + s^3[\lambda^{n+4} d_0 + \lambda^{n+5} d_0] \\ &\quad + \cdots + s^m \lambda^{n+2m} d_0 \\ &\preceq s\lambda^n [1 + s\lambda^2 + s^2\lambda^4 + \cdots] d_0 + s\lambda^{n+1} [1 + s\lambda^2 + s^2\lambda^4 + \cdots] d_0 \\ &= \frac{1 + \lambda}{1 - s\lambda^2} s\lambda^n d_0 \quad (\text{note that } s\lambda^2 < 1). \end{aligned}$$

Therefore,

$$d(x_n, x_{n+2m+1}) \preceq \frac{1 + \lambda}{1 - s\lambda^2} s\lambda^n d_0.$$

Let  $0 \ll c$  be given. Choose  $\delta > 0$  such that  $c + N_\delta(0) \subseteq P$ , where  $N_\delta(0) = \{y \in E : \|y\| < \delta\}$ . Also choose a natural number  $N_1$  such that  $\frac{1+\lambda}{1-s\lambda^2} s\lambda^n d_0 \in N_\delta(0)$ , for all  $n \geq N_1$ . Then  $\frac{1+\lambda}{1-s\lambda^2} s\lambda^n d_0 \ll c$  for all  $n \geq N_1$ . Thus  $d(x_n, x_{n+2m+1}) \preceq \frac{1+\lambda}{1-s\lambda^2} s\lambda^n d_0 \ll c$  for all  $n \geq N_1$ .

If  $p$  is even, say  $p = 2m$ , then using (3.2) and (3.3) we obtain

$$\begin{aligned} d(x_n, x_{n+2m}) &\preceq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m})] \\ &\preceq s[d_n + d_{n+1}] + s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) \\ &\quad + d(x_{n+4}, x_{n+2m})] \\ &\preceq s[d_n + d_{n+1}] + s^2[d_{n+2} + d_{n+3}] + s^3[d_{n+4} + d_{n+5}] \\ &\quad + \cdots + s^{m-1}[d_{2m-4} + d_{2m-3}] + s^{m-1}d(x_{n+2m-2}, x_{n+2m}) \\ &\preceq s[\lambda^n d_0 + \lambda^{n+1} d_0] + s^2[\lambda^{n+2} d_0 + \lambda^{n+3} d_0] + s^3[\lambda^{n+4} d_0 + \lambda^{n+5} d_0] \\ &\quad + \cdots + s^{m-1}[\lambda^{2m-4} d_0 + \lambda^{2m-3} d_0] + s^{m-1} \lambda^{n+2m-2} d_0^* \\ &\preceq s\lambda^n [1 + s\lambda^2 + s^2\lambda^4 + \cdots] d_0 + s\lambda^{n+1} [1 + s\lambda^2 + s^2\lambda^4 + \cdots] d_0 \\ &\quad + s^{m-1} \lambda^{n+2m-2} d_0^*, \end{aligned}$$

i.e.

$$\begin{aligned} d(x_n, x_{n+2m}) &\preceq \frac{1+\lambda}{1-s\lambda^2} s\lambda^n d_0 + s^{m-1} \lambda^{n+2m-2} d_0^* \\ &< \frac{1+\lambda}{1-s\lambda^2} s\lambda^n d_0 + (s\lambda)^{2m} \lambda^{n-2} d_0^* \quad (\text{as } 1 < s) \\ &\preceq \frac{1+\lambda}{1-s\lambda^2} s\lambda^n d_0 + \lambda^{n-2} d_0^* \quad (\text{as } \lambda \leq \frac{1}{s}). \end{aligned}$$

Therefore,

$$d(x_n, x_{n+2m}) \preceq \frac{1+\lambda}{1-s\lambda^2} s\lambda^n d_0 + \lambda^{n-2} d_0^*$$

Now choose a natural number  $N_2$  such that  $\frac{1+\lambda}{1-s\lambda^2} s\lambda^n d_0 + \lambda^{n-2} d_0^* \in N_\delta(0)$ , for all  $n \geq N_2$ . Then  $\frac{1+\lambda}{1-s\lambda^2} s\lambda^n d_0 + \lambda^{n-2} d_0^* \ll c$  for all  $n \geq N_2$ . Thus  $d(x_n, x_{n+2m}) \preceq \frac{1+\lambda}{1-s\lambda^2} s\lambda^n d_0 + \lambda^{n-2} d_0^* \ll c$  for all  $n \geq N_2$ . Let  $N_0 = \max\{N_1, N_2\}$ . Then for all  $n \geq N_0$  we have

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+p}) \ll c$$

Thus  $\{x_n\}$  is a Cauchy sequence in  $X$ . By the completeness of  $(X, d)$  there exists  $u \in X$  such that

$$\lim_{n \rightarrow \infty} x_n = u.$$

We shall show that  $u$  is a fixed point of  $T$ .

For any  $n \in \mathbb{N}$  we have

$$\begin{aligned} d(u, Tu) &\preceq s[d(u, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tu)] \\ &= s[d(u, x_n) + d_n + d(Tx_n, Tu)] \\ &\preceq s[d(u, x_n) + d_n + \lambda d(x_n, u)] \\ &\preceq s[(1+\lambda)d(x_n, u) + \lambda^n d_0]. \end{aligned}$$

Now choose  $N_3, N_4$  such that  $d(x_n, u) \ll \frac{c}{2s(1+\lambda)}$  for all  $n \geq N_3$  and  $\lambda^n d_0 \ll \frac{c}{2s}$  for all  $n \geq N_4$  and let  $N_0 = \max\{N_3, N_4\}$ . Then for all  $n \geq N_0$ ,  $d(u, Tu) \ll c$ . It follows that  $d(u, Tu) = 0$ , i.e.,  $Tu = u$ . Thus  $u$  is a fixed point of  $T$ .

For uniqueness, let  $v$  be another fixed point of  $T$ . Then it follows from (3.1) that  $d(u, v) = d(Tu, Tv) \preceq \lambda d(u, v) < d(u, v)$ , a contradiction. Therefore, we must have  $d(u, v) = 0$ , i.e.,  $u = v$ . Thus the fixed point is unique. ■

**THEOREM 3.2.** *Let  $(X, d)$  be a complete generalized cone b-metric space with coefficient  $s > 1$ ,  $P$  be a solid cone and  $T: X \rightarrow X$  be a mapping satisfying*

$$d(Tx, Ty) \preceq \lambda[d(x, Tx) + d(y, Ty)] + L \cdot \alpha(x, y) \tag{3.4}$$

for all  $x, y \in X$ , where  $\lambda \in [0, \frac{1}{s+1})$ ,  $L \geq 0$  and  $\alpha(x, y) = \min\{d(x, Tx) + d(y, Ty), d(x, Ty), d(y, Tx)\}$ . Then  $T$  has a unique fixed point.

*Proof.* Let  $x_0 \in X$  be arbitrary. We define a sequence  $\{x_n\}$  by  $x_{n+1} = Tx_n$  for all  $n \geq 0$ . We shall show that  $\{x_n\}$  is Cauchy sequence. If  $x_n = x_{n+1}$  then

$x_n$  is a fixed point of  $T$ . So, suppose that  $x_n \neq x_{n+1}$  for all  $n \geq 0$ . Setting  $d(x_n, x_{n+1}) = d_n$ , it follows from (3.4) that

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \preceq \lambda[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] \\ &\quad + L \cdot \min\{d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\} \\ &\preceq \lambda[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\ &\quad + L \cdot \min\{d(x_{n-1}, x_n) + d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), d(x_n, x_n)\} \\ d_n &\preceq \lambda[d_{n-1} + d_n] \\ d_n &\preceq \frac{\lambda}{1-\lambda} d_{n-1} = \beta d_{n-1}, \end{aligned}$$

where  $\beta = \frac{\lambda}{1-\lambda} < \frac{1}{s}$  (as  $\lambda < \frac{1}{s+1}$ ). Repeating this process we obtain

$$d_n \preceq \beta^n d_0. \quad (3.5)$$

Also, we can assume that  $x_0$  is not a periodic point of  $T$ . Indeed, if  $x_0 = x_n$  then using (3.5), for any  $n \geq 2$ , we have

$$d_0 = d(x_0, x_1) = d(x_0, Tx_0) = d(x_n, Tx_n) = d(x_n, x_{n+1}) = d_n \preceq \beta^n d_0.$$

Therefore, we must have  $-d_0 \in P$ , i.e.,  $x_0 = x_1$ , and so  $x_0$  is a fixed point of  $T$ . Thus we assume that  $x_n \neq x_m$  for all distinct  $n, m \in \mathbb{N}$ . Again using (3.4) and (3.5) for any  $n \in \mathbb{N}$ , we obtain

$$\begin{aligned} d(x_n, x_{n+2}) &= d(Tx_{n-1}, Tx_{n+1}) \preceq \lambda[d(x_{n-1}, Tx_{n-1}) + d(x_{n+1}, Tx_{n+1})] \\ &\quad + L \cdot \min\{d(x_{n-1}, Tx_{n-1}) + d(x_{n+1}, Tx_{n+1}), d(x_{n-1}, Tx_{n+1}), \\ &\quad d(x_{n+1}, Tx_{n-1})\} \\ &= \lambda[d(x_{n-1}, x_n) + d(x_{n+1}, x_{n+2})] \\ &\quad + L \cdot \min\{d(x_{n-1}, x_n) + d(x_{n+1}, x_{n+2}), d(x_{n-1}, x_{n+2}), d(x_{n+1}, x_n)\} \\ &\preceq \lambda[d_{n-1} + d_{n+1}] + L \cdot \min\{d_{n-1} + d_{n+1}, d_n\}. \end{aligned}$$

The following three cases arise:

Case 1.  $\min\{d_{n-1} + d_{n+1}, d_n\} = d_{n-1} + d_{n+1}$ . Then we have

$$\begin{aligned} d(x_n, x_{n+2}) &\preceq \lambda[\beta^{n-1} d_0 + \beta^{n+1} d_0] + L[\beta^{n-1} d_0 + \beta^{n+1} d_0] \\ &\preceq \lambda\beta^{n-1}[1 + \beta^2]d_0 + L \cdot \beta^{n-1}[1 + \beta^2]d_0 \\ &= \gamma\beta^{n-1}d_0, \end{aligned}$$

where  $\gamma = [\lambda + L][1 + \beta^2] > 0$ .

Case 2.  $\min\{d_{n-1} + d_{n+1}, d_n\} = d_n$ . Then we have

$$\begin{aligned} d(x_n, x_{n+2}) &\preceq \lambda[\beta^{n-1} d_0 + \beta^{n+1} d_0] + L[\beta^n d_0] \\ &\preceq \lambda\beta^{n-1}[1 + \beta^2]d_0 + L \cdot \beta^{n-1}\beta d_0 \preceq \gamma\beta^{n-1}d_0. \end{aligned}$$

Case 3.  $\min\{d_{n-1} + d_{n+1}, d_n\} = 0$ . Then we have

$$d(x_n, x_{n+2}) \preceq \lambda[\beta^{n-1} d_0 + \beta^{n+1} d_0] \preceq \lambda\beta^{n-1}[1 + \beta^2]d_0 \preceq \gamma\beta^{n-1}d_0.$$

Therefore, in all cases we have

$$d(x_n, x_{n+2}) \preceq \gamma \beta^{n-1} d_0. \quad (3.6)$$

For the obtained sequence  $\{x_n\}$ , we consider two possible cases for  $d(x_n, x_{n+p})$ .

If  $p$  is odd, say  $p = 2m + 1$ , then using (3.5) we obtain

$$\begin{aligned} d(x_n, x_{n+2m+1}) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m+1})] \\ &\leq s[d_n + d_{n+1}] + s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) \\ &\quad + d(x_{n+4}, x_{n+2m+1})] \\ &\preceq s[d_n + d_{n+1}] + s^2[d_{n+2} + d_{n+3}] + s^3[d_{n+4} + d_{n+5}] \\ &\quad + \cdots + s^m d_{n+2m} \\ &\leq s[\beta^n d_0 + \beta^{n+1} d_0] + s^2[\beta^{n+2} d_0 + \beta^{n+3} d_0] + s^3[\beta^{n+4} d_0 + \beta^{n+5} d_0] \\ &\quad + \cdots + s^m \beta^{n+2m} d_0 \\ &\preceq s\beta^n [1 + s\beta^2 + s^2\beta^4 + \cdots] d_0 + s\beta^{n+1} [1 + s\beta^2 + s^2\beta^4 + \cdots] d_0 \\ &= \frac{1 + \beta}{1 - s\beta^2} s\beta^n d_0 \quad (\text{note that } s\beta^2 < 1). \end{aligned}$$

Therefore,

$$d(x_n, x_{n+2m+1}) \preceq \frac{1 + \beta}{1 - s\beta^2} s\beta^n d_0.$$

Let  $0 \ll c$  be given. Choose  $\delta > 0$  such that  $c + N_\delta(0) \subseteq P$  where  $N_\delta(0) = \{y \in E : \|y\| < \delta\}$ . Also choose a natural number  $N_1$  such that  $\frac{1+\beta}{1-s\beta^2} s\beta^n d_0 \in N_\delta(0)$ , for all  $n \geq N_1$ . Then  $\frac{1+\beta}{1-s\beta^2} s\beta^n d_0 \ll c$  for all  $n \geq N_1$ . Thus  $d(x_n, x_{n+2m+1}) \leq \frac{1+\beta}{1-s\beta^2} s\beta^n d_0 \ll c$  for all  $n \geq N_1$ .

If  $p$  is even, say  $p = 2m$ , then using (3.5) and (3.6) we obtain

$$\begin{aligned} d(x_n, x_{n+2m}) &\preceq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m})] \\ &\preceq s[d_n + d_{n+1}] + s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) \\ &\quad + d(x_{n+4}, x_{n+2m})] \\ &\preceq s[d_n + d_{n+1}] + s^2[d_{n+2} + d_{n+3}] + s^3[d_{n+4} + d_{n+5}] \\ &\quad + \cdots + s^{m-1}[d_{2m-4} + d_{2m-3}] + s^{m-1}d(x_{n+2m-2}, x_{n+2m}) \\ &\preceq s[\beta^n d_0 + \beta^{n+1} d_0] + s^2[\beta^{n+2} d_0 + \beta^{n+3} d_0] + s^3[\beta^{n+4} d_0 + \beta^{n+5} d_0] \\ &\quad + \cdots + s^{m-1}[\beta^{2m-4} d_0 + \beta^{2m-3} d_0] + s^{m-1}\gamma\beta^{n+2m-3} d_0 \\ &\preceq s\beta^n [1 + s\beta^2 + s^2\beta^4 + \cdots] d_0 + s\beta^{n+1} [1 + s\beta^2 + s^2\beta^4 + \cdots] d_0 \\ &\quad + s^{m-1}\gamma\beta^{n+2m-3} d_0, \end{aligned}$$

i.e.

$$\begin{aligned} d(x_n, x_{n+2m}) &\preceq \frac{1+\beta}{1-s\beta^2} s\beta^n d_0 + s^{m-1} \gamma \beta^{n+2m-3} d_0 \\ &< \frac{1+\beta}{1-s\beta^2} s\beta^n d_0 + \gamma (s\beta)^{2m} \beta^{n-3} d_0 \quad (\text{as } 1 < s) \\ &\preceq \frac{1+\beta}{1-s\beta^2} s\beta^n d_0 + \gamma \beta^{n-3} d_0 \quad (\text{as } \beta \leq \frac{1}{s}). \end{aligned}$$

Therefore,

$$d(x_n, x_{n+2m}) \preceq \frac{1+\beta}{1-s\beta^2} s\beta^n d_0 + \gamma \beta^{n-3} d_0.$$

Now choose a natural number  $N_2$  such that  $\frac{1+\beta}{1-s\beta^2} s\beta^n d_0 + \gamma \beta^{n-3} d_0 \in N_\delta(0)$ , for all  $n \geq N_2$ . Then  $\frac{1+\beta}{1-s\beta^2} s\beta^n d_0 + \gamma \beta^{n-3} d_0 \ll c$  for all  $n \geq N_2$ . Thus  $d(x_n, x_{n+2m}) \leq \frac{1+\beta}{1-s\beta^2} s\beta^n d_0 + \gamma \beta^{n-3} d_0 \ll c$  for all  $n \geq N_2$ . Let  $N_0 = \max\{N_1, N_2\}$ . Then for all  $n \geq N_0$  we have

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+p}) \ll c.$$

Thus  $\{x_n\}$  is a Cauchy sequence in  $X$ . By completeness of  $(X, d)$  there exists  $u \in X$  such that  $\lim_{n \rightarrow \infty} x_n = u$ .

We shall show that  $u$  is a fixed point of  $T$ . Again, for any  $n \in \mathbb{N}$  we have

$$\begin{aligned} d(u, Tu) &\preceq s[d(u, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tu)] \\ &= s[d(u, x_n) + d_n + d(Tx_n, Tu)] \\ &\preceq s[d(u, x_n) + d_n + \lambda\{d(x_n, Tx_n) + d(u, Tu)\} + L \cdot \alpha(x_n, u)] \\ &= s[d(u, x_n) + d_n + \lambda\{d(x_n, x_{n+1}) + d(u, Tu)\} + L \cdot \alpha(x_n, u)], \end{aligned}$$

$$(1-s\lambda)d(u, Tu) \preceq s[d(u, x_n) + (1+\lambda)\beta^n d_0 + L \cdot \alpha(x_n, u)],$$

where

$$\begin{aligned} \alpha(x_n, u) &= \text{Min}\{d(x_n, Tx_n) + d(u, Tu), d(x_n, Tu), d(u, Tx_n)\} \\ &\preceq \min\{\lambda^n d_0 + d(u, Tu), d(x_n, Tu), d(x_{n+1}, u)\} \\ &\rightarrow 0, \text{ as } n \rightarrow \infty \end{aligned}$$

Choose a natural number  $N_3$  such that  $\frac{s}{1-\lambda s}[d(u, x_n) + (1+\lambda)\beta^n d_0 + L \cdot \alpha(x_n, u)] \in N_\delta(0)$ , for all  $n \geq N_3$ . Then  $\frac{1+\beta}{1-s\beta^2} s\beta^n d_0 \ll c$  for all  $n \geq N_3$ . It follows that  $d(u, Tu) \ll c$ , for all  $n \geq N_3$ , i.e.  $Tu = u$ . Thus  $u$  is a fixed point of  $T$ .

For uniqueness, let  $v$  be another fixed point of  $T$ . Then it follows from (3.4) that  $d(u, v) = d(Tu, Tv) \leq \lambda[d(u, Tu) + d(v, Tv)] = \lambda[d(u, u) + d(v, v)] = 0$ . Therefore, we have  $d(u, v) = 0$ , i.e.,  $u = v$ . Thus the fixed point is unique. ■

Taking  $L = 0$  in Theorem 3.2, we get an analogue of Kannan contraction principle in a generalized cone b-metric space as follows.

**THEOREM 3.3.** *Let  $(X, d)$  be a complete generalized cone b-metric space with coefficient  $s > 1$ ,  $P$  be a solid cone and  $T: X \rightarrow X$  be a mapping satisfying*

$$d(Tx, Ty) \preceq \lambda[d(x, Tx) + d(y, Ty)]$$

*for all  $x, y \in X$ , where  $\lambda \in [0, \frac{1}{s+1})$ . Then  $T$  has a unique fixed point.*

EXAMPLE 3.4. Let  $X = A \cup B$ , where  $A = [0, \frac{1}{2}]$ ,  $B = [1, 2]$  and let  $E = C_R(X)$ ,  $P = \{f \in E : f(t) \geq 0, t \in X\}$ . It is known that  $P$  is a nonnormal cone. (see [18]) Let  $\phi : X \rightarrow R$  such that  $\phi(t) = 1 + t$ . Define  $d : X \times X \rightarrow E$  such that for all  $x, y, t \in X$ ,  $d(x, y) = d(y, x)$  and

$$\left\{ \begin{array}{ll} d(0, \frac{1}{2}) = d(\frac{1}{3}, \frac{1}{4}) = d(\frac{1}{5}, \frac{1}{6}) = 0.6\phi(t) \\ d(0, \frac{1}{3}) = d(\frac{1}{2}, \frac{1}{5}) = d(\frac{1}{4}, \frac{1}{5}) = 0.2\phi(t) \\ d(0, \frac{1}{4}) = d(\frac{1}{2}, \frac{1}{3}) = d(\frac{1}{4}, \frac{1}{6}) = 0.4\phi(t) \\ d(0, \frac{1}{5}) = d(\frac{1}{2}, \frac{1}{6}) = d(\frac{1}{3}, \frac{1}{6}) = 0.5\phi(t) \\ d(0, \frac{1}{6}) = d(\frac{1}{2}, \frac{1}{4}) = d(\frac{1}{3}, \frac{1}{5}) = 0.3\phi(t) \\ d(x, y) = 0, & \text{if } x = y \\ d(x, y) = \phi(t), & \text{if } x, y \in A - \{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}\} \\ d(x, y) = \frac{1}{2n}\phi(t), & \text{if } x = \frac{1}{n} (n \geq 2) \in A \text{ and } y \in \{1, 2\}; \\ d(x, y) = |x - y|^2\phi(t), & \text{otherwise.} \end{array} \right.$$

Clearly  $d$  is a generalized cone b-metric with  $s = 2$ . However, there does not exist  $s > 1$  satisfying  $d(x, y) \leq s[d(x, z) + d(z, y)]$  for all  $x, y, z \in X$ , and so  $(X, d)$  is not a cone b-metric space. Also  $(X, d)$  is not a cone metric as  $d(\frac{1}{3}, \frac{1}{4}) = 0.6(1 + t) > d(\frac{1}{3}, \frac{1}{5}) + d(\frac{1}{5}, \frac{1}{4}) = 0.5(1 + t)$ . Define  $T$  by

$$Tx = \begin{cases} \frac{9}{20}, & x \in [1, 2] \cup \{\frac{1}{6}\} \\ \frac{1}{2} - x, & x \in C = \{\frac{1}{n} : n \geq 3, n \neq 6\} \\ \frac{1}{4}, & x \in A - \{C \cup \frac{1}{6}\} \end{cases}$$

Then  $T$  satisfies contraction (3.4). In fact, if  $\alpha(x, y) \neq 0$  then by choosing  $L$  sufficiently large,  $T$  will satisfy condition (3.4). We will discuss some cases when  $\alpha(x, y) = 0$ .

Case (i):  $x \in [1, 2] \cup \{\frac{1}{6}\}$ ,  $y \in C = \{\frac{1}{n} : n \geq 3, n \neq 6\}$ .

$$d(Tx, Ty) = d(\frac{9}{20}, \frac{1}{2} - y), d(x, Tx) = d(x, \frac{9}{20}), d(y, Ty) = d(y, \frac{1}{2} - y), d(x, Ty) = d(x, \frac{1}{2} - y), d(y, Tx) = d(y, \frac{9}{20}).$$

$\alpha(x, y) = 0$  if  $x + y = \frac{1}{2}$  or  $y = \frac{9}{20}$ . Since  $y \in C = \{\frac{1}{n} : n \geq 3, n \neq 6\}$ ,  $y = \frac{9}{20}$  is not possible.

$x + y = \frac{1}{2}$  only at  $x = \frac{1}{6}$  and  $y = \frac{1}{3}$ . Then  $d(Tx, Ty) = d(\frac{9}{20}, \frac{1}{6}) = 0.080278\phi(t)$ ,  $d(x, Tx) = d(\frac{1}{6}, \frac{9}{20}) = 0.080278\phi(t)$ ,  $d(y, Ty) = d(\frac{1}{3}, \frac{1}{6}) = 0.5\phi(t)$ . Then clearly we can find  $\lambda \in (0, \frac{1}{3})$  satisfying  $d(Tx, Ty) \leq \lambda[d(x, Tx) + d(y, Ty)]$ .

Case (ii):  $x \in [1, 2] \cup \{\frac{1}{6}\}$ ,  $y \in A - \{C \cup \frac{1}{6}\}$ .

$$d(Tx, Ty) = d(\frac{9}{20}, \frac{1}{4}), d(x, Tx) = d(x, \frac{9}{20}), d(y, Ty) = d(y, \frac{1}{4}), d(x, Ty) = d(x, \frac{1}{4}), d(y, Tx) = d(y, \frac{9}{20}).$$

$\alpha(x, y) = 0$  only at  $x = \frac{1}{4}$  or  $y = \frac{9}{20}$ .  $x = \frac{1}{4}$  is not possible.

Let  $y = \frac{9}{20}$  and  $x \in [1, 2] \cup \{\frac{1}{6}\}$ .  $d(Tx, Ty) = d(\frac{9}{20}, \frac{1}{4}) = 0.04\phi(t)$ ,  $d(y, Ty) = d(\frac{9}{20}, \frac{1}{4}) = 0.04\phi(t)$ ,  $d(x, Tx) = d(x, \frac{9}{20}) = |x - \frac{9}{20}|^2\phi(t)$ ; when  $x = 1$ ,  $d(x, Tx) =$

$0.3025\phi(t)$ , when  $x = 2$ ,  $d(x, Tx) = 2.4025\phi(t)$ , when  $x = \frac{1}{6}$ ,  $d(x, Tx) = 0.0802777\phi(t)$ .

Again clearly we can find  $\lambda \in (0, \frac{1}{3})$  satisfying  $d(Tx, Ty) \leq \lambda[d(x, Tx) + d(y, Ty)]$ .

Case (iii):  $x \in C = \{\frac{1}{n} : n \geq 3, n \neq 6\}$ ,  $y \in A - \{C \cup \frac{1}{6}\}$ .

$d(Tx, Ty) = d(\frac{1}{2} - x, \frac{1}{4})$ ,  $d(x, Tx) = d(x, \frac{1}{2} - x)$ ,  $d(y, Ty) = d(y, \frac{1}{4})$ ,  $d(x, Ty) = d(x, \frac{1}{4})$ ,  $d(y, Tx) = d(y, \frac{1}{2} - x)$ .

$\alpha(x, y) = 0$  at  $x = \frac{1}{4}$  and  $x + y = \frac{1}{2}$ .

At  $x = \frac{1}{4}$  and  $y \in A - \{C \cup \frac{1}{6}\}$ ,  $d(Tx, Ty) = (\frac{1}{4}, \frac{1}{4}) = 0$ . Hence contraction is satisfied.

At  $x = \frac{1}{n}$  and  $y = \frac{1}{2} - \frac{1}{n}$ ,  $d(Tx, Ty) = d(\frac{1}{2} - \frac{1}{n}, \frac{1}{4}) = |\frac{1}{2} - \frac{1}{n} - \frac{1}{4}|^2 \phi(t) = |\frac{1}{4} - \frac{1}{n}|^2 \phi(t)$ ,  $d(x, Tx) = d(\frac{1}{n}, \frac{1}{2} - \frac{1}{n}) = |\frac{1}{n} - \frac{1}{2} + \frac{1}{n}|^2 \phi(t) = |\frac{2}{n} - \frac{1}{2}|^2 \phi(t)$ ,  $d(y, Ty) = d(\frac{1}{2} - \frac{1}{n}, \frac{1}{4}) = |\frac{1}{4} - \frac{1}{n}|^2 \phi(t)$ .

We have to show that  $d(Tx, Ty) < \lambda[d(x, Tx) + d(y, Ty)]$  for some  $\lambda \in (0, \frac{1}{3})$ , i.e.,

$$\left| \frac{1}{4} - \frac{1}{n} \right|^2 \phi(t) < 5\lambda \left| \frac{1}{4} - \frac{1}{n} \right|^2 \phi(t).$$

The above inequality is clearly true for  $\lambda = \frac{1}{4} < \frac{1}{3}$ .

Similarly in all other cases  $T$  satisfies condition (3.4). Thus  $T$  satisfies all the conditions of Theorem 3.2 and  $\frac{1}{4}$  is the unique fixed point. However  $T$  does not satisfy contractions of Theorem 3.3 at  $x = \frac{1}{6}$  and  $y = \frac{1}{4}$  as  $d(Tx, Ty) = d(\frac{9}{20}, \frac{1}{4}) = 0.04\phi(t) > \frac{1}{3}[d(x, Tx) + d(y, Ty)] = \frac{1}{3}[d(\frac{1}{6}, \frac{9}{20}) + d(\frac{1}{4}, \frac{1}{4})] = 0.02676\phi(t)$ .

REMARK 3.5. Theorems 3.1 and 3.2 are proper generalizations of the results of [1,2,12,19,21] and many others.

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