МАТЕМАТИЧКИ ВЕСНИК
67, 4 (2015), 258–268
December 2015

оригинални научни рад
research paper

ON SOLVING PARABOLIC EQUATION WITH HOMOGENEOUS BOUNDARY AND INTEGRAL INITIAL CONDITIONS

Mladen Ignjatović

Abstract. In this paper we consider the second order parabolic partial differential equation with constant coefficients subject to homogeneous Dirichlet boundary conditions and initial condition containing nonlocal integral term. We derive first and second order finite difference schemes for the parabolic problem, combining implicit and Crank-Nicolson methods with two discretizations of the integral term. One numerical example is presented to test and illustrate the proposed algorithm.

1. Introduction

The first paper devoted to partial differential equations with nonlocal integral condition goes back to J. R. Cannon [3]. For parabolic partial differential equations with nonlocal condition the reader can see Bouziani [2], Dehghan [6], Ionkin [8], Kamynin [10]. Problems related to elliptic equations were considered by (among others) Ashyralyev [1], Gushin [7]. Deghan [5] and Pulkina [11] dealt with hyperbolic equations. In this article, we consider parabolic differential equation with homogeneous Dirichlet boundary conditions and nonlocal weighted integral condition

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad (x,t) \in \Omega, \tag{1}$$

$$u(0,t) = u(1,t) = 0, \quad 0 < t < T, \tag{2}$$

$$u(x,0) = \int_0^T \alpha(s) u(x,s)\, ds + g(x), \tag{3}$$

where $g(x)$ and $f(x,t)$ are given smooth functions on $(0,1)$ and $\Omega = [0,1] \times [0,T]$ respectively, under assumption $\int_0^T |\alpha(\rho)|\, d\rho < 1$, $\alpha(t) > 0$ for $t \in [0,T]$.

The paper is organized as follows: In Section 2 we propose that a solution of the problem (1)–(3) exists. In Sections 2 and 3, we present, respectively, first

2010 Mathematics Subject Classification: 65M12, 65M06, 65M22

Keywords and phrases: Finite difference method; stability estimate; parabolic equation; non-local condition; second-order of convergence.

and second order of convergence Euler implicit scheme for solving this problem. In Section 4, we prove stability and give error analysis of second order of convergence Euler difference scheme. Numerical results that illustrate theoretical discussion are presented in Section 5.

2. Existence of solution

Let us suppose that the functions f and g can be expressed in terms of their Fourier sine series in Ω,

$$f(x,t) = \sum_{k=1}^{\infty} \phi_k(t)\sin(k\pi x), \quad g(x) = \sum_{k=1}^{\infty} \gamma_k \sin(k\pi x)$$

and search for a solution of the problem (1)–(3) in the form

$$u(x,t) = \sum_{k=1}^{\infty} v_k(t)\sin(k\pi x). \tag{4}$$

For any given $n \in \mathbb{N}$, we define functions

$$f_n(x,t) = \sum_{k=1}^{n} \phi_k(t)\sin(k\pi x), \quad g_n(x) = \sum_{k=1}^{n} \gamma_k \sin(k\pi x)$$

and problem P_n,

$$\frac{\partial u_n}{\partial t} = \frac{\partial^2 u_n}{\partial x^2} + f_n(x,t), \quad (x,t) \in \Omega, \tag{5}$$

$$u_n(0,t) = u_n(1,t) = 0, \quad 0 < t < T,$$

$$u_n(x,0) = \int_0^T \alpha(s)u_n(x,s)\,ds + g_n(x), \quad x \in [0,1]. \tag{6}$$

We will try to find solution u_n of the problem P_n in the form

$$u_n(x,t) = \sum_{k=1}^{n} v_k(t)\sin(k\pi x). \tag{7}$$

From (5) we have

$$\sum_{k=1}^{n} (v_k'(t) + k^2\pi^2 v_k)\sin(k\pi x) = \sum_{k=1}^{n} \phi_k(t)\sin(k\pi x),$$

and since the set of functions $\{\sin(k\pi x)\}_{k=1}^{n}$ is linearly independent on $[0,1]$,

$$v_k'(t) + k^2\pi^2 v_k = \phi_k(t), \quad k = 1,\dots,n, \ \ t \in [0,T]. \tag{8}$$

General solution to linear differential equations (8) is

$$v_k(t) = e^{-k^2\pi^2 t}\left(C_k + \int_0^t \phi_k(r)e^{k^2\pi^2 r}\,dr\right), \quad k = 1,\dots,n. \tag{9}$$

Now $v_k(0) = C_k$. From (6) and (7) we have

$$u_n(x,0) = \sum_{k=1}^{n} \left(\int_0^T \alpha(s) v_k(s)\, ds + \gamma_k \right) \sin(k\pi x),$$
$$u_n(x,0) = \sum_{k=1}^{n} v_k(0) \sin(k\pi x) = \sum_{k=1}^{n} C_k \sin(k\pi x),$$

that is

$$C_k = \int_0^T \alpha(s) v_k(s)\, ds + \gamma_k, \quad k = 1, \dots, n.$$

Now substituting (9) into the last equation we have

$$C_k = C_k \int_0^T \alpha(s) e^{-k^2\pi^2 s}\, ds + \int_0^T \alpha(s) e^{-k^2\pi^2 s} \int_0^s \phi_k(r) e^{k^2\pi^2 r}\, dr\, ds + \gamma_k,$$

$k = 1, \dots, n$, and finally

$$C_k = \frac{\int_0^T \alpha(s) e^{-k^2\pi^2 s} \int_0^s \phi_k(r) e^{k^2\pi^2 r}\, dr\, ds + \gamma_k}{1 - \int_0^T \alpha(s) e^{-k^2\pi^2 s}\, ds} \quad k = 1, \dots, n. \tag{10}$$

In order to prove the convergence of the series (4) we will use a well known result which says that if $F(t) \in C^m[0,1]$ then for its sine and cosine Fourier coefficients the following assessment stands:

$$|a_k|, |b_k| = o(k^{-m}).$$

So if $g(x)$ and $f(x,t)$ are functions of class $C^2[0,1]$, then

$$|\gamma_k| \le \frac{\text{const}}{k^2} \text{ and } |\phi_k| \le \frac{\text{const}}{k^2}.$$

Now using (10), $|C_k| \le \frac{\text{const}}{k^2}$ which is a sufficient condition for convergence of the sequence $u_n(x,t)$ and thereby (4). Now we have shown that for any $n \in \mathbb{N}$ the problem P_n has a solution u_n, so when $n \to \infty$, by construction, we conclude that there exists a solution u to the problem (1)–(3).

3. First order of convergence finite difference scheme

Let us discretize the problem (1)–(3) with a first order Euler implicit scheme: find U_i^j, $i = 0, \dots, N, \quad j = 0, \dots, M$, such that

$$\frac{U_i^{k+1} - U_i^k}{\tau} - \frac{U_{i+1}^{k+1} - 2U_i^{k+1} + U_{i-1}^{k+1}}{h^2} = f(ih, (k+1)\tau), \tag{11}$$
$$1 \le i \le N-1, \quad \frac{1}{N} = h; \quad 0 \le k \le M-1, \qquad \frac{T}{M} = \tau, \quad i, k, M, N \in \mathbb{N},$$

with boundary conditions

$$U_0^k = U_N^k = 0, \quad 0 \le k \le M. \tag{12}$$

The integral condition is discretized by rectangular rule

$$U^0(x) = \sum_{m=1}^{M} \alpha(m\tau) U^m(x)\tau + g(x), \quad x = x_i = ih, \tag{13}$$

where U_i^j represents numerical approximation of $u(x_i, t_j)$, the value of the analytical solution u at mesh-point (x_i, t_j) and $x_i = ih, t_j = j\tau$.

The discretized problem (11)–(13) can be written in the matrix form

$$AU_{i+1} + BU_i + AU_{i-1} = \varphi_i, \quad i = 1, \ldots, N-1, \tag{14}$$

where vector U_i contains approximated values on the different time levels on the space point $x_i = ih$:

$$U_i = \begin{bmatrix} U_i^0 \\ U_i^1 \\ \vdots \\ U_i^M \end{bmatrix}_{(M+1)\times 1}, \quad i = 0, 1, \ldots, N. \tag{15}$$

Note that, due to the boundary condition (12),

$$U_0 = U_N = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{(M+1)\times 1}. \tag{16}$$

The matrices A and B are defined as follows:

$$A = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & -\frac{1}{h^2} & 0 & \ldots & 0 & 0 \\ 0 & 0 & -\frac{1}{h^2} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -\frac{1}{h^2} & 0 \\ 0 & 0 & 0 & \ldots & 0 & -\frac{1}{h^2} \end{bmatrix}_{(M+1)^2}$$

$$B = \begin{bmatrix} 1 & -\tau\alpha(1\tau) & -\tau\alpha(2\tau) & \ldots & -\tau\alpha((M-1)\tau) & -\tau\alpha(M\tau) \\ -\frac{1}{\tau} & \frac{1}{\tau} + \frac{2}{h^2} & 0 & \ldots & 0 & 0 \\ 0 & -\frac{1}{\tau} & \frac{1}{\tau} + \frac{2}{h^2} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -\frac{1}{\tau} & \frac{1}{\tau} + \frac{2}{h^2} \end{bmatrix}_{(M+1)^2}.$$

Vector φ_i is defined in the following way:

$$\varphi_i = \begin{bmatrix} g_i \\ f_i^1 \\ \vdots \\ f_i^M \end{bmatrix}_{(M+1)\times 1}.$$

The system of matrix equations (14) can be solved using the simplified form of Gaussian elimination for tridiagonal systems of linear equations, known as tridiagonal matrix algorithm (or Thomas algorithm) adapted for solving tridiagonal systems of linear matrix equations.

We will find unknown vectors U_i using the following formula:

$$U_i = \alpha_{i+1} U_{i+1} + \beta_{i+1}, \quad i = N-1, \dots, 1, \tag{17}$$

while due to boundary conditions

$$U_N = U_0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{(M+1)\times 1}, \tag{18}$$

the coefficients α_i and β_i of dimension $(M+1)^2$ i $(M+1)\times 1$ are given by

$$\alpha_{i+1} = -(B + A\alpha_i)^{-1} A \tag{19}$$

$$\beta_{i+1} = (B + A\alpha_i)^{-1}(\varphi_i - A\beta_i), \tag{20}$$

where α_1 is the zero matrix and β_1 is the vector with all zeros. The order of algorithm complexity is $\mathcal{O}(M^3 N)$.

4. Second order of convergence finite difference scheme

In this part we present the second order difference scheme. The Crank-Nicolson [4] second order difference scheme for the problem (1)–(3) is given by: find U_i^j, $i = 0, \dots, N$, $j = 0, \dots, M$, such that

$$\frac{U_i^{k+1} - U_i^k}{\tau} - \frac{1}{2}\frac{U_{i+1}^{k+1} - 2U_i^{k+1} + U_{i-1}^{k+1}}{h^2} - \frac{1}{2}\frac{U_{i+1}^{k} - 2U_i^{k} + U_{i-1}^{k}}{h^2} = f(ih, (k+0.5)\tau), \tag{21}$$

$$1 \le i \le N-1, \quad \frac{1}{N} = h; \quad 0 \le k \le M-1, \qquad \frac{T}{M} = \tau, \quad i, k, M, N \in \mathbb{N},$$

with boundary conditions

$$U_0^k = U_N^k = 0, \quad 0 \le k \le M. \tag{22}$$

For the integral condition approximation, we use the trapezoidal rule

$$U^0(x) = \sum_{m=1}^{M} \frac{\tau}{2}\big(\alpha(m\tau)U^m(x) + \alpha((m-1)\tau)U^{m-1}(x)\big) + g(x). \tag{23}$$

Similarly as in the first order scheme, the second order difference scheme can be written in the matrix form

$$AU_{i+1} + BU_i + AU_{i-1} = \varphi_i, \quad i = 1, \dots, N-1.$$

The vectors U_i, $i = 0, 1, \dots, N$ are given in the same way as in (15) and (16), while the matrices A and B are defined as follows

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ -\frac{1}{2h^2} & -\frac{1}{2h^2} & 0 & \dots & 0 & 0 \\ 0 & -\frac{1}{2h^2} & -\frac{1}{2h^2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -\frac{1}{2h^2} & -\frac{1}{2h^2} \end{bmatrix}_{(M+1)^2}$$

$$B = \begin{bmatrix} 1 - \frac{\tau}{2}\alpha(0) & -\tau\alpha(1\tau) & \dots & -\tau\alpha((M-1)\tau) & -\frac{\tau}{2}\alpha(M\tau) \\ -\frac{1}{\tau} + \frac{1}{h^2} & \frac{1}{\tau} + \frac{1}{h^2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -\frac{1}{\tau} + \frac{1}{h^2} & \frac{1}{\tau} + \frac{1}{h^2} \end{bmatrix}_{(M+1)^2}.$$

Vector φ_i is defined by

$$\varphi_i = \begin{bmatrix} g_i \\ f_i^{0.5} \\ \vdots \\ f_i^{M-0.5} \end{bmatrix}_{(M+1)\times 1}.$$

This system of matrix equations can be solved using formulas (17)–(20). The order of algorithm complexity is also $\mathcal{O}(M^3N)$.

5. Stability and error analysis of the second order difference scheme

The following lemma proves stability of the difference scheme (21)–(23).

Lemma 5.1. *Assume that $g(x)$ and $f(x,t)$ are given continuous functions on $[0,1]$ and $\Omega = [0,1] \times [0,T]$ respectively. Let $\alpha(t)$ also be a known function, such that $\int_0^T |\alpha(\rho)|\, d\rho < 1$, $\alpha(t) > 0$ and $\alpha''(t) < \infty$. Then the solution of the scheme (21)–(23) satisfies the following stability estimate:*

$$\max_{1\le k\le M} \|U^k\|_h^2 \le \frac{2}{1-C}\|g\|_h^2 + \left[1 + \frac{2C^2}{(1-C)^2}\right] \frac{\tau}{16} \sum_{m=0}^{M-1} \|f^{m+0.5}\|_h^2,$$

where the constant C depends only on $\int_0^T |\alpha(\rho)|\, d\rho$.

Proof. From $\int_0^T |\alpha(\rho)|\, d\rho < 1$ we get

$$\int_0^T |\alpha(\rho)|\, d\rho = 1 - \delta, \text{ where } \delta \in (0,1).$$

Using the error formula for trapezoidal rule we have

$$\left| \int_0^T |\alpha(\rho)|\, d\rho - \frac{\tau}{2} \sum_{m=1}^{M} [\alpha(m\tau) + \alpha((m-1)\tau)] \right| \le \frac{1}{12} TD\tau^2, \tag{24}$$

where D stands for $D = \max_{0\le t\le T} |\alpha''(t)|$. If we choose τ so that

$$\frac{1}{12}TD\tau^2 \le \frac{\delta}{2},$$

that is $\tau \le \sqrt{\frac{6\delta}{TD}}$, we have that

$$\begin{aligned}
&\frac{\tau}{2}\sum_{m=1}^{M}[\alpha(m\tau)+\alpha((m-1)\tau)] \\
&\qquad = \int_0^T \alpha(\rho)\,d\rho + \frac{\tau}{2}\sum_{m=1}^{M}[\alpha(m\tau)+\alpha((m-1)\tau)] - \int_0^T \alpha(\rho)\,d\rho \\
&\qquad \le \int_0^T \alpha(\rho)\,d\rho + \Big|\frac{\tau}{2}\sum_{m=1}^{M}[\alpha(m\tau)+\alpha((m-1)\tau)] - \int_0^T \alpha(\rho)\,d\rho\Big| \\
&\qquad\qquad \le 1-\delta+\frac{\delta}{2} = 1-\frac{\delta}{2} = C < 1.
\end{aligned}$$

Approximating the integral term in initial condition using the trapezoidal rule we have

$$U^0(x) = \frac{\tau}{2}\sum_{m=1}^{M}[\alpha(m\tau)U^m(x)+\alpha((m-1)\tau)U^{m-1}(x)] + g(x),$$

whereby it follows that

$$\|U^0\|_h \le C \max_{0\le k\le M}\|U^k\|_h + \|g\|_h, \tag{25}$$

where $\|\cdot\|_h$ is defined as $\|U^i\|_h = (\sum_{j=0}^N h{U_j^i}^2)^{\frac{1}{2}}$. Using the inequality for stability of weighted scheme, which can be found in [9] (for weight parameter $\theta = 0.5$ weighted scheme is Crank-Nicolson scheme).

$$\max_{0\le k\le M}\|U^k\|_h^2 \le \|U^0\|_h^2 + \frac{\tau}{16}\sum_{m=0}^{M-1}\|f^{m+0.5}\|_h^2, \tag{26}$$

and the well-known inequality $\sqrt{a+b} \le \sqrt{a}+\sqrt{b}$, $a,b\ge 0$ we obtain

$$\max_{0\le k\le M}\|U^k\|_h \le \|U^0\|_h + \sqrt{\frac{\tau}{16}\sum_{m=0}^{M-1}\|f^{m+0.5}\|_h^2}. \tag{27}$$

From inequalities (25), (27) we get

$$\|U^0\|_h \le C\|U^0\|_h + C\sqrt{\frac{\tau}{16}\sum_{m=0}^{M-1}\|f^{m+0.5}\|_h^2} + \|g\|_h,$$

and the final estimate for $\|U^0\|_h$

$$\|U^0\|_h \le \frac{C}{1-C}\sqrt{\frac{\tau}{16}\sum_{m=0}^{M-1}\|f^{m+0.5}\|_h^2} + \frac{1}{1-C}\|g\|_h.$$

After substitution in (26) and using the obvious inequality $2ab \le a^2 + b^2$, we get

$$\max_{1\le k\le M} \|U^k\|_h^2 \le \frac{2}{1-C}\|g\|_h^2 + \left[1 + \frac{2C^2}{(1-C)^2}\right] \frac{\tau}{16} \sum_{m=0}^{M-1} \|f^{m+0.5}\|_h^2,$$

which proves stability of the algorithm for sufficiently small τ. ■

Now we will analyze the error of the finite difference scheme (21)–(23) proposed for solution of the problem (1)–(3). In order to do that, let us first define the global error

$$z_i^j = u_i^j - U_i^j,$$

for $i = 0, \dots, N$, $j = 0, \dots, M$, where $u_i^j = u(x_i, t_j)$. It can easily be seen that the global error satisfies the finite difference scheme of the form

$$\frac{z_i^{k+1} - z_i^k}{\tau} - \frac{1}{2}\frac{z_{i+1}^{k+1} - 2z_i^{k+1} + z_{i-1}^{k+1}}{h^2} - \frac{1}{2}\frac{z_{i+1}^k - 2z_i^k + z_{i-1}^k}{h^2} = \psi_i^{k+\frac{1}{2}},$$
$$i = 1, \dots, N-1, : k = 0, \dots, M-1, \quad (28)$$

$$z_0^k = z_N^k = 0, \quad 0 \le k \le M.$$

$$z_i^0 = \sum_{m=1}^{M} \tfrac{\tau}{2}\left(\alpha(m\tau) z_i^m + \alpha((m-1)\tau) z_i^{m-1}\right) + \chi_i, \quad 0 \le i \le N \quad (29)$$

where the terms $\psi_i^{k+\frac{1}{2}}$ and χ_i will be determined later. So, according to Lemma 5.1,

$$\max_{1\le k\le M} \|z^k\|_h^2 \le \frac{2}{1-C}\|\chi_i\|_h^2 + \left[1 + \frac{2C^2}{(1-C)^2}\right] \frac{\tau}{16} \sum_{m=0}^{M-1} \|\psi_i^{m+\frac{1}{2}}\|_h^2. \quad (30)$$

In order to estimate the global error, we need to estimate $\|\psi_i^{m+\frac{1}{2}}\|_h$. If we substitute (30) into (28), we have

$$\psi_i^{k+\frac{1}{2}} = \frac{u_i^{k+1} - u_i^k}{\tau} - \frac{1}{2}\frac{u_{i+1}^{k+1} - 2u_i^{k+1} + u_{i-1}^{k+1}}{h^2} - \frac{1}{2}\frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{h^2}$$
$$- \left(\frac{U_i^{k+1} - U_i^k}{\tau} - \frac{1}{2}\frac{U_{i+1}^{k+1} - 2U_i^{k+1} + U_{i-1}^{k+1}}{h^2} - \frac{1}{2}\frac{U_{i+1}^k - 2U_i^k + U_{i-1}^k}{h^2}\right). \quad (31)$$

According to (21) the part of the right-hand side between the brackets of the last equation, is equal to $f(x_i, t_{k+\frac{1}{2}})$ that is, taking (11) into account, further equal to $\frac{\partial u}{\partial t}(x_i, t_{k+\frac{1}{2}}) - \frac{\partial^2 u}{\partial x^2}(x_i, t_{k+\frac{1}{2}})$, so

$$\psi_i^{k+\frac{1}{2}} = \left[\frac{u_i^{k+1} - u_i^k}{\tau} - \frac{\partial u}{\partial t}(x_i, t_{k+\frac{1}{2}})\right] +$$
$$\left[\frac{\partial^2 u}{\partial x^2}(x_i, t_{k+\frac{1}{2}}) - \frac{1}{2}\left(\frac{u_{i+1}^{k+1} - 2u_i^{k+1} + u_{i-1}^{k+1}}{h^2} + \frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{h^2}\right)\right]. \quad (32)$$

If we now expand function u in a Taylor series about the point $(x_i, t_{k+\frac{1}{2}})$ in t-direction we have

$$\frac{u_i^{k+1} - u_i^k}{\tau} = \frac{\partial u}{\partial t}(x_i, t_{k+\frac{1}{2}}) + \frac{\tau^2}{24}\frac{\partial^3 u}{\partial t^3}(x_i, t_{k+\frac{1}{2}}) + \cdots .$$

By expanding function u in a Taylor series first about the point $(x_i, t_{k+\frac{1}{2}})$ in x-direction and then expanding that again about the point $(x_i, t_{k+\frac{1}{2}})$ in t-direction yields

$$\begin{aligned}\frac{u_{i+1}^{k+1} - 2u_i^{k+1} + u_{i-1}^{k+1}}{h^2} &= \left(\frac{\partial^2 u}{\partial x^2}(x_i, t_{k+\frac{1}{2}}) + \frac{1}{12}h^2\frac{\partial^4 u}{\partial x^4}(x_i, t_{k+\frac{1}{2}}) + \cdots\right) \\ &+ \frac{\tau}{2}\left(\frac{\partial^3 u}{\partial x^2 \partial t}(x_i, t_{k+\frac{1}{2}}) + \frac{1}{12}h^2\frac{\partial^5 u}{\partial x^4 \partial t}(x_i, t_{k+\frac{1}{2}}) + \cdots\right) \\ &+ \frac{1}{2}\left(\frac{\tau}{2}\right)^2\left(\frac{\partial^4 u}{\partial x^2 \partial t^2}(x_i, t_{k+\frac{1}{2}}) + \frac{1}{12}h^2\frac{\partial^6 u}{\partial x^4 \partial t^2}(x_i, t_{k+\frac{1}{2}}) + \cdots\right) + \cdots\end{aligned}$$

There is a similar expansion for $\frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{h^2}$

$$\begin{aligned}\frac{u_{i+1}^{k} - 2u_i^{k} + u_{i-1}^{k}}{h^2} &= \left(\frac{\partial^2 u}{\partial x^2}(x_i, t_{k+\frac{1}{2}}) + \frac{1}{12}h^2\frac{\partial^4 u}{\partial x^4}(x_i, t_{k+\frac{1}{2}}) + \cdots\right) \\ &- \frac{\tau}{2}\left(\frac{\partial^3 u}{\partial x^2 \partial t}(x_i, t_{k+\frac{1}{2}}) + \frac{1}{12}h^2\frac{\partial^5 u}{\partial x^4 \partial t}(x_i, t_{k+\frac{1}{2}}) + \cdots\right) \\ &+ \frac{1}{2}\left(\frac{\tau}{2}\right)^2\left(\frac{\partial^4 u}{\partial x^2 \partial t^2}(x_i, t_{k+\frac{1}{2}}) + \frac{1}{12}h^2\frac{\partial^6 u}{\partial x^4 \partial t^2}(x_i, t_{k+\frac{1}{2}}) + \cdots\right) + \cdots\end{aligned}$$

Now substituting the last three equations in (32) gives us

$$\psi_i^{k+\frac{1}{2}} = \frac{\tau^2}{24}\frac{\partial^3 u}{\partial t^3}(x_i, t_{k+\frac{1}{2}}) + \cdots - \frac{1}{12}h^2\frac{\partial^4 u}{\partial x^4}(x_i, t_{k+\frac{1}{2}}) - \frac{\tau^2}{8}\frac{\partial^4 u}{\partial x^2 \partial t^2}(x_i, t_{k+\frac{1}{2}}) + \cdots , \tag{33}$$

and the final estimate for $\psi_i^{k+\frac{1}{2}}$

$$|\psi_i^{k+\frac{1}{2}}| \le \frac{h^2}{12}M_{4x} + \frac{\tau^2}{24}(3M_{2x2t} + M_{3t}) + H.O.T.$$

where $H.O.T.$ signifies terms of higher order than h^2 and τ^2 and

$$M_{mxnt} = \max_{(x,t)\in\Omega}\left|\frac{\partial^{m+n}}{\partial x^m \partial t^n}u(x,t)\right| .$$

In order to give final estimate for the global error, we also need to estimate χ_i. From (29), (23) and (13),

$$\chi_i = \int_0^T \alpha(s)u(x_i, s)\, ds - \frac{\tau}{2}\sum_{m=1}^{M}[\alpha(m\tau)u^m(x) + \alpha((m-1)\tau)u^{m-1}(x)].$$

So using (24) we have that

$$|\chi_i| \leq \frac{1}{12}TD\tau^2. \tag{34}$$

Now if we substitute (34), (33) into (30) we get the final error estimate of the Crank-Nicolson scheme

$$\max_{1\leq k\leq M} \|u^k - U^k\|_h^2 \leq c(\tau^2 + h^2),$$

where c is a positive constant, independent of h and τ. Thus, we deduce that Crank-Nicolson scheme unconditionally converges in the norm $\|\cdot\|_h$, with the convergence rate $\mathcal{O}(\tau^2 + h^2)$.

6. Numerical results

Let us observe the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2e^t \sin x \tag{35}$$

in the area $[0, \pi] \times [0, 1]$, with boundary conditions

$$u(0, t) = u(\pi, t) = 0, \tag{36}$$

and an initial condition

$$u(x, 0) = \int_0^1 e^{-s} u(x, s)\, ds. \tag{37}$$

The analytical solution of given problem (35)–(37) is

$$u(x, t) = e^t \sin x.$$

Errors for the first and second order scheme are given in Tables 1 and 2, respectively. Note that errors ε_2 and ε_{max} are determined, respectively, by formulas:

$$\varepsilon_2 = \left(\frac{\sum_{i=0}^{N} \sum_{j=0}^{M} (U_i^j - u(ih, j\tau))^2}{MN} \right)^{\frac{1}{2}},$$

$$\varepsilon_{max} = \max_{\substack{0\leq i\leq N \\ 0\leq j\leq M}} |U_i^j - u(ih, j\tau)|.$$

N, M	ε_2	ε_{max}
$N = M = 10$	$4.3308e-2$	$8.7449e-2$
$N = M = 20$	$1.8524e-2$	$3.8582e-2$
$N = M = 40$	$8.5570e-3$	$1.8108e-2$
$N = M = 80$	$4.1111e-3$	$8.7704e-3$

Table 1. Errors of the first order scheme

N, M	ε_2	ε_{max}
$N = M = 10$	$4.8817e - 3$	$9.8572e - 3$
$N = M = 20$	$1.1249e - 3$	$2.3429e - 3$
$N = M = 40$	$2.6992e - 4$	$5.7120e - 4$
$N = M = 80$	$6.6104e - 5$	$1.4102e - 4$

Table 2. Errors of the second order scheme

Overall, it can be concluded that the proposed finite difference schemes for solving this problem are stable and that they satisfy the given order of convergence.

ACKNOWLEDGEMENT. The author is indebted to Professor B. Jovanović for introducing him to the problem, helpful comments and for the references.

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(received 14.08.2014; in revised form 13.05.2015; available online 20.06.2015)

Faculty of Technology, University of East Sarajevo, Karakaj bb, 75400 Zvornik, Bosnia and Herzegovina

E-mail: ignjatovic@gmail.com