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KOROVKIN TYPE APPROXIMATION THEOREM IN $A_2^{\mathcal{I}}$-STATISTICAL SENSE

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Abstract. In this paper we consider the notion of $A_2^{\mathcal{I}}$-statistical convergence for real double sequences which is an extension of the notion of $A^{\mathcal{I}}$-statistical convergence for real single sequences introduced by Savas, Das and Dutta. We primarily apply this new notion to prove a Korovkin type approximation theorem. In the last section, we study the rate of $A_2^{\mathcal{I}}$-statistical convergence.

1. Introduction and background

Throughout the paper $\mathbb{N}$ will denote the set of all positive integers. Approximation theory has important applications in the theory of polynomial approximation in various areas of functional analysis. For a sequence $\{T_n\}_{n\in\mathbb{N}}$ of positive linear operators on $C(X)$, the space of real valued continuous functions on a compact subset X of real numbers, Korovkin [20] first established the necessary and sufficient conditions for the uniform convergence of $\{T_n f\}_{n\in\mathbb{N}}$ to a function f by using the test functions $e_0 = 1,\ e_1 = x,\ e_2 = x^2$ [1]. The study of the Korovkin type approximation theory has a long history and is a well-established area of research. As is mentioned in [12] in particular, the matrix summability methods of Cesáro type are strong enough to correct the lack of convergence of various sequences of positive linear operators such as the interpolation operators of Hermite-Fejér [6]. In recent years, using the concept of uniform statistical convergence various statistical approximation results have been proved [9,10]. Erkuş and Duman [15] studied a Korovkin type approximation theorem via A-statistical convergence in the space $H_w(I^2)$ where $I^2 = [0,\infty)\times[0,\infty)$ which was extended for double sequences of positive linear operators of two variables in A-statistical sense by Demirci and Dirik in [12]. Our primary interest in this paper is to obtain a general Korovkin type approximation theorem for double sequences of positive linear operators of two variables from $H_w(\mathcal{K})$ to $C(\mathcal{K})$ where $\mathcal{K} = [0,A]\times[0,B]$, $A, B \in (0,1)$, in the sense of $A_2^{\mathcal{I}}$-statistical convergence.

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The concept of convergence of a sequence of real numbers was extended to statistical convergence by Fast [17]. Further investigations started in this area after the pioneering works of Šalát [31] and Fridy [18]. The notion of $\mathcal{I}$-convergence of real sequences was introduced by Kostyrko et al. [23] as a generalization of statistical convergence using the notion of ideals (see [3,4,5] for further references). Later the idea of $\mathcal{I}$-convergence was also studied in topological spaces in [24]. On the other hand statistical convergence was generalized to A-statistical convergence by Kolk [21,22]. Later a lot of works have been done on matrix summability and A-statistical convergence (see [2,7,8,11,16,19,21,22,25,29]). In particular, very recently in [33] and [34] the two above mentioned approaches were unified and the very general notion of $A^{\mathcal{I}}$-statistical convergence was introduced and studied. In this paper we consider an extension of this notion to double sequences, namely $A_2^{\mathcal{I}}$-statistical convergence.

A real double sequence $\{x_{mn}\}_{m,n\in\mathbb{N}}$ is said to be convergent to L in Pringsheim's sense if for every $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that $|x_{mn} - L| < \varepsilon$ for all $m, n > N(\varepsilon)$ and denoted by $\lim_{m,n}\ x_{mn} = L$. A double sequence is called bounded if there exists a positive number M such that $|x_{mn}| \leq M$ for all $(m, n) \in \mathbb{N}\times\mathbb{N}$. A real double sequence $\{x_{mn}\}_{m,n\in\mathbb{N}}$ is statistically convergent to L if for every $\varepsilon > 0$,

$$\lim_{j,k} \frac{|\{m \leq j, n \leq k : |x_{mn} - L| \geq \varepsilon\}|}{jk} = 0$$

[27,28].

Recall that a family $\mathcal{I} \subset 2^Y$ of subsets of a nonempty set Y is said to be an ideal in Y if $(i) A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$; $(ii) A \in \mathcal{I}, B \subset A$ implies $B \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$. If $\mathcal{I}$ is a non-trivial proper ideal in Y (i.e. $Y \notin \mathcal{I}, \mathcal{I} \neq \{\emptyset\}$) then the family of sets $F(\mathcal{I}) = \{M \subset Y :$ there exists $A \in \mathcal{I} : M = Y \setminus A\}$ is a filter in Y. It is called the filter associated with the ideal $\mathcal{I}$. A non-trivial ideal $\mathcal{I}$ of $\mathbb{N}\times\mathbb{N}$ is called strongly admissible if $\{i\}\times\mathbb{N}$ and $\mathbb{N}\times\{i\}$ belong to $\mathcal{I}$ for each $i \in \mathbb{N}$. It is evident that a strongly admissible ideal is admissible also. Let $\mathcal{I}_0 = \{A \subset \mathbb{N}\times\mathbb{N} :\exists\ m(A) \in \mathbb{N}$ such that $i, j \geq m(A) \Longrightarrow (i, j) \notin A\}$. Then $\mathcal{I}_0$ is a non-trivial strongly admissible ideal [14]. Let $A = (a_{nk})$ be a non-negative regular matrix. For an ideal $\mathcal{I}$ of $\mathbb{N}$ a sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be $A^{\mathcal{I}}$-statistically convergent to L if for any $\varepsilon > 0$ and $\delta > 0$,

$$\left\{n \in \mathbb{N} : \sum_{k\in K(\varepsilon)} a_{nk} \geq \delta\right\} \in \mathcal{I},$$

where $K(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ [33,34].

Let $A = (a_{jkmn})$ be a four dimensional summability matrix. For a given double sequence $\{x_{mn}\}_{m,n\in\mathbb{N}}$, the A-transform of x, denoted by $Ax := ((Ax)_{jk})$, is given by

$$(Ax)_{jk} = \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} x_{mn}$$

provided the double series converges in Pringsheim sense for every $(j, k) \in \mathbb{N}^2$. In 1926, Robison [30] presented a four dimensional analog of the regularity by considering an additional assumption of boundedness. This assumption was made because a convergent double sequence is not necessarily bounded.

Recall that a four dimensional matrix $A = (a_{jkmn})$ is said to be RH-regular if it maps every bounded convergent double sequence into a convergent double sequence with the same limit. The Robison-Hamilton conditions state that a four dimensional matrix $A = (a_{jkmn})$ is RH-regular if and only if

(i) $\lim_{j,k} a_{jkmn} = 0$ for each $(m,n) \in \mathbb{N}^2$,

(ii) $\lim_{j,k} \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} = 1$,

(iii) $\lim_{j,k} \sum_{m\in\mathbb{N}} |a_{jkmn}| = 0$ for each $n \in \mathbb{N}$,

(iv) $\lim_{j,k} \sum_{n\in\mathbb{N}} |a_{jkmn}| = 0$ for each $m \in \mathbb{N}$,

(v) $\sum_{(m,n)\in\mathbb{N}^2} |a_{jkmn}|$ is convergent,

(vi) there exist finite positive integers M_0 and N_0 such that $\sum_{m,n>N_0} |a_{jkmn}| < M_0$ holds for every $(j,k) \in \mathbb{N}^2$.

Let $A = (a_{jkmn})$ be a non-negative RH-regular summability matrix and let $K \subset \mathbb{N}^2$. Then the A-density of K is given by

$$\delta_A^{(2)}\{K\} = \lim_{j,k} \sum_{(m,n)\in K} a_{jkmn}$$

provided the limit exists. A real double sequence $x = \{x_{mn}\}_{m,n\in\mathbb{N}}$ is said to be A-statistically convergent to a number L if for every $\varepsilon > 0$

$$\delta_A^{(2)}\{(m,n) \in \mathbb{N}^2 : |x_{mn} - L| \geq \varepsilon\} = 0.$$

We denote $\mathcal{I}_{\delta_A^{(2)}} = \left\{C \subset \mathbb{N}^2 : \delta_A^{(2)}\{C\} = 0\right\}$ which is an admissible ideal in $\mathbb{N} \times \mathbb{N}$. Throughout we use $\mathcal{I}$ as a non-trivial strongly admissible ideal on $\mathbb{N} \times \mathbb{N}$.

2. A Korovkin type approximation theorem

Recently the concept of $\mathcal{I}$-statistical convergence for real single sequences has been introduced by Das and Savas as a notion of convergence which is strictly weaker than the notion of statistical convergence (see [32] for details). Consequently this notion has been further investigated in [13]. Very recently it has been further generalized by using a summability matrix A into $A^{\mathcal{I}}$-statistical convergence for real single sequences by Savas, Das and Dutta [33,34]. In this paper we consider the following natural extension of these convergence for real double sequences.

The following definition is due to E. Savas (who has informed about it in a personal communication).

Definition 2.1. A real double sequence $\{x_{m,n}\}_{m,n\in\mathbb{N}}$ is said to be $\mathcal{I}_2$-statistically convergent to L if for each $\varepsilon > 0$ and $\delta > 0$,

$$\left\{(j,k) \in \mathbb{N}^2 : \tfrac{1}{jk}|\{m \leq j, n \leq k : |x_{mn} - L| \geq \varepsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

We now introduce the main definition of this paper.

Definition 2.2. Let $A = (a_{jkmn})$ be a non-negative RH-regular summability matrix. Then a real double sequence $\{x_{mn}\}_{m,n\in\mathbb{N}}$ is said to be $A_2^{\mathcal{I}}$-statistically convergent to a number L if for every $\varepsilon > 0$ and $\delta > 0$,

$$\Big\{(j,k) \in \mathbb{N}^2 : \sum_{(m,n)\in K_2(\varepsilon)} a_{jkmn} \geq \delta\Big\} \in \mathcal{I},$$

where $K_2(\varepsilon) = \{(m,n) \in \mathbb{N}^2 : \ |x_{mn} - L| \geq \varepsilon\}$. In this case, we write $A_2^{\mathcal{I}}$-*st*-$\lim_{m,n} x_{mn} = L$.

It should be noted that, if we take $A = C(1,1)$, the double Cesáro matrix [26] defined as follows

$$a_{jkmn} = \begin{cases} \frac{1}{jk} & \text{for } m \leq j, n \leq k; \\ 0 & \text{otherwise,} \end{cases}$$

then $A_2^{\mathcal{I}}$-statistical convergence coincides with the notion of $\mathcal{I}_2$-statistical convergence. Again if we replace the matrix A by the identity matrix for four dimensional matrices and $\mathcal{I} = \mathcal{I}_0$ then $A_2^{\mathcal{I}}$-statistical convergence reduces to the Pringsheim convergence for double sequences. For the ideal $\mathcal{I} = \mathcal{I}_0$, $A_2^{\mathcal{I}}$-statistical convergence implies A-statistical convergence for double sequences. The basic properties of $A_2^{\mathcal{I}}$-statistically convergent double sequences are similar to $A^{\mathcal{I}}$-statistical convergent single sequences and can be obtained analogously as in [32,33]. So our main aim here is to present an application of this notion in approximation theory.

Throughout this section, let $\mathcal{K} = [0,A] \times [0,B]$ $A, B \in (0,1)$ and denote the space of all real valued continuous functions on $\mathcal{K}$ by $C(\mathcal{K})$. This space is endowed with the supremum norm $\|f\| = \sup_{(x,y)\in\mathcal{K}}|f(x,y)|$, $f \in C(\mathcal{K})$. Consider the space $H_w(\mathcal{K})$ of real valued functions f on $\mathcal{K}$ satisfying

$$|f(u,v) - f(x,y)| \leq w\Big(f; \sqrt{(\frac{u}{1-u} - \frac{x}{1-x})^2 + (\frac{v}{1-v} - \frac{y}{1-y})^2}\Big)$$

where w is the modulus of continuity for $\delta > 0$ given by

$$w(f;\delta) = \sup\{|f(u,v) - f(x,y)| : \ (u,v),\ (x,y) \in \mathcal{K},\ \sqrt{(u-x)^2 + (v-y)^2} \leq \delta\}.$$

Then it is clear that any function in $H_w(\mathcal{K})$ is continuous and bounded on $\mathcal{K}$.

We will use the following test functions $f_0(x,y) = 1$, $f_1(x,y) = \frac{x}{1-x}$, $f_2 = \frac{y}{1-y}$, $f_3(x,y) = (\frac{x}{1-x})^2 + (\frac{y}{1-y})^2$ and we denote the value of Tf at a point $(u,v) \in \mathcal{K}$ by $T(f;u,v)$.

Now we establish the Korovkin type approximation theorem in $A_2^{\mathcal{I}}$-statistical sense.

Theorem 2.1. *Let $\{T_{mn}\}_{m,n\in\mathbb{N}}$ be a sequence of positive linear operators from $H_w(\mathcal{K})$ into $C(\mathcal{K})$ and let $A = (a_{jkmn})$ be a non-negative RH-regular summability matrix. Then for any $f \in H_w(\mathcal{K})$,*

$$A_2^{\mathcal{I}}\text{-}st\text{-}\lim_{m,n} \|T_{mn}f - f\| = 0 \tag{1}$$

is satisfied if the following holds

$$A_2^{\mathcal{I}}\text{-}st\text{-}\lim_{m,n} \|T_{mn}f_i - f_i\| = 0,\ \ i = 0,1,2,3. \tag{2}$$

Proof. Assume that (2) holds. Let $f \in H_w(\mathcal{K})$. Our objective is to show that for given $\varepsilon > 0$ there exist constants $C_0,\ C_1,\ C_2,\ C_3$ (depending on $\varepsilon > 0$) such that

$$\|T_{mn}f - f\| \leq \varepsilon + C_3\|T_{mn}f_3 - f_3\| + C_2\|T_{mn}f_2 - f_2\| + C_1\|T_{mn}f_1 - f_1\| + C_0\|T_{mn}f_0 - f_0\|.$$

If this is done then our hypothesis implies that for any $\varepsilon > 0,\ \delta > 0$,

$$\left\{(j,k) \in \mathbb{N}^2 : \sum_{(m,n)\in K_2(\varepsilon)} a_{jkmn} \geq \delta\right\} \in \mathcal{I},$$

where $K_2(\varepsilon) = \{(m,n) \in \mathbb{N}^2 :\ \|T_{mn}f - f\| \geq \varepsilon\}$.

To this end, start by observing that for each $(u,v) \in \mathcal{K}$ the function $0 \leq g_{uv} \in H_w(\mathcal{K})$ defined by $g_{uv}(s,t) = (\frac{s}{1-s} - \frac{u}{1-u})^2 + (\frac{t}{1-t} - \frac{v}{1-v})^2$ satisfies $g_{uv} = (\frac{x}{1-x})^2 + (\frac{y}{1-y})^2 - \frac{2u}{1-u}\frac{x}{1-x} - \frac{2v}{1-v}\frac{y}{1-y} + (\frac{u}{1-u})^2 + (\frac{v}{1-v})^2$. Since each T_{mn} is a positive operator, $T_{mn}g_{uv}$ is a positive function. In particular, we have for each $(u,v) \in \mathcal{K}$,

$$\begin{aligned}
0 &\leq T_{mn}g_{uv}(u,v)\\
&= [T_{mn}((\tfrac{x}{1-x})^2 + (\tfrac{y}{1-y})^2 - \tfrac{2u}{1-u}\tfrac{x}{1-x} - \tfrac{2v}{1-v}\tfrac{y}{1-y} + (\tfrac{u}{1-u})^2 + (\tfrac{v}{1-v})^2; u,v)]\\
&= [T_{mn}((\tfrac{x}{1-x})^2 + (\tfrac{y}{1-y})^2; u,v) - (\tfrac{u}{1-u})^2 - (\tfrac{v}{1-v})^2]\\
&\quad - \tfrac{2u}{1-u}[T_{mn}(\tfrac{x}{1-x}; u,v) - \tfrac{u}{1-u}] - \tfrac{2v}{1-v}[T_{mn}(\tfrac{y}{1-y}; u,v) - \tfrac{v}{1-v}]\\
&\quad + \{(\tfrac{u}{1-u})^2 + (\tfrac{v}{1-v})^2\}[T_{mn}f_0 - f_0]\\
&\leq \|T_{mn}f_3 - f_3\| + \tfrac{2u}{1-u}\|T_{mn}f_1 - f_1\|\\
&\quad + \tfrac{2v}{1-v}\|T_{mn}f_2 - f_2\| + \{(\tfrac{u}{1-u})^2 + (\tfrac{v}{1-v})^2\}\|T_{mn}f_0 - f_0\|.
\end{aligned}$$

Let $M = \|f\|$ and $\varepsilon > 0$. By the uniform continuity of f on $\mathcal{K}$ there exists a $\delta > 0$ such that $-\varepsilon < f(s,t) - f(u,v) < \varepsilon$ holds whenever

$$\sqrt{(\frac{s}{1-s} - \frac{u}{1-u})^2 + (\frac{t}{1-t} - \frac{v}{1-v})^2} < \delta,$$

$(s,t), (u,v) \in \mathcal{K}$. Next observe that

$$\begin{aligned}
-\varepsilon - \frac{2M}{\delta^2}&\left\{\left(\frac{s}{1-s} - \frac{u}{1-u}\right)^2 + \left(\frac{t}{1-t} - \frac{v}{1-v}\right)^2\right\}\\
&\leq f(s,t) - f(u,v)\\
&\leq \varepsilon + \frac{2M}{\delta^2}\left\{\left(\frac{s}{1-s} - \frac{u}{1-u}\right)^2 + \left(\frac{t}{1-t} - \frac{v}{1-v}\right)^2\right\} \qquad (3)
\end{aligned}$$

Indeed, if $\sqrt{(\frac{s}{1-s} - \frac{u}{1-u})^2 + (\frac{t}{1-t} - \frac{v}{1-v})^2} < \delta$ then (3) follows from

$$-\varepsilon < f(s,t) - f(u,v) < \varepsilon.$$

On the other hand, if $\sqrt{(\frac{s}{1-s}-\frac{u}{1-u})^2+(\frac{t}{1-t}-\frac{v}{1-v})^2}\geq\delta$ then (3) follows from

$$\begin{aligned}-\varepsilon-\frac{2M}{\delta^2}&\left\{\left(\frac{s}{1-s}-\frac{u}{1-u}\right)^2+\left(\frac{t}{1-t}-\frac{v}{1-v}\right)^2\right\}\\&\leq-2M\leq f(s,t)-f(u,v)\leq 2M\\&\leq\varepsilon+\frac{2M}{\delta^2}\left\{\left(\frac{s}{1-s}-\frac{u}{1-u}\right)^2+\left(\frac{t}{1-t}-\frac{v}{1-v}\right)^2\right\}.\end{aligned}$$

Since each T_{mn} is positive and linear it follows from (3) that

$$-\varepsilon T_{mn}f_0-\frac{2M}{\delta^2}T_{mn}g_{uv}\leq T_{mn}f-f(u,v)T_{mn}f_0\leq\varepsilon T_{mn}f_0+\frac{2M}{\delta^2}T_{mn}g_{uv}.$$

Therefore

$$\begin{aligned}|T_{mn}(f;u,v)&-f(u,v)T_{mn}(f_0;u,v)|\\&\leq\varepsilon+\varepsilon\left[T_{mn}(f_0;u,v)-f_0(u,v)\right]+\frac{2M}{\delta^2}T_{mn}g_{uv}\\&\leq\varepsilon+\varepsilon\|T_{mn}f_0-f_0\|+\frac{2M}{\delta^2}T_{mn}g_{uv}\end{aligned}$$

In particular, note that

$$\begin{aligned}|T_{mn}&(f;u,v)-f(u,v)|\\&\leq|T_{mn}(f;u,v)-f(u,v)T_{mn}(f_0;u,v)|+|f(u,v)|\,|T_{mn}(f_0;u,v)-f_0(u,v)|\\&\leq\varepsilon+(M+\varepsilon)\|T_{mn}f_0-f_0\|+\frac{2M}{\delta^2}T_{mn}g_{uv}\end{aligned}$$

which implies

$$\begin{aligned}\|T_{mn}f-f\|\leq\varepsilon&+C_3\|T_{mn}f_3-f_3\|+C_2\|T_{mn}f_2-f_2\|\\&+C_1\|T_{mn}f_1-f_1\|+C_0\|T_{mn}f_0-f_0\|,\end{aligned}$$

where $C_0=\left[\frac{2M}{\delta^2}\{(\frac{A}{1-A})^2+(\frac{B}{1-B})^2\}+M+\varepsilon\right]$, $C_1=\frac{4M}{\delta^2}\frac{A}{1-A}$, $C_2=\frac{4M}{\delta^2}\frac{B}{1-B}$ and $C_3=\frac{2M}{\delta^2}$, i.e.,

$$\|T_{mn}f-f\|\leq\varepsilon+C\sum_{i=0}^{3}\|T_{mn}f_i-f_i\|,\ i=0,1,2,3,$$

where $C=\max\{C_0,\ C_1,\ C_2,\ C_3\}$.

For a given $\gamma>0$, choose $\varepsilon>0$ such that $\varepsilon<\gamma$. Now let

$$U=\{(m,n):\|T_{mn}f-f\|\geq\gamma\}$$

and

$$U_i=\left\{(m,n):\|T_{mn}f_i-f_i\|\geq\frac{\gamma-\varepsilon}{4C}\right\},\ i=0,1,2,3.$$

It follows that $U\subset\bigcup_{i=0}^{3}U_i$ and consequently for all $(j,k)\in\mathbb{N}^2$

$$\sum_{(m,n)\in U}a_{jkmn}\leq\sum_{i=0}^{3}\sum_{(m,n)\in U_i}a_{jkmn},$$

which implies that for any $\sigma > 0$ and $(m,n) \in U$,

$$\left\{(j,k) \in \mathbb{N}^2 : \sum_{(m,n)\in U} a_{jkmn} \geq \sigma\right\} \subseteq \bigcup_{i=0}^{3} \left\{(j,k) \in \mathbb{N}^2 : \sum_{(m,n)\in U_i} a_{jkmn} \geq \frac{\sigma}{3}\right\}.$$

Therefore from hypothesis, $\{(j,k) \in \mathbb{N}^2 : \sum_{(m,n)\in U} a_{jkmn} \geq \sigma\} \in \mathcal{I}$. This completes the proof of the theorem. ■

We now show that our theorem is stronger than the A-statistical version [12] (and so the classical version). Let $\mathcal{I}$ be a non-trivial strongly admissible ideal of $\mathbb{N} \times \mathbb{N}$. Choose an infinite subset $C = \{(p_i, q_i) : i \in \mathbb{N}\}$, from $\mathcal{I}$ such that $p_i \neq q_i$ for all i, $p_1 < p_2 < \cdots$ and $q_1 < q_2 < \cdots$. Let $\{u_{mn}\}_{m,n\in\mathbb{N}}$ be given by

$$u_{mn} = \begin{cases} 1 & m, n \text{ are even} \\ 0 & \text{otherwise.} \end{cases}$$

Let $A = (a_{jkmn})$ be given by

$$a_{jkmn} = \begin{cases} 1 & \text{if } j = p_i, k = q_i, m = 2p_i, n = 2q_i \text{ for some } i \in \mathbb{N} \\ 1 & \text{if } (j,k) \neq (p_i, q_i), \text{ for any } i, m = 2j+1, n = 2k+1 \\ 0 & \text{otherwise.} \end{cases}$$

Now for $0 < \varepsilon < 1$, $K_2(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : |u_{mn} - 0| \geq \varepsilon\} = \{(m,n) : m, n \text{ are even}\}$. Observe that

$$\sum_{(m,n)\in K_2(\varepsilon)} a_{jkmn} = \begin{cases} 1 & \text{if } j = p_i, k = q_i \text{ for some } i \in \mathbb{N} \\ 0 & \text{if } (j,k) \neq (p_i, q_i), \text{ for any } i \in \mathbb{N}. \end{cases}$$

Thus for any $\delta > 0$,

$$\left\{(j,k) \in \mathbb{N} \times \mathbb{N} : \sum_{(m,n)\in K_2(\varepsilon)} a_{jkmn} \geq \delta\right\} = C \in \mathcal{I},$$

which shows that $\{u_{mn}\}_{m,n\in\mathbb{N}}$ is $A_2^{\mathcal{I}}$-statistically convergent to 0. Evidently this sequence is not A-statistically convergent to 0.

Consider the following Meyer-König and Zeler operators

$$M_{mn}(f; x, y) = (1-x)^{m+1}(1-y)^{n+1} \times \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} f\left(\frac{k}{k+m+1}, \frac{l}{l+m+1}\right) \binom{m+k}{k} \binom{n+l}{l} x^k y^l$$

where $f \in H_w(\mathcal{K})$ and $\mathcal{K} = [0, A] \times [0, B]$, $A, B \in (0,1)$. Then $M_{mn}(f_0; x, y) = f_0(x, y)$, $M_{mn}(f_1; x, y) = \frac{x}{1-x}$, $M_{mn}(f_2; x, y) = \frac{y}{1-y}$ and

$$M_{mn}(f_3; x, y) = \frac{m+2}{m+1}\left(\frac{x}{1-x}\right)^2 + \frac{1}{m+1}\frac{x}{1-x} + \frac{n+2}{n+1}\left(\frac{y}{1-y}\right)^2 + \frac{1}{n+1}\frac{y}{1-y}.$$

Then $\lim_{m,n} \|M_{mn}f - f\| = 0$.

Now consider the following positive linear operator T_{mn} on $H_w(\mathcal{K})$ defined by $T_{mn}(f;x,y) = (1+u_{mn})M_{mn}(f;x,y)$. It is easy to observe that $\|T_{mn}f_i - f_i\| = u_{mn}$ for $i = 0,1,2$ which imply that $A_2^{\mathcal{I}}$-st-$\lim_{m,n} \|T_{mn}f_i - f_i\| = 0,\ i = 0,1,2$. Again,

$$\begin{aligned}
\|T_{mn}f_3 - f_3\| &= \Big\|(1+u_{mn})\Big\{\frac{m+2}{m+1}\Big(\frac{x}{1-x}\Big)^2 + \frac{1}{m+1}\frac{x}{1-x} + \frac{n+2}{n+1}\Big(\frac{y}{1-y}\Big)^2 \\
&\quad + \frac{1}{n+1}\frac{y}{1-y}\Big\} - \Big(\frac{x}{1-x}\Big)^2 - \Big(\frac{y}{1-y}\Big)^2\Big\| \\
&\le D\left\{\frac{2}{m+1} + \frac{2}{n+1} + u_{mn}\frac{m+3}{m+1} + u_{mn}\frac{n+3}{n+1}\right\},
\end{aligned}$$

where $D = \max\left\{(\frac{A}{1-A})^2, (\frac{B}{1-B})^2, (\frac{A}{1-A}), (\frac{A}{1-A})\right\}$. Therefore

$$\begin{aligned}
&\big\{(m,n)\in\mathbb{N}^2 : \|T_{mn}(f_3) - f_3\| \ge \varepsilon\big\} \\
&\quad\subseteq \left\{(m,n)\in\mathbb{N}^2 : \frac{1}{m+1} + \frac{1}{n+1} \ge \frac{\varepsilon}{4D}\right\} \\
&\qquad\cup \left\{(m,n)\in\mathbb{N}^2 : u_{mn}\frac{m+3}{m+1} + u_{mn}\frac{n+3}{n+1} \ge \frac{\varepsilon}{2D}\right\} \\
&\quad\subseteq \left\{(m,n)\in\mathbb{N}^2 : \frac{1}{m+1} + \frac{1}{n+1} \ge \frac{\varepsilon}{4D}\right\} \\
&\qquad\cup \left\{(m,n)\in\mathbb{N}^2 : u_{mn} + \frac{m+3}{m+1} + \frac{n+3}{n+1} \ge 2\sqrt{\frac{\varepsilon}{2D}}\right\} \\
&\quad\subseteq \left\{(m,n)\in\mathbb{N}^2 : \frac{1}{m+1} + \frac{1}{n+1} \ge \frac{\varepsilon}{4D}\right\} \cup \left\{(m,n)\in\mathbb{N}^2 : u_{mn} \ge \sqrt{\frac{\varepsilon}{2D}}\right\} \\
&\qquad\cup \left\{(m,n)\in\mathbb{N}^2 : \frac{m+3}{m+1} + \frac{n+3}{n+1} \ge \sqrt{\frac{\varepsilon}{2D}}\right\} \\
&\quad\subseteq \left\{(m,n)\in\mathbb{N}^2 : \frac{1}{m+1} + \frac{1}{n+1} \ge \frac{\varepsilon}{4D}\right\} \cup \left\{(m,n)\in\mathbb{N}^2 : u_{mn} \ge \sqrt{\frac{\varepsilon}{2D}}\right\} \\
&\qquad\cup \left\{(m,n)\in\mathbb{N}^2 : \frac{1}{m+1} + \frac{1}{n+1} \ge \frac{1}{6}\sqrt{\frac{\varepsilon}{2D}}\right\} \\
&\qquad\cup \left\{(m,n)\in\mathbb{N}^2 : \frac{m}{m+1} + \frac{n}{n+1} \ge \frac{1}{2}\sqrt{\frac{\varepsilon}{2D}}\right\},
\end{aligned}$$

which implies that $A_2^{\mathcal{I}}$-st-$\lim_{m,n} \|T_{mn}f_3 - f_3\| = 0$. Hence from previous theorem it follows that $A_2^{\mathcal{I}}$-st-$\lim_{m,n} \|T_{mn}f - f\| = 0$ for any $f \in H_w(\mathcal{K})$. But since $\{u_{mn}\}_{m,n\in\mathbb{N}}$ is not A-statistically convergent so the sequence $\{T_{mn}(f;x,y)\}_{m,n\in\mathbb{N}}$ considered above does not converge A-statistically to the function $f \in H_w(\mathcal{K})$.

3. Rate of $A_2^{\mathcal{I}}$-statistical convergence

In this section we present a way to compute the rate of $A_2^{\mathcal{I}}$-statistical convergence in Theorem 2.1. We will need the following definitions.

Definition 3.1. Let $A=(a_{jkmn})$ be a non-negative RH-regular summability matrix and let $\{\alpha_{mn}\}_{m,n\in\mathbb{N}}$ be a positive non-increasing double sequence. Then a real double sequence $\{x_{mn}\}_{m,n\in\mathbb{N}}$ is said to be $A_2^{\mathcal{I}}$-statistically convergent to a number L with the rate of $o(\alpha_{mn})$ if for every $\varepsilon>0$ and $\delta>0$,

$$\left\{(j,k)\in\mathbb{N}^2:\frac{1}{\alpha_{jk}}\sum_{(m,n)\in K_2(\varepsilon)}a_{jkmn}\geq\delta\right\}\in\mathcal{I},$$

where $K_2(\varepsilon)=\{(m,n)\in\mathbb{N}^2:\ |x_{mn}-L|\geq\varepsilon\}$. In this case, we write $A_2^{\mathcal{I}}$-*st*-$o(\alpha_{mn})$-$\lim_{m,n}x_{mn}=L$.

Definition 3.2. Let $A=(a_{jkmn})$ be a non-negative RH-regular summability matrix and let $\{\alpha_{mn}\}_{m,n\in\mathbb{N}}$ be a positive non-increasing double sequence. Then a real double sequence $\{x_{mn}\}_{m,n\in\mathbb{N}}$ is said to be $A_2^{\mathcal{I}}$-statistically convergent to a number L with the rate of $o_m(\alpha_{mn})$ if for every $\varepsilon>0$ and $\delta>0$,

$$\left\{(j,k)\in\mathbb{N}^2:\sum_{(m,n)\in K_2(\varepsilon)}a_{jkmn}\geq\delta\right\}\in\mathcal{I},$$

where $K_2(\varepsilon)=\{(m,n)\in\mathbb{N}^2:\ |x_{mn}-L|\geq\varepsilon\alpha_{mn}\}$. In this case, we write $A_2^{\mathcal{I}}$-*st*-$o_m(\alpha_{mn})$-$\lim_{m,n}x_{mn}=L$.

Lemma 3.1. *Let $\{x_{mn}\}_{m,n\in\mathbb{N}}$ and $\{y_{mn}\}_{m,n\in\mathbb{N}}$ be double sequences. Assume that $A=(a_{jkmn})$ is a non-negative RH-regular summability matrix and let $\{\alpha_{mn}\}_{m,n\in\mathbb{N}}$ and $\{\beta_{mn}\}_{m,n\in\mathbb{N}}$ be positive non-increasing double sequences. If*

$$A_2^{\mathcal{I}}\text{-}st\text{-}o(\alpha_{mn})\text{-}\lim_{m,n}x_{mn}=L_1\ \text{ and }\ A_2^{\mathcal{I}}\text{-}st\text{-}o(\beta_{mn})\text{-}\lim_{m,n}x_{mn}=L_2$$

then we have

(*i*) $A_2^{\mathcal{I}}$-*st*-$o(\gamma_{mn})$-$\lim\limits_{m,n}(x_{mn}\pm y_{mn})=L_1\pm L_2$ *where* $\gamma_{mn}=\max\{\alpha_{mn},\beta_{mn}\}$,

(*ii*) $A_2^{\mathcal{I}}$-*st*-$o(\alpha_{mn})$-$\lim\limits_{m,n}\lambda x_{mn}=\lambda L_1$ *for any real number* λ.

Proof. The proof is straightforward and so is omitted. ∎

Lemma 3.2. *Let $\{x_{mn}\}_{m,n\in\mathbb{N}}$ and $\{y_{mn}\}_{m,n\in\mathbb{N}}$ be double sequences. Assume that $A=(a_{jkmn})$ is a non-negative RH-regular summability matrix and let $\{\alpha_{mn}\}_{m,n\in\mathbb{N}}$ and $\{\beta_{mn}\}_{m,n\in\mathbb{N}}$ be positive non-increasing double sequences. If*

$$A_2^{\mathcal{I}}\text{-}st\text{-}o_m(\alpha_{mn})\text{-}\lim_{m,n}x_{mn}=L_1\ \text{ and }\ A_2^{\mathcal{I}}\text{-}st\text{-}o_m(\beta_{mn})\text{-}\lim_{m,n}x_{mn}=L_2$$

then we have

(*i*) $A_2^{\mathcal{I}}$-*st*-$o_m(\gamma_{mn})$-$\lim\limits_{m,n}(x_{mn}\pm y_{mn})=L_1\pm L_2$ *where* $\gamma_{mn}=\max\{\alpha_{mn},\beta_{mn}\}$,

(*ii*) $A_2^{\mathcal{I}}$-*st*-$o_m(\alpha_{mn})$-$\lim\limits_{m,n}\lambda x_{mn}=\lambda L_1$ *for any real number* λ.

Proof. The proof is straightforward and so is omitted. ■

Now we prove the following theorem.

THEOREM 3.1. *Let $\{T_{mn}\}_{m,n\in\mathbb{N}}$ be a sequence of positive linear operators from $H_w(\mathcal{K})$ into $C(\mathcal{K})$. Let $A=(a_{jkmn})$ be a non-negative RH-regular summability matrix and $\{\alpha_{mn}\}_{m,n\in\mathbb{N}}$ and $\{\beta_{mn}\}_{m,n\in\mathbb{N}}$ be positive non-increasing double sequences. Assume that the following conditions hold*

$$(i)\ A_2^{\mathcal{I}}\text{-}st\text{-}o(\alpha_{mn})\text{-}\lim_{m,n}\|T_{mn}f_0-f_0\|=0,$$

$$(ii)\ A_2^{\mathcal{I}}\text{-}st\text{-}o(\beta_{mn})\text{-}\lim_{m,n} w(f;\delta_{mn})=0,$$

where $\delta:=\delta_{mn}=\sqrt{\|T_{mn}(\psi)\|}$ with $\psi(u,v)=(\frac{x}{1-x}-\frac{u}{1-u})^2+(\frac{y}{1-y}-\frac{v}{1-v})^2$. Then for any $f\in H_w(\mathcal{K})$,

$$A_2^{\mathcal{I}}\text{-}st\text{-}o(\gamma_{mn})\text{-}\lim_{m,n}\|T_{mn}f-f\|=0,$$

where $\gamma_{mn}=\max\{\alpha_{mn},\beta_{mn}\}$ for each $(m,n)\in\mathbb{N}^2$.

Proof. Let $\{T_{mn}\}_{m,n\in\mathbb{N}}$ be a sequence of positive linear operators from $H_w(\mathcal{K})$ into $C(\mathcal{K})$ and let $A=(a_{jkmn})$ be a non-negative RH-regular summability matrix and $N=\|f\|$. Then for any $f\in H_w(\mathcal{K})$,

$$\begin{aligned}
&|T_{mn}(f;u,v)-f(u,v)|\\
&\le T_{mn}(|f(x,y)-f(u,v)|;u,v)+|f(u,v)||T_{mn}(f_0;u,v)-f_0(u,v)|\\
&\le w(f;\delta)T_{mn}\left(1+\frac{\sqrt{(\frac{u}{1-u}-\frac{x}{1-x})^2+(\frac{v}{1-v}-\frac{y}{1-y})^2}}{\delta};u,v\right)\\
&\qquad+N|T_{mn}(f_0;u,v)-f_0(u,v)|\\
&=w(f;\delta)T_{mn}(f_0;u,v)+w(f;\delta)T_{mn}\left(\frac{\sqrt{(\frac{u}{1-u}-\frac{x}{1-x})^2+(\frac{v}{1-v}-\frac{y}{1-y})^2}}{\delta};u,v\right)\\
&\quad+N|T_{mn}(f_0;u,v)-f_0(u,v)|\\
&=w(f;\delta)T_{mn}(f_0;u,v)-w(f;\delta)f_0(u,v)+w(f;\delta)+\frac{w(f;\delta)}{\delta^2}T_{mn}(\psi;u,v)\\
&\quad+N|T_{mn}(f_0;u,v)-f_0(u,v)|\\
&\le w(f;\delta)|T_{mn}(f_0;u,v)-f_0(u,v)|+w(f;\delta)+\frac{w(f;\delta)}{\delta^2}T_{mn}(\psi;u,v)\\
&\quad+N|T_{mn}(f_0;u,v)-f_0(u,v)|.
\end{aligned}$$

Taking supremum over $(u,v)\in\mathcal{K}$,

$$\|T_{mn}f-f\|\le w(f;\delta)\|T_{mn}f_0-f_0\|+w(f;\delta)+\frac{w(f;\delta)}{\delta^2}\|T_{mn}\psi\|+N\|T_{mn}f_0-f_0\|.$$

If we take $\delta:=\delta_{mn}=\sqrt{\|T_{mn}\psi\|}$ then

$$\begin{aligned}
\|T_{mn}f-f\|&\le w(f;\delta)\|T_{mn}f_0-f_0\|+2w(f;\delta)+N\|T_{mn}f_0-f_0\|\\
&\le M\{w(f;\delta)\|T_{mn}f_0-f_0\|+w(f;\delta)+\|T_{mn}f_0-f_0\|\},
\end{aligned}$$

where $M = \max\{2, N\}$. Let $\mu > 0$ be given. Now consider the following sets

$$\begin{aligned} U &= \{(m,n) : \|T_{mn}f - f\| \geq \mu\}, \\ U_1 &= \{(m,n) : w(f;\delta) \geq \frac{\mu}{3M}\}, \\ U_2 &= \{(m,n) : \|T_{mn}f_0 - f_0\| \geq \frac{\mu}{3M}\}, \\ U_3 &= \{(m,n) : w(f;\delta)\|T_{mn}f_0 - f_0\| \geq \frac{\mu}{3M}\}. \end{aligned}$$

Then $U \subset U_1 \cup U_2 \cup U_3$. Now define

$$\begin{aligned} U_3' &= \{(m,n) : w(f;\delta) \geq \sqrt{\frac{\mu}{3M}}\}, \\ U_3'' &= \{(m,n) : \|T_{mn}f_0 - f_0\| \geq \sqrt{\frac{\mu}{3M}}\}. \end{aligned}$$

Then $U \subset U_1 \cup U_2 \cup U_3' \cup U_3''$. Now since $\gamma_{mn} = \max\{\alpha_{mn}, \beta_{mn}\}$ for each $(m,n) \in \mathbb{N}^2$ then for all $(j,k) \in \mathbb{N}^2$,

$$\begin{aligned} \frac{1}{\gamma_{j,k}} \sum_{(m,n)\in U} a_{jkmn} \leq \frac{1}{\beta_{j,k}} \sum_{(m,n)\in U_1} a_{jkmn} + \frac{1}{\alpha_{j,k}} \sum_{(m,n)\in U_2} a_{jkmn} \\ + \frac{1}{\beta_{j,k}} \sum_{(m,n)\in U_3'} a_{jkmn} + \frac{1}{\alpha_{j,k}} \sum_{(m,n)\in U_3''} a_{jkmn}. \end{aligned}$$

Then for any $\sigma > 0$

$$\begin{aligned} &\left\{(j,k) \in \mathbb{N}^2 : \frac{1}{\gamma_{j,k}} \sum_{(m,n)\in U} a_{jkmn} \geq \sigma\right\} \\ &\subseteq \left\{(j,k) \in \mathbb{N}^2 : \frac{1}{\beta_{j,k}} \sum_{(m,n)\in U_1} a_{jkmn} \geq \frac{\sigma}{4}\right\} \cup \left\{(j,k) \in \mathbb{N}^2 : \frac{1}{\alpha_{j,k}} \sum_{(m,n)\in U_2} a_{jkmn} \geq \frac{\sigma}{4}\right\} \\ &\cup \left\{(j,k) \in \mathbb{N}^2 : \frac{1}{\beta_{j,k}} \sum_{(m,n)\in U_3'} a_{jkmn} \geq \frac{\sigma}{4}\right\} \cup \left\{(j,k) \in \mathbb{N}^2 : \frac{1}{\alpha_{j,k}} \sum_{(m,n)\in U_3''} a_{jkmn} \geq \frac{\sigma}{4}\right\}. \end{aligned}$$

Now from hypothesis the sets on the right-hand side belong to $\mathcal{I}$ and consequently

$$\left\{(j,k) \in \mathbb{N}^2 : \frac{1}{\gamma_{j,k}} \sum_{(m,n)\in U} a_{jkmn} \geq \sigma\right\} \in \mathcal{I}$$

for any $\sigma > 0$. This completes the proof. ∎

The proof of the following theorem is analogous to the proof of Theorem 3.1 and so is omitted.

Theorem 3.2. *Let $\{T_{mn}\}_{m,n\in\mathbb{N}}$ be a sequence of positive linear operators from $H_w(\mathcal{K})$ into $C(\mathcal{K})$. Let $A = (a_{jkmn})$ be a non-negative RH-regular summability matrix and $\{\alpha_{mn}\}_{m,n\in\mathbb{N}}$ and $\{\beta_{mn}\}_{m,n\in\mathbb{N}}$ be positive non-increasing double sequences.*

Assume that the following conditions hold

$$(i)\ A_2^{\mathcal{I}}\text{-}st\text{-}o_m(\alpha_{mn})\text{-}\lim_{m,n} \|T_{mn}f_0 - f_0\| = 0,$$

$$(ii)\ A_2^{\mathcal{I}}\text{-}st\text{-}o_m(\beta_{mn})\text{-}\lim_{m,n} w(f;\delta_{mn}) = 0,$$

where $\delta_{mn} = \sqrt{\|T_{mn}(\psi)\|}$ *with* $\psi(u,v) = (\frac{x}{1-x} - \frac{u}{1-u})^2 + (\frac{y}{1-y} - \frac{v}{1-v})^2$. *Then for any* $f \in H_w(\mathcal{K})$,

$$A_2^{\mathcal{I}}\text{-}st\text{-}o_m(\gamma_{mn})\text{-}\lim_{m,n} \|T_{mn}(f) - f\| = 0,$$

where $\gamma_{mn} = \max\{\alpha_{mn}, \beta_{mn}\}$ *for each* $(m,n) \in \mathbb{N}^2$.

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