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INTEGRAL INEQUALITIES OF JENSEN TYPE FOR λ-CONVEX FUNCTIONS

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Abstract. Some integral inequalities of Jensen type for λ-convex functions defined on real intervals are given.

1. Introduction

Assume that I and J are intervals in $\mathbb{R}, (0,1) \subseteq J$ and functions h and f are real non-negative functions defined in J and I, respectively.

Definition 1. [20] Let $h: J \to [0,\infty)$ with h not identical to 0. We say that $f: I \to [0,\infty)$ is an h-convex function if for all $x, y \in I$ we have

$$f(tx + (1-t)y) \le h(t)f(x) + h(1-t)f(y) \tag{1.1}$$

for all $t \in (0,1)$.

For some results concerning this class of functions see [3, 13, 17–20].

This class of functions contains the class of Godunova-Levin type functions [9, 10, 14, 16]. It also contains the class of P functions and quasi-convex functions. For some results on P-functions see [15] while for quasi convex functions, the reader can consult [11].

Definition 2. [4] Let s be a real number, $s \in (0,1]$. A function $f: [0,\infty) \to [0,\infty)$ is said to be s-convex (in the second sense) or Breckner s-convex if

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0,\infty)$ and $t \in [0,1]$.

For some properties of this class of functions see [1, 2, 4, 7, 8, 12].

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We can introduce now another class of functions defined on a convex subset C of a linear space X that contains as limiting cases the classes of Godunova-Levin and P-functions.

Definition 3. We say that the function $f\colon C \subseteq X \to [0,\infty)$ is of s-Godunova-Levin type, with $s \in [0,1]$, if

$$f(tx+(1-t)y) \le \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y), \tag{1.2}$$

for all $t \in (0,1)$ and $x, y \in C$.

We observe that for $s = 0$ we obtain the class of P-functions while for $s = 1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(C)$ the class of s-Godunova-Levin functions defined on C, then we obviously have

$$P(C) = Q_0(C) \subseteq Q_{s_1}(C) \subseteq Q_{s_2}(C) \subseteq Q_1(C) = Q(C)$$

for $0 \le s_1 \le s_2 \le 1$.

For different inequalities of Hermite-Hadamard or Jensen type related to these classes of functions, see [1, 3, 13, 15–19].

A function $h\colon J \to \mathbb{R}$ is said to be *supermultiplicative* if

$$h(ts) \ge h(t)h(s) \text{ for any } t, s \in J. \tag{1.3}$$

If the inequality (1.3) is reversed, then h is said to be *submultiplicative*. If the equality holds in (1.3) then h is said to be a multiplicative function on J.

In [15], we introduced the following concept of functions:

Definition 4. Let $\lambda\colon [0,\infty) \to [0,\infty)$ be a function with the property that $\lambda(t) > 0$ for all $t > 0$. A mapping $f\colon C \to \mathbb{R}$ defined on convex subset C of a linear space X is called λ-convex on C if

$$f\left(\frac{\alpha x + \beta y}{\alpha+\beta}\right) \le \frac{\lambda(\alpha) f(x) + \lambda(\beta) f(y)}{\lambda(\alpha+\beta)} \tag{1.4}$$

for all $\alpha, \beta \ge 0$ with $\alpha + \beta > 0$ and $x, y \in C$.

We observe that if $f\colon C \to \mathbb{R}$ is λ-convex on C, then f is h-convex on C with $h(t) = \frac{\lambda(t)}{\lambda(1)}$, $t \in [0,1]$. If $f\colon C \to [0,\infty)$ is h-convex function with h supermultiplicative on $[0,\infty)$, then f is λ-convex with $\lambda = h$.

We have the following result providing many examples of subadditive functions $\lambda\colon [0,\infty) \to [0,\infty)$.

Theorem 1. [5] *Let $h(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients $a_n \ge 0$ for all $n \in \mathbb{N}$ and convergent on the open disk $D(0,R)$ with $R > 0$ or $R = \infty$. If $r \in (0,R)$ then the function $\lambda_r\colon [0,\infty) \to [0,\infty)$ given by*

$$\lambda_r(t) := \ln\left[\frac{h(r)}{h(r\exp(-t))}\right] \tag{1.5}$$

is nonnegative, increasing and subadditive on $[0,\infty)$.

Now, if we take $h(z) = \frac{1}{1-z}$, $z \in D(0,1)$, then

$$\lambda_r(t) = \ln\left[\frac{1 - r\exp(-t)}{1-r}\right] \tag{1.6}$$

is nonnegative, increasing and subadditive on $[0,\infty)$ for any $r \in (0,1)$.

If we take $h(z) = \exp(z)$, $z \in \mathbb{C}$ then

$$\lambda_r(t) = r[1 - \exp(-t)] \tag{1.7}$$

is nonnegative, increasing and subadditive on $[0,\infty)$ for any $r > 0$.

Corollary 1. [5] *Let $h(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients $a_n \geq 0$ for all $n \in \mathbb{N}$ and convergent on the open disk $D(0,R)$ with $R > 0$ or $R = \infty$ and $r \in (0,R)$. For a mapping $f\colon C \to \mathbb{R}$ defined on convex subset C of a linear space X, the following statements are equivalent:*

(i) The function f is λ_r-convex with $\lambda_r\colon [0,\infty) \to [0,\infty)$,

$$\lambda_r(t) := \ln\left[\frac{h(r)}{h(r\exp(-t))}\right];$$

(ii) We have the inequality

$$\left[\frac{h(r)}{h(r\exp(-\alpha-\beta))}\right]^{f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)} \leq \left[\frac{h(r)}{h(r\exp(-\alpha))}\right]^{f(x)} \left[\frac{h(r)}{h(r\exp(-\beta))}\right]^{f(y)} \tag{1.8}$$

for any $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$ and $x, y \in C$.

(iii) We have the inequality

$$\frac{[h(r\exp(-\alpha))]^{f(x)}[h(r\exp(-\beta))]^{f(y)}}{[h(r\exp(-\alpha-\beta))]^{f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)}} \leq [h(r)]^{f(x)+f(y)-f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)} \tag{1.9}$$

for any $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$ and $x, y \in C$.

We observe that, in the case when

$$\lambda_r(t) = r[1 - \exp(-t)],\ t \geq 0$$

then the function f is λ_r-convex on convex subset C of a linear space X iff

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right) \leq \frac{[1-\exp(-\alpha)]f(x) + [1-\exp(-\beta)]f(y)}{1-\exp(-\alpha-\beta)} \tag{1.10}$$

for any $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$ and $x, y \in C$. Notice that this definition is independent of $r > 0$.

The inequality (1.10) is equivalent to

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right) \leq \frac{\exp(\beta)[\exp(\alpha)-1]f(x) + \exp(\alpha)[\exp(\beta)-1]f(y)}{\exp(\alpha+\beta)-1} \tag{1.11}$$

for any $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$ and $x, y \in C$.

Motivated by the large interest on Jensen and Hermite-Hadamard inequalities that has been materialized in the last two decades by the publication of hundreds of papers, we establish here some inequalities of these types for λ-convex functions defined on real intervals.

2. Unweighted Jensen integral inequalities

The following discrete inequality of Jensen type has been obtained in [6]:

Theorem 2. *Let $\lambda\colon[0,\infty)\to[0,\infty)$ be a function with the property that $\lambda(t)>0$ for all $t>0$ and a mapping $f\colon C\to\mathbb{R}$ defined on convex subset C of a linear space X. The following statements are equivalent:*

(i) f is λ-convex on C;

(ii) For all $x_i\in C$ and $p_i\geq 0$ with $i\in\{1,\dots,n\}$, $n\geq 2$ so that $P_n>0$, we have the inequality

$$f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)\leq \frac{1}{\lambda(P_n)}\sum_{i=1}^n \lambda(p_i)f(x_i). \tag{2.1}$$

The proof can be done by induction over $n\geq 2$.

Corollary 2. *Let $f\colon C\to\mathbb{R}$ be a λ-convex function on C and $\alpha_i\in[0,1]$, $i\in\{1,\dots,n\}$ with $\sum_{i=1}^n\alpha_i=1$. Then for any $x_i\in C$ with $i\in\{1,\dots,n\}$ we have the inequality*

$$f\left(\sum_{i=1}^n \alpha_i x_i\right)\leq \frac{1}{\lambda(1)}\sum_{i=1}^n \lambda(\alpha_i)f(x_i). \tag{2.2}$$

In particular, we have

$$f\left(\frac{x_1+\dots+x_n}{n}\right)\leq c(n)\frac{f(x_1)+\dots+f(x_n)}{n}, \tag{2.3}$$

where

$$c(n):=\frac{n\lambda(\frac{1}{n})}{\lambda(1)}.$$

We have the following version of Jensen's inequality as well:

Corollary 3. *Let $f\colon C\to\mathbb{R}$ be a λ-convex function on C and $x_i\in C$ and $p_i\geq 0$ with $i\in\{1,\dots,n\}$, $n\geq 2$ so that $P_n>0$. Then we have the inequality*

$$f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)\leq \frac{1}{\lambda(1)}\sum_{i=1}^n \lambda\left(\frac{p_i}{P_n}\right)f(x_i). \tag{2.4}$$

The proof follows by (2.2) for $\alpha_i=\frac{p_i}{P_n}$, $i\in\{1,\dots,n\}$.

We are able now to state and prove the following unweighted Jensen inequality for Riemann integral:

Theorem 3. *Let $u\colon[a,b]\to[m,M]$ be a Riemann integrable function on $[a,b]$. Let $\lambda\colon[0,\infty)\to[0,\infty)$ be a function with the property that $\lambda(t)>0$ for all $t>0$*

and the function $f\colon [m,M] \to [0,\infty)$ *is* λ *-convex and Riemann integrable on the interval* $[m,M]$*. If the following limit exists*

$$\lim_{t\to 0+} \frac{\lambda(t)}{t} = k \in (0,\infty) \tag{2.5}$$

then

$$f\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right) \le \frac{k}{\lambda(b-a)}\int_a^b f(u(t))\,dt. \tag{2.6}$$

Proof. Consider the sequence of divisions

$$d_n:\ x_i^{(n)} = a + \frac{i}{n}(b-a),\ i\in\{0,\dots,n\}$$

and the intermediate points

$$\xi_i^{(n)} = a + \frac{i}{n}(b-a),\ i\in\{0,\dots,n\}.$$

We observe that the norm of the division $\Delta_n := \max_{i\in\{0,\dots,n-1\}}(x_{i+1}^{(n)} - x_i^{(n)}) = \frac{b-a}{n} \to 0$ as $n\to\infty$ and since u is Riemann integrable on $[a,b]$, then

$$\int_a^b u(t)\,dt = \lim_{n\to\infty}\sum_{i=0}^{n-1} u(\xi_i^{(n)})[x_{i+1}^{(n)} - x_i^{(n)}] = \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=0}^{n-1} u\left(a+\frac{i}{n}(b-a)\right).$$

Also, since $f\colon [m,M]\to[0,\infty)$ is Riemann integrable, then $f\circ u$ is Riemann integrable and

$$\int_a^b f(u(t))\,dt = \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=0}^{n-1} f\left[u(a+\frac{i}{n}(b-a))\right].$$

Utilising the inequality (2.1) for $p_i := \frac{b-a}{n}$ and $x_i := u(a+\frac{i}{n}(b-a))$ we have

$$\begin{aligned} f&\left(\frac{1}{b-a}\frac{b-a}{n}\sum_{i=0}^{n-1} u\left(a+\frac{i}{n}(b-a)\right)\right)\\ &\le \frac{n}{\lambda(b-a)(b-a)}\lambda\left(\frac{b-a}{n}\right)\frac{b-a}{n}\sum_{i=0}^{n-1} f\left(u\left(a+\frac{i}{n}(b-a)\right)\right) \end{aligned} \tag{2.7}$$

for any $n\ge 1$.

Observe that

$$\lim_{n\to\infty}\frac{\lambda(\frac{b-a}{n})}{\frac{b-a}{n}} = \lim_{t\to 0+}\frac{\lambda(t)}{t} = k\in(0,\infty),$$

and by taking the limit over $n\to\infty$ in the inequality (2.7), we deduce the desired result (2.6). ■

Corollary 4. *Let* $u\colon [a,b]\to[m,M]$ *be a Riemann integrable function on* $[a,b]$ *and* $h(z) = \sum_{n=0}^{\infty} a_n z^n$ *be a power series with nonnegative coefficients* $a_n \ge 0$

for all $n \in \mathbb{N}$ *and convergent on the open disk* $D(0,R)$ *with* $R > 0$ *or* $R = \infty$ *and* $r \in (0,R)$. *Let* $\lambda_r\colon [0,\infty) \to [0,\infty)$ *be given by*

$$\lambda_r(t) := \ln\left[\frac{h(r)}{h(r\exp(-t))}\right]$$

and the function $f\colon [m,M] \to [0,\infty)$ *be* λ_r*-convex and Riemann integrable on the interval* $[m,M]$. *Then*

$$f\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right) \le \frac{rh'(r)}{h(r)\ln\left[\frac{h(r)}{h(r\exp(-(b-a)))}\right]}\int_a^b f(u(t))\,dt. \tag{2.8}$$

Proof. We observe that λ_r is differentiable on $(0,\infty)$ and

$$\lambda_r'(t) := \frac{r\exp(-t)h'(r\exp(-t))}{h(r\exp(-t))}$$

for $t \in (0,\infty)$, where $h'(z) = \sum_{n=1}^{\infty} na_n z^{n-1}$. Since $\lambda_r(0) = 0$, therefore

$$k = \lim_{s\to 0+}\frac{\lambda(s)}{s} = \lambda_+'(0) = \frac{rh'(r)}{h(r)} > 0 \text{ for } r \in (0,R).$$

Utilising (2.6) we get the desired result (2.8). ∎

The following Hermite-Hadamard type inequality holds:

Corollary 5. *With the assumptions of Theorem 3 for* f *and* λ *and if* $[a,b] = [m,M]$*, we have the Hermite-Hadamard type inequality*

$$f\left(\frac{a+b}{2}\right) \le \frac{k}{\lambda(b-a)}\int_a^b f(t)\,dt. \tag{2.9}$$

Remark 1. Assume that the function $f\colon [m,M] \to [0,\infty)$ is λ-convex and Riemann integrable on the interval $[m,M]$ with

$$\lambda(t) = 1 - \exp(-t),\ t \ge 0.$$

If $u\colon [a,b] \to [m,M]$ is a Riemann integrable function on $[a,b]$, then

$$f\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right) \le \frac{1}{1-\exp(-(b-a))}\int_a^b f(u(t))\,dt.$$

In particular, for $[a,b] = [m,M]$ and $u(t) = t$ we have the Hermite-Hadamard type inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{1-\exp(-(b-a))}\int_a^b f(t)\,dt.$$

The proof follows from (2.6) observing that

$$k = \lim_{t\to 0+}\frac{\lambda(t)}{t} = \lambda_+'(0) = 1.$$

Utilising a similar argument and the inequality (2.4) we can state the following result as well:

Theorem 4. *Let* $u\colon [a,b] \to [m,M]$ *be a Riemann integrable function on* $[a,b]$*. Let* $\lambda\colon [0,\infty) \to [0,\infty)$ *be a function with the property that* $\lambda(t) > 0$ *for all* $t > 0$ *and the function* $f\colon [m,M] \to [0,\infty)$ *is* λ *-convex and Riemann integrable on the interval* $[m,M]$*. If the limit* (2.5) *exists, then*

$$f\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right) \le \frac{k}{\lambda(1)(b-a)}\int_a^b f(u(t))\,dt. \tag{2.10}$$

Examples of such inequalities are incorporated below:

Corollary 6. *Let* $u\colon [a,b] \to [m,M]$ *be a Riemann integrable function on* $[a,b]$ *and* $h(z) = \sum_{n=0}^{\infty} a_n z^n$ *be a power series with nonnegative coefficients* $a_n \ge 0$ *for all* $n \in \mathbb{N}$ *and convergent on the open disk* $D(0,R)$ *with* $R > 0$ *or* $R = \infty$ *and* $r \in (0,R)$*. Let* $\lambda_r\colon [0,\infty) \to [0,\infty)$ *be given by*

$$\lambda_r(t) := \ln\left[\frac{h(r)}{h(r\exp(-t))}\right]$$

and the function $f\colon [m,M] \to [0,\infty)$ *be* λ_r*-convex and Riemann integrable on the interval* $[m,M]$*. Then*

$$f\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right) \le \frac{rh'(r)}{(b-a)h(r)\ln\left[\frac{h(r)}{h(re^{-1})}\right]}\int_a^b f(u(t))\,dt. \tag{2.11}$$

We also have the Hermite-Hadamard type inequality:

Corollary 7. *With the assumptions of Theorem 4 for* f *and* λ *and if* $[a,b] = [m,M]$*, we have the Hermite-Hadamard type inequality*

$$f\left(\frac{a+b}{2}\right) \le \frac{k}{\lambda(1)(b-a)}\int_a^b f(t)\,dt. \tag{2.12}$$

Remark 2. Assume that the function $f\colon [m,M] \to [0,\infty)$ is λ-convex and Riemann integrable on the interval $[m,M]$ with $\lambda(t) = 1 - \exp(-t)$, $t \ge 0$. If $u\colon [a,b] \to [m,M]$ is a Riemann integrable function on $[a,b]$, then

$$f\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right) \le \frac{e}{e-1}\cdot\frac{1}{b-a}\int_a^b f(u(t))\,dt.$$

In particular, for $[a,b] = [m,M]$ and $u(t) = t$ we have the Hermite-Hadamard type inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{e}{e-1}\cdot\frac{1}{b-a}\int_a^b f(t)\,dt.$$

3. Weighted Jensen integral inequalities

We can prove now a weighted version of Jensen inequality.

Theorem 5. *Let $u, w\colon [a,b] \to [m,M]$ be Riemann integrable functions on $[a,b]$ and $w(t) \geq 0$ for any $t \in [a,b]$ with $\int_a^b w(t)\,dt > 0$. Let $\lambda\colon [0,\infty) \to [0,\infty)$ be a function with the property that $\lambda(t) > 0$ for all $t > 0$ and the function $f\colon [m,M] \to [0,\infty)$ is λ-convex and Riemann integrable on the interval $[m,M]$. If the following limit exists, is finite and*

$$\lim_{t\to\infty} \frac{t}{\lambda(t)} = \ell > 0, \tag{3.1}$$

then

$$f\left(\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)u(t)\,dt\right) \leq \ell \frac{1}{\int_a^b w(t)\,dt}\int_a^b \lambda(w(t))f(u(t))\,dt. \tag{3.2}$$

Proof. Consider the sequence of divisions

$$d_n:\ x_i^{(n)} = a + \frac{i}{n}(b-a),\ i \in \{0,\dots,n\}$$

and the intermediate points

$$\xi_i^{(n)} = a + \frac{i}{n}(b-a),\ i \in \{0,\dots,n\}.$$

We observe that the norm of the division $\Delta_n := \max_{i\in\{0,\dots,n-1\}}(x_{i+1}^{(n)} - x_i^{(n)}) = \frac{b-a}{n} \to 0$ as $n \to \infty$.

If we write the inequality (2.1) for the sequences

$$p_i = w\left(a + \frac{i}{n}(b-a)\right) \text{ and } x_i = u\left(a + \frac{i}{n}(b-a)\right),\ i \in \{0,\dots,n\}$$

we get

$$\begin{aligned} &f\left(\frac{1}{\sum_{i=0}^{n-1} w(a+\frac{i}{n}(b-a))}\sum_{i=0}^{n-1} w\left(a+\frac{i}{n}(b-a)\right)u\left(a+\frac{i}{n}(b-a)\right)\right) \\ &\quad\leq \frac{1}{\lambda(\sum_{i=0}^{n-1} w(a+\frac{i}{n}(b-a)))} \\ &\quad\quad\times \sum_{i=0}^{n-1} \lambda\left(w\left(a+\frac{i}{n}(b-a)\right)\right) f\left(u\left(a+\frac{i}{n}(b-a)\right)\right), \end{aligned} \tag{3.3}$$

for $n \geq 1$.

Observe that

$$\begin{aligned} &f\left(\frac{1}{\sum_{i=0}^{n-1} w(a+\frac{i}{n}(b-a))}\sum_{i=0}^{n-1} w\left(a+\frac{i}{n}(b-a)\right)u\left(a+\frac{i}{n}(b-a)\right)\right) \\ &= f\left(\frac{\frac{b-a}{n}}{\frac{b-a}{n}\sum_{i=0}^{n-1} w(a+\frac{i}{n}(b-a))}\sum_{i=0}^{n-1} w\left(a+\frac{i}{n}(b-a)\right)u\left(a+\frac{i}{n}(b-a)\right)\right) \end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\lambda(\sum_{i=0}^{n-1} w(a+\frac{i}{n}(b-a)))} \\
&\quad \times \sum_{i=0}^{n-1} \lambda\left(w\left(a+\frac{i}{n}(b-a)\right)\right) f\left(u\left(a+\frac{i}{n}(b-a)\right)\right) \\
&= \frac{\sum_{i=0}^{n-1} w(a+\frac{i}{n}(b-a))}{\lambda(\sum_{i=0}^{n-1} w(a+\frac{i}{n}(b-a)))} \times \frac{1}{\frac{b-a}{n}\sum_{i=0}^{n-1} w(a+\frac{i}{n}(b-a))} \\
&\quad \times \frac{b-a}{n}\sum_{i=0}^{n-1} \lambda\left(w\left(a+\frac{i}{n}(b-a)\right)\right) f\left(u\left(a+\frac{i}{n}(b-a)\right)\right).
\end{aligned}$$

Then from (3.3) we get

$$\begin{aligned}
&f\left(\frac{\frac{b-a}{n}}{\frac{b-a}{n}\sum_{i=0}^{n-1} w(a+\frac{i}{n}(b-a))}\sum_{i=0}^{n-1} w\left(a+\frac{i}{n}(b-a)\right) u\left(a+\frac{i}{n}(b-a)\right)\right) \\
&\leq \frac{\sum_{i=0}^{n-1} w(a+\frac{i}{n}(b-a))}{\lambda(\sum_{i=0}^{n-1} w(a+\frac{i}{n}(b-a)))} \times \frac{1}{\frac{b-a}{n}\sum_{i=0}^{n-1} w(a+\frac{i}{n}(b-a))} \\
&\quad \times \frac{b-a}{n}\sum_{i=0}^{n-1} \lambda\left(w\left(a+\frac{i}{n}(b-a)\right)\right) f\left(u\left(a+\frac{i}{n}(b-a)\right)\right) \qquad (3.4)
\end{aligned}$$

for all $n \geq 1$. Since

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=0}^{n-1} w\left(a+\frac{i}{n}(b-a)\right) = \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=0}^{n-1} w\left(a+\frac{i}{n}(b-a)\right) \times \lim_{n\to\infty}\frac{n}{b-a} \\
&= \int_a^b w(t)\,dt \times \infty = \infty,
\end{aligned}$$

then

$$\lim_{n\to\infty}\frac{\sum_{i=0}^{n-1} w(a+\frac{i}{n}(b-a))}{\lambda(\sum_{i=0}^{n-1} w(a+\frac{i}{n}(b-a)))} = \lim_{n\to\infty}\frac{t}{\lambda(t)} = \ell$$

and by letting $n \to \infty$ in (3.4) we get the desired result (3.2). ■

The following unweighted version of Jensen inequality holds:

Corollary 8. *Let $u\colon [a,b] \to [m,M]$ be a Riemann integrable function on $[a,b]$. Let $\lambda\colon [0,\infty) \to [0,\infty)$ be a function with the property that $\lambda(t) > 0$ for all $t > 0$ and the function $f\colon [m,M] \to [0,\infty)$ be λ -convex and Riemann integrable on the interval $[m,M]$. If the limit* (3.1) *exists, then*

$$f\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right) \leq \ell\lambda(1)\frac{1}{b-a}\int_a^b f(u(t))\,dt. \qquad (3.5)$$

Moreover, if $[a,b] = [m,M]$, then by taking $u(t) = t$, $t \in [a,b]$, we have the Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \ell\lambda(1)\frac{1}{b-a}\int_a^b f(t)\,dt. \qquad (3.6)$$

Remark 3. In order to give examples of subadditive functions $\lambda\colon[0,\infty)\to[0,\infty)$ with the property that $\lambda(t)>0$ for all $t>0$ and for which the following limit exists, is finite and

$$\lim_{t\to\infty}\frac{t}{\lambda(t)}=\ell>0, \tag{3.7}$$

we consider the power series $h(z)=\sum_{n=1}^{\infty}a_nz^n$ with nonnegative coefficients $a_n\geq 0$ for all $n\geq 1$, $a_1>0$ and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$.

Let $\lambda_r\colon[0,\infty)\to[0,\infty)$ be given by

$$\lambda_r(t):=\ln\left[\frac{h(r)}{h(r\exp(-t))}\right].$$

We know that λ_r is differentiable on $(0,\infty)$ and

$$\lambda_r'(t)=\frac{r\exp(-t)h'(r\exp(-t))}{h(r\exp(-t))}$$

for $t\in(0,\infty)$, where $h'(z)=\sum_{n=1}^{\infty}na_nz^{n-1}$. By l'Hospital's rule we have

$$\lim_{t\to\infty}\frac{t}{\lambda_r(t)}=\lim_{t\to\infty}\frac{1}{\lambda_r'(t)}.$$

Since for the power series $h(z)=a_1z+a_2z^2+a_3z^3+\cdots$ we have $h'(z)=a_1+2a_2z+3a_3z^2+\cdots$, then

$$\begin{aligned}\lambda_r'(t)&=\frac{r\exp(-t)(a_1+2a_2r\exp(-t)+3a_3(r\exp(-t))^2+\cdots)}{r\exp(-t)(a_1+a_2r\exp(-t)+a_3(r\exp(-t))^2+\cdots)}\\&=\frac{a_1+2a_2r\exp(-t)+3a_3(r\exp(-t))^2+\cdots}{a_1+a_2r\exp(-t)+a_3(r\exp(-t))^2+\cdots},\ t\in(0,\infty).\end{aligned}$$

Therefore $\lim_{t\to\infty}\lambda_r'(t)=1$ and $\lim_{t\to\infty}\frac{t}{\lambda_r(t)}=1$.

Corollary 9. *Let $u,w\colon[a,b]\to[m,M]$ be Riemann integrable functions on $[a,b]$ and $w(t)\geq 0$ for any $t\in[a,b]$ with $\int_a^b w(t)\,dt>0$. Consider the power series $h(z)=\sum_{n=1}^{\infty}a_nz^n$ with nonnegative coefficients $a_n\geq 0$ for all $n\geq 1$, $a_1>0$ and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$. Let $r\in(0,R)$ and assume that the function $f\colon[m,M]\to[0,\infty)$ is λ_r-convex and Riemann integrable on the interval $[m,M]$ with*

$$\lambda_r(t):=\ln\left[\frac{h(r)}{h(r\exp(-t))}\right].$$

Then we have the inequality

$$f\left(\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)u(t)\,dt\right)\leq\frac{1}{\int_a^b w(t)\,dt}\int_a^b\ln\left[\frac{h(r)}{h(r\exp(-w(t)))}\right]f(u(t))\,dt. \tag{3.8}$$

The proof follows by Theorem 5 observing that $\ell = 1$.

Remark 4. With the assumptions of Corollary 9 for u, h and f we have

$$f\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right) \le \ln\left[\frac{h(r)}{h(re^{-1})}\right]\frac{1}{b-a}\int_a^b f(u(t))\,dt. \tag{3.9}$$

In particular, for $[a,b] = [m,M]$ we have the Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \le \ln\left[\frac{h(r)}{h(re^{-1})}\right]\frac{1}{b-a}\int_a^b f(t)\,dt. \tag{3.10}$$

4. Interval dependency

Let $u\colon [a,b] \to [m,M]$ be a Riemann integrable function on $[a,b]$. Let $\lambda\colon [0,\infty) \to [0,\infty)$ be a function with the property that $\lambda(t) > 0$ for all $t > 0$ and the function $f\colon [m,M] \to [0,\infty)$ be λ-convex and Riemann integrable on the interval $[m,M]$. Assume also that the following limit exists

$$\lim_{t\to 0+}\frac{\lambda(t)}{t} = k \in (0,\infty).$$

By Theorem 3 we have that

$$\Delta(f,u,\lambda;[a,b]) := \int_a^b f(u(t))\,dt - \frac{1}{k}\lambda(b-a) f\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right) \ge 0. \tag{4.1}$$

Theorem 6. *With the above assumptions for u, λ, f and k we have:*

(i) For any $c \in (a,b)$ we have

$$\Delta(f,u,\lambda;[a,b]) \ge \Delta(f,u,\lambda;[a,c]) + \Delta(f,u,\lambda;[c,b]) \ge 0, \tag{4.2}$$

i.e., $\Delta(f,u,\lambda;\cdot)$ is a superadditive function of intervals.

(ii) For any $c,d \in (a,b)$ with $c < d$ we have

$$\Delta(f,u,\lambda;[a,b]) \ge \Delta(f,u,\lambda;[c,d]) \ge 0, \tag{4.3}$$

i.e., $\Delta(f,u,\lambda;\cdot)$ is a monotonic nondecreasing function of intervals.

Proof. (i) By the λ-convexity of f we have for $c \in (a,b)$ that

$$\begin{aligned}
&f\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right)\\
&\quad = f\left(\frac{c-a}{b-a}\left(\frac{1}{c-a}\int_a^c u(t)\,dt\right) + \frac{b-c}{b-a}\left(\frac{1}{b-c}\int_c^b u(t)\,dt\right)\right)\\
&\quad \le \frac{\lambda(c-a) f(\frac{1}{c-a}\int_a^c u(t)\,dt) + \lambda(b-c) f(\frac{1}{b-c}\int_c^b u(t)\,dt)}{\lambda(b-a)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\Delta(f,u,\lambda;[a,b]) \\
&= \int_a^c f(u(t))\,dt + \int_c^b f(u(t))\,dt - \frac{1}{k}\lambda(b-a) f\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right) \\
&\geq \int_a^c f(u(t))\,dt + \int_c^b f(u(t))\,dt - \frac{1}{k}\lambda(b-a) \\
&\quad\times \left[\frac{\lambda(c-a) f(\frac{1}{c-a}\int_a^c u(t)\,dt) + \lambda(b-c) f(\frac{1}{b-c}\int_c^b u(t)\,dt)}{\lambda(b-a)}\right] \\
&= \Delta(f,u,\lambda;[a,c]) + \Delta(f,u,\lambda;[c,b])
\end{aligned}$$

and the inequality (4.2) is proved.

(ii) Obvious by the property (4.2). ∎

REMARK 5. If $[a,b]=[m,M]$ and $u(t)=t$, $t\in[a,b]$ then the functional

$$\delta(f,\lambda;[a,b]) := \int_a^b f(t)\,dt - \frac{1}{k}\lambda(b-a) f\left(\frac{a+b}{2}\right) \geq 0$$

is a superadditive and monotonic nondecreasing function of intervals.

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