МАТЕМАТИЧКИ ВЕСНИК
68, 2 (2016), 119–129
June 2016

оригинални научни рад
research paper

RADICAL TRANSVERSAL SCREEN SEMI-SLANT LIGHTLIKE SUBMANIFOLDS OF INDEFINITE KAEHLER MANIFOLDS

S. S. Shukla and Akhilesh Yadav

Abstract. In this paper, we introduce the notion of radical transversal screen semi-slant lightlike submanifolds of indefinite Kaehler manifolds giving characterization theorem with some non-trivial examples of such submanifolds. Integrability conditions of distributions D_1, D_2 and $RadTM$ on radical transversal screen semi-slant lightlike submanifolds of indefinite Kaehler manifolds have been obtained. Further, we obtain necessary and sufficient conditions for foliations determined by above distributions to be totally geodesic.

1. Introduction

The theory of lightlike submanifolds of a semi-Riemannian manifold was introduced by Duggal and Bejancu [2]. A submanifold M of a semi-Riemannian manifold $\overline{M}$ is said to be lightlike submanifold if the induced metric g on M is degenerate, i.e., there exists a non-zero $X \in \Gamma(TM)$ such that $g(X,Y) = 0$, $\forall Y \in \Gamma(TM)$. Various classes of lightlike submanifolds of indefinite Kaehler manifolds have been defined according to the behaviour of distributions on these submanifolds with respect to the action of (1,1) tensor field $\overline{J}$ in Kaehler structure of the ambient manifolds. Such submanifolds have been studied in [3, 7].

The geometry of slant submanifolds of Kaehler manifolds was studied by B. Y. Chen in [1] and the geometry of semi-slant submanifolds of Kaehler manifolds was studied by N. Papaghuic in [5]. In [6], Sahin studied screen-slant lightlike submanifolds of an indefinite Hermitian manifold. The theory of radical transversal, transversal, semi-transversal lightlike submanifolds has been studied in [8]. In [9–11], the authors studied lightlike submanifolds, radical transversal lightlike submanifolds and radical transversal screen semi-slant lightlike submanifolds. In this paper, we introduce the notion of radical transversal screen semi-slant lightlike submanifolds of indefinite Kaehler manifolds. This new class of lightlike submanifolds of an indefinite Kaehler manifold includes radical transversal and transversal lightlike submanifolds as its sub-cases.

2010 Mathematics Subject Classification: 53C15, 53C40, 53C50

Keywords and phrases: Semi-Riemannian manifold; degenerate metric; radical distribution; screen distribution; screen transversal vector bundle; lightlike transversal vector bundle; Gauss and Weingarten formulae.

The paper is arranged as follows. There are some basic results in Section 2. In Section 3, we introduce radical transversal screen semi-slant lightlike submanifolds of an indefinite Kaehler manifold, giving some examples. Section 4 is devoted to the study of foliations determined by distributions on radical transversal screen semi-slant lightlike submanifolds of indefinite Kaehler manifolds.

2. Preliminaries

A submanifold (M^m, g) immersed in a semi-Riemannian manifold $(\overline{M}^{m+n}, \overline{g})$ is called a lightlike submanifold [2] if the metric g induced from $\overline{g}$ is degenerate and the radical distribution $RadTM$ is of rank r, where $1 \leq r \leq m$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $RadTM$ in TM, that is

$$TM = RadTM \oplus_{orth} S(TM).$$

Now consider a screen transversal vector bundle $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $RadTM$ in $TM^{\perp}$. Since for any local basis $\{\xi_i\}$ of $RadTM$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\overline{g}(\xi_i, N_j) = \delta_{ij}$ and $\overline{g}(N_i, N_j) = 0$, it follows that there exists a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$. Let $tr(TM)$ be complementary (but not orthogonal) vector bundle to TM in $T\overline{M}|_M$. Then

$$tr(TM) = ltr(TM) \oplus_{orth} S(TM^{\perp}),$$
$$T\overline{M}|_M = TM \oplus tr(TM),$$
$$T\overline{M}|_M = S(TM) \oplus_{orth} [RadTM \oplus ltr(TM)] \oplus_{orth} S(TM^{\perp}).$$

Following are four cases of a lightlike submanifold $\big(M, g, S(TM), S(TM^{\perp})\big)$:

Case 1. r-lightlike if $r < \min(m, n)$,

Case 2. co-isotropic if $r = n < m$, $S\left(TM^{\perp}\right) = \{0\}$,

Case 3. isotropic if $r = m < n$, $S\left(TM\right) = \{0\}$,

Case 4. totally lightlike if $r = m = n$, $S(TM) = S(TM^{\perp}) = \{0\}$.

The Gauss and Weingarten formulae are given as

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$
$$\overline{\nabla}_X V = -A_V X + \nabla^t_X V, \tag{2.2}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(tr(TM))$, where $\nabla_X Y, A_V X$ belong to $\Gamma(TM)$ and $h(X, Y), \nabla^t_X V$ belong to $\Gamma(tr(TM))$. ∇ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$ respectively. The second fundamental form h is a symmetric $F(M)$-bilinear form on $\Gamma(TM)$ with values in $\Gamma(tr(TM))$ and the shape operator A_V is a linear endomorphism of $\Gamma(TM)$. From (2.1) and (2.2), for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$, we have

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \tag{2.3}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N), \tag{2.4}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W), \tag{2.5}$$

where $h^l(X,Y) = L(h(X,Y))$, $h^s(X,Y) = S(h(X,Y))$, $D^l(X,W) = L(\nabla^t_X W)$, $D^s(X,N) = S(\nabla^t_X N)$. L and S are the projection morphisms of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$ respectively. ∇^l and ∇^s are linear connections on $ltr(TM)$ and $S(TM^{\perp})$ called the lightlike connection and screen transversal connection on M respectively.

Now by using (2.1), (2.3)–(2.5) and metric connection $\overline{\nabla}$, we obtain

$$\overline{g}(h^s(X,Y),W) + \overline{g}(Y, D^l(X,W)) = g(A_W X, Y),$$
$$\overline{g}(D^s(X,N),W) = \overline{g}(N, A_W X).$$

Denote the projection of TM on $S(TM)$ by $\overline{P}$. Then from the decomposition of the tangent bundle of a lightlike submanifold, for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$, we have

$$\nabla_X \overline{P}Y = \nabla^*_X \overline{P}Y + h^*(X, \overline{P}Y),$$
$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi,$$

By using the above equations, we obtain

$$\overline{g}(h^l(X,\overline{P}Y),\xi) = g(A^*_\xi X, \overline{P}Y),$$
$$\overline{g}(h^*(X,\overline{P}Y),N) = g(A_N X, \overline{P}Y),$$
$$\overline{g}(h^l(X,\xi),\xi) = 0, \qquad A^*_\xi \xi = 0.$$

It is important to note that in general ∇ is not a metric connection. Since $\overline{\nabla}$ is metric connection, by using (2.3), we get

$$(\nabla_X g)(Y,Z) = \overline{g}(h^l(X,Y),Z) + \overline{g}(h^l(X,Z),Y).$$

An indefinite almost Hermitian manifold $(\overline{M}, \overline{g}, \overline{J})$ is a $2m$-dimensional semi-Riemannian manifold $\overline{M}$ with semi-Riemannian metric $\overline{g}$ of constant index q, $0 < q < 2m$ and a (1, 1) tensor field $\overline{J}$ on $\overline{M}$ such that following conditions are satisfied:

$$\overline{J}^2 X = -X,$$
$$\overline{g}(\overline{J}X, \overline{J}Y) = \overline{g}(X,Y), \tag{2.6}$$

for all $X, Y \in \Gamma(T\overline{M})$.

An indefinite almost Hermitian manifold $(\overline{M}, \overline{g}, \overline{J})$ is called an indefinite Kaehler manifold if $\overline{J}$ is parallel with respect to $\overline{\nabla}$, i.e.,

$$(\overline{\nabla}_X \overline{J})Y = 0, \tag{2.7}$$

for all $X, Y \in \Gamma(T\overline{M})$, where $\overline{\nabla}$ is Levi-Civita connection with respect to $\overline{g}$.

3. Radical transversal screen semi-slant lightlike submanifolds

In this section, we introduce the notion of radical transversal screen semi-slant lightlike submanifolds of indefinite Kaehler manifolds. At first, we state the following lemma for later use:

LEMMA 3.1. *Let M be a $2q$-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$, of index $2q$ such that $2q < dim(M)$. Then the screen distribution $S(TM)$ on lightlike submanifold M is Riemannian.*

The proof of above Lemma follows as in Lemma 3.1 of [6], so we omit it.

DEFINITION 3.1. Let M be a $2q$-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$ of index $2q$ such that $2q < dim(M)$. Then we say that M is a radical transversal screen semi-slant lightlike submanifold of $\overline{M}$ if the following conditions are satisfied:

(i) $\overline{J}(RadTM) = ltr(TM)$,

(ii) there exist non-degenerate orthogonal distributions D_1 and D_2 on M such that $S(TM) = D_1 \oplus_{orth} D_2$,

(iii) the distribution D_1 is an invariant, i.e. $\overline{J}D_1 = D_1$,

(iv) the distribution D_2 is slant with angle $\theta(\neq 0)$, i.e. for each $x \in M$ and each non-zero vector $X \in (D_2)_x$, the angle θ between $\overline{J}X$ and the vector subspace $(D_2)_x$ is a non-zero constant, which is independent of the choice of $x \in M$ and $X \in (D_2)_x$.

This constant angle θ is called the slant angle of distribution D_2. A radical transversal screen semi-slant lightlike submanifold is said to be proper if $D_1 \neq \{0\}$, $D_2 \neq \{0\}$ and $\theta \neq \frac{\pi}{2}$.

From the above definition, we have the following decomposition

$$TM = RadTM \oplus_{orth} D_1 \oplus_{orth} D_2.$$

Let $(\mathbb{R}^{2m}_{2q}, \overline{g}, \overline{J})$ denote the manifold $\mathbb{R}^{2m}_{2q}$ with its usual Kaehler structure given by

$$\overline{g} = \tfrac{1}{4}\big(-\textstyle\sum_{i=1}^{q} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=q+1}^{m} dx^i \otimes dx^i + dy^i \otimes dy^i\big),$$
$$\overline{J}\big(\textstyle\sum_{i=1}^{m}(X_i\partial x_i + Y_i\partial y_i)\big) = \sum_{i=1}^{m}(Y_i\partial x_i - X_i\partial y_i),$$

where (x^i, y^i) are the cartesian coordinates on $\mathbb{R}^{2m}_{2q}$. Now we construct some examples of radical transversal screen semi-slant lightlike submanifolds of an indefinite Kaehler manifold.

EXAMPLE 1. Let $(\mathbb{R}^{12}_2, \overline{g}, \overline{J})$ be an indefinite Kaehler manifold, where $\overline{g}$ is of signature $(-,+,+,+,+,+,-,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6\}$.

Suppose M is a submanifold of $\mathbb{R}^{12}_2$ given by $x^1 = -y^2 = u_1$, $x^2 = -y^1 = u_2$, $x^3 = u_3 \cos\beta$, $y^3 = -u_4 \cos\beta$, $x^4 = u_4 \sin\beta$, $y^4 = u_3 \sin\beta$, $x^5 = u_5 \sin u_6$, $y^5 = u_5 \cos u_6$, $x^6 = \sin u_5$, $y^6 = \cos u_5$.

The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6\}$, where

$Z_1 = 2(\partial x_1 - \partial y_2)$, $Z_2 = 2(\partial x_2 - \partial y_1)$,

$Z_3 = 2(\cos\beta\partial x_3 + \sin\beta\partial y_4)$, $Z_4 = 2(\sin\beta\partial x_4 - \cos\beta\partial y_3)$,

$Z_5 = 2(\sin u_6\partial x_5 + \cos u_6\partial y_5 + \cos u_5\partial x_6 - \sin u_5\partial y_6)$,

$Z_6 = 2(u_5\cos u_6\partial x_5 - u_5\sin u_6\partial y_5)$.

Hence $RadTM = span\,\{Z_1, Z_2\}$ and $S(TM) = span\,\{Z_3, Z_4, Z_5, Z_6\}$.

Now $ltr(TM)$ is spanned by $N_1 = -\partial x_1 - \partial y_2$, $N_2 = -\partial x_2 - \partial y_1$ and $S(TM^\perp)$ is spanned by

$W_1 = 2(\sin\beta\partial x_3 - \cos\beta\partial y_4)$, $W_2 = 2(\cos\beta\partial x_4 + \sin\beta\partial y_3)$,

$W_3 = 2(\sin u_6\partial x_5 + \cos u_6\partial y_5 - \cos u_5\partial x_6 + \sin u_5\partial y_6)$,

$W_4 = 2(u_5\sin u_5\partial x_6 + u_5\cos u_5\partial y_6)$.

It follows that $\overline{J}Z_1 = 2N_2$ and $\overline{J}Z_2 = 2N_1$, which implies that $\overline{J}RadTM = ltr(TM)$. On the other hand, we can see that $D_1 = span\,\{Z_3, Z_4\}$ such that $\overline{J}Z_3 = Z_4$ and $\overline{J}Z_4 = -Z_3$, which implies that D_1 is invariant with respect to $\overline{J}$ and $D_2 = span\,\{Z_5, Z_6\}$ is a slant distribution with slant angle $\pi/4$. Hence M is a radical transversal screen semi-slant 2-lightlike submanifold of $\mathbb{R}^{12}_2$.

EXAMPLE 2. Let $(\mathbb{R}^{12}_2, \overline{g}, \overline{J})$ be an indefinite Kaehler manifold, where $\overline{g}$ is of signature $(-,+,+,+,+,+,-,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6\}$.

Suppose M is a submanifold of $\mathbb{R}^{12}_2$ given by $x^1 = u_1$, $y^1 = -u_2$, $x^2 = u_1\cos\alpha - u_2\sin\alpha$, $y^2 = u_1\sin\alpha + u_2\cos\alpha$, $x^3 = y^4 = u_3$, $x^4 = -y^3 = u_4$, $x^5 = u_5\cos\theta$, $y^5 = u_6\cos\theta$, $x^6 = u_6\sin\theta$, $y^6 = u_5\sin\theta$.

The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6\}$, where

$Z_1 = 2(\partial x_1 + \cos\alpha\partial x_2 + \sin\alpha\partial y_2)$, $Z_2 = 2(-\partial y_1 - \sin\alpha\partial x_2 + \cos\alpha\partial y_2)$,

$Z_3 = 2(\partial x_3 + \partial y_4)$, $Z_4 = 2(\partial x_4 - \partial y_3)$,

$Z_5 = 2(\cos\theta\partial x_5 + \sin\theta\partial y_6)$, $Z_6 = 2(\sin\theta\partial x_6 + \cos\theta\partial y_5)$.

Hence $RadTM = span\,\{Z_1, Z_2\}$ and $S(TM) = span\,\{Z_3, Z_4, Z_5, Z_6\}$.

Now $ltr(TM)$ is spanned by $N_1 = -\partial x_1 + \cos\alpha\partial x_2 + \sin\alpha\partial y_2$, $N_2 = \partial y_1 - \sin\alpha\partial x_2 + \cos\alpha\partial y_2$ and $S(TM^\perp)$ is spanned by

$W_1 = 2(\partial x_3 - \partial y_4)$, $W_2 = 2(\partial x_4 + \partial y_3)$,

$W_3 = 2(\sin\theta\partial x_5 - \cos\theta\partial y_6)$, $W_4 = 2(\cos\theta\partial x_6 - \sin\theta\partial y_5)$.

It follows that $\overline{J}Z_1 = -2N_2$, $\overline{J}Z_2 = 2N_1$, which implies that $\overline{J}RadTM = ltr(TM)$. On the other hand, we can see that $D_1 = span\,\{Z_3, Z_4\}$ such that $\overline{J}Z_3 = Z_4$, $\overline{J}Z_4 = -Z_3$, which implies that D_1 is invariant with respect to $\overline{J}$ and $D_2 = span\,\{Z_5, Z_6\}$ is a slant distribution with slant angle 2θ. Hence M is a radical transversal screen semi-slant 2-lightlike submanifold of $\mathbb{R}^{12}_2$.

Now, for any vector field X tangent to M, we put $\overline{J}X = PX + FX$, where PX and FX are tangential and transversal parts of $\overline{J}X$ respectively. We denote the projections on $RadTM$, D_1 and D_2 in TM by P_1, P_2 and P_3 respectively. Then for any $X \in \Gamma(TM)$, we get

$$X = P_1X + P_2X + P_3X. \tag{3.1}$$

Now applying $\overline{J}$ to (3.1), we have

$$\overline{J}X = \overline{J}P_1X + \overline{J}P_2X + \overline{J}P_3X,$$

which gives

$$\overline{J}X = \overline{J}P_1X + \overline{J}P_2X + fP_3X + FP_3X, \tag{3.2}$$

where fP_3X (resp. FP_3X) denotes the tangential (resp. transversal) component of $\overline{J}P_3X$. Thus we get $\overline{J}P_1X \in \Gamma(ltr(TM)), \overline{J}P_2X \in \Gamma(D_1), fP_3X \in \Gamma(D_2)$ and $FP_3X \in \Gamma(S(TM^\perp))$.

Similarly, we denote the projections of $tr(TM)$ on $ltr(TM)$ and $S(TM^\perp)$ by Q_1 and Q_2 respectively. Then for any $W \in \Gamma(tr(TM))$, we have

$$W = Q_1W + Q_2W. \tag{3.3}$$

Applying $\overline{J}$ to (3.3), we obtain

$$\overline{J}W = \overline{J}Q_1W + \overline{J}Q_2W,$$

which gives

$$\overline{J}W = \overline{J}Q_1W + BQ_2W + CQ_2W, \tag{3.4}$$

where BQ_2W (resp. CQ_2W) denotes the tangential (resp. transversal) component of $\overline{J}Q_2W$. Thus we get $\overline{J}Q_1W \in \Gamma(RadTM)$, $BQ_2W \in \Gamma(D_2)$ and $CQ_2W \in \Gamma(S(TM^\perp))$.

Now, by using (2.7), (3.2), (3.4) and (2.3)–(2.5) and identifying the components on $RadTM$, D_1, D_2, $ltr(TM)$ and $S(TM^\perp)$, we obtain

$$\begin{aligned} P_1(\nabla_X\overline{J}P_2Y) + P_1(\nabla_X fP_3Y) &= P_1(A_{\overline{J}P_1Y}X) + P_1(A_{FP_3Y}X) + \overline{J}h^l(X,Y),\\ P_2(\nabla_X\overline{J}P_2Y) + P_2(\nabla_X fP_3Y) &= P_2(A_{FP_3Y}X) + P_2(A_{\overline{J}P_1Y}X) + \overline{J}P_2\nabla_XY, \end{aligned} \tag{3.5}$$

$$\begin{aligned} &P_3(\nabla_X\overline{J}P_2Y) + P_3(\nabla_X fP_3Y)\\ &\qquad = P_3(A_{FP_3Y}X) + P_3(A_{\overline{J}P_1Y}X) + fP_3\nabla_XY + Bh^s(X,Y), \end{aligned} \tag{3.6}$$

$$\nabla^l_X\overline{J}P_1Y + h^l(X,\overline{J}P_2Y) + h^l(X,fP_3Y) = \overline{J}P_1\nabla_XY - D^l(X,FP_3Y), \tag{3.7}$$

$$\begin{aligned} &D^s(X,\overline{J}P_1Y) + h^s(X,\overline{J}P_2Y) + h^s(X,fP_3Y)\\ &\qquad = Ch^s(X,Y) - \nabla^s_X FP_3Y + FP_3\nabla_XY. \end{aligned} \tag{3.8}$$

THEOREM 3.2. *Let M be a $2q$-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then M is a radical transversal screen semi-slant lightlike submanifold if and only if*

(i) *$\overline{J}ltr(TM)$ is a distribution on M such that $\overline{J}ltr(TM) = RadTM$,*

(ii) *distribution D_1 is invariant with respect to $\overline{J}$, i.e. $\overline{J}D_1 = D_1$,*

(iii) *there exists a constant $\lambda \in [0, 1)$ such that $P^2X = -\lambda X$.*

Moreover, there also exists a constant $\mu \in (0, 1]$ such that $BFX = -\mu X$, for all $X \in \Gamma(D_2)$, where D_1 and D_2 are non-degenerate orthogonal distributions on M such that $S(TM) = D_1 \oplus_{orth} D_2$ and $\lambda = \cos^2\theta$, θ is slant angle of D_2.

Proof. Let M be a radical transversal screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then distribution D_1 is invariant with respect to $\overline{J}$ and $\overline{J}RadTM = ltr(TM)$. Thus $\overline{J}X \in ltr(TM)$, for all $X \in \Gamma(RadTM)$. Hence $\overline{J}(\overline{J}X) \in \overline{J}(ltr(TM))$, which implies $-X \in \overline{J}(ltr(TM))$, for all $X \in \Gamma(RadTM)$, which proves (i) and (ii).

Now for any $X \in \Gamma(D_2)$, we have $|PX| = |\overline{J}X|\cos\theta$, which implies

$$\cos\theta = \frac{|PX|}{|\overline{J}X|}. \tag{3.9}$$

In view of (3.9), we get $\cos^2\theta = \frac{|PX|^2}{|\overline{J}X|^2} = \frac{g(PX,PX)}{g(\overline{J}X,\overline{J}X)} = \frac{g(X,P^2X)}{g(X,\overline{J}^2X)}$, which gives

$$g(X, P^2X) = \cos^2\theta\, g(X, \overline{J}^2X). \tag{3.10}$$

Since M is radical transversal screen semi-slant lightlike submanifold, $\cos^2\theta = \lambda(\text{constant}) \in [0, 1)$ and therefore from (3.14), we get $g(X, P^2X) = \lambda g(X, \overline{J}^2X) = g(X, \lambda\overline{J}^2X)$, which implies

$$g(X, (P^2 - \lambda\overline{J}^2)X) = 0.$$

Now for any $X \in \Gamma(D_2)$, we have $\overline{J}^2(X) = P^2X + FPX + BFX + CFX$. Taking the tangential component, we get $P^2X = -X - BFX \in \Gamma(D_2)$, for any $X \in \Gamma(D_2)$. Thus $(P^2 - \lambda\overline{J}^2)X \in \Gamma(D_2)$. Since the induced metric $g = g|_{D_2\times D_2}$ is non-degenerate(positive definite), by the facts above, we have $(P^2 - \lambda\overline{J}^2)X = 0$, which implies

$$P^2X = \lambda\overline{J}^2X = -\lambda X. \tag{3.11}$$

Now, for any vector field $X \in \Gamma(D_2)$, we have

$$\overline{J}X = PX + FX, \tag{3.12}$$

where PX and FX are tangential and transversal parts of $\overline{J}X$ respectively.

Applying $\overline{J}$ to (3.12) and taking tangential component, we get

$$-X = P^2X + BFX. \tag{3.13}$$

From (3.11) and (3.13), we get $BFX = -\mu X$, where $1 - \lambda = \mu(\text{constant}) \in (0, 1]$. This proves (iii).

Conversely suppose that conditions (i), (ii) and (iii) are satisfied. From (i), we have $\overline{J}N \in RadTM$, for all $N \in \Gamma(ltr(TM))$. Hence $\overline{J}(\overline{J}N) \in \overline{J}(RadTM)$, which implies $-N \in \overline{J}(RadTM)$, for all $N \in \Gamma(ltr(TM))$. Thus $\overline{J}RadTM = ltr(TM)$. From (3.18), for any $X \in \Gamma(D_2)$, we get $-X = P^2X - \mu X$, which implies $P^2X = -\lambda X$, where $1 - \mu = \lambda(\text{constant}) \in [0, 1)$.

Now $\cos\theta = \frac{g(\overline{J}X,PX)}{|\overline{J}X||PX|} = -\frac{g(X,\overline{J}PX)}{|\overline{J}X||PX|} = -\frac{g(X,P^2X)}{|\overline{J}X||PX|} = -\lambda\frac{g(X,\overline{J}^2X)}{|\overline{J}X||PX|} = \lambda\frac{g(\overline{J}X,\overline{J}X)}{|\overline{J}X||PX|}$.

From the above equation, we get

$$\cos\theta = \lambda\frac{|\overline{J}X|}{|PX|}. \tag{3.14}$$

Therefore (3.9) and (3.14) give $\cos^2\theta = \lambda(\text{constant})$.

Hence M is a radical transversal screen semi-slant lightlike submanifold. ■

Corollary 3.1. *Let M be a radical transversal screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$ with slant angle θ, then for any $X, Y \in \Gamma(D_2)$, we have*

(i) $g(PX, PY) = \cos^2\theta\, g(X, Y)$,

(ii) $g(FX, FY) = \sin^2\theta\, g(X, Y)$.

The proof of above Corollary follows by using similar steps as in proof of Corollary 3.2 of [6].

Theorem 3.3. *Let M be a radical transversal screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$ with structure vector field tangent to M. Then $RadTM$ is integrable if and only if*

(i) $P_2(A_{\overline{J}P_1Y}X) = P_2(A_{\overline{J}P_1X}Y)$ *and* $P_3(A_{\overline{J}P_1Y}X) = P_3(A_{\overline{J}P_1X}Y)$,

(ii) $D^s(Y, \overline{J}P_1X) = D^s(X, \overline{J}P_1Y)$, *for all* $X, Y \in \Gamma(RadTM)$.

Proof. Let M be a radical transversal screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Let $X, Y \in \Gamma(RadTM)$. From (3.8), we have $D^s(X, \overline{J}P_1Y) = Ch^s(X, Y) + FP_3\nabla_XY$, which gives $D^s(X, \overline{J}P_1Y) - D^s(Y, \overline{J}P_1X) = FP_3[X, Y]$. In view of (3.5), we have $P_2(A_{\overline{J}P_1Y}X) + \overline{J}P_2\nabla_XY = 0$, which implies $P_2(A_{\overline{J}P_1X}Y) - P_2(A_{\overline{J}P_1Y}X) = \overline{J}P_2[X, Y]$. Also from (3.6), we have $P_3(A_{\overline{J}P_1Y}X) + Bh^s(X, Y) + fP_3\nabla_XY = 0$, which gives $P_3(A_{\overline{J}P_1X}Y) - P_3(A_{\overline{J}P_1Y}X) = fP_3[X, Y]$. This concludes the theorem. ■

Theorem 3.4. *Let M be a radical transversal screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then D_1 is integrable if and only if*

(i) $h^l(Y, \overline{J}P_2X) = h^l(X, \overline{J}P_2Y)$ *and* $h^s(Y, \overline{J}P_2X) = h^s(X, \overline{J}P_2Y)$,

(ii) $P_3(\nabla_X\overline{J}P_2Y) = P_3(\nabla_Y\overline{J}P_2X)$, *for all* $X, Y \in \Gamma(D_1)$.

Proof. Let M be a radical transversal screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Let $X, Y \in \Gamma(D_1)$. From (3.8), we have $h^s(X, \overline{J}P_2Y) = Ch^s(X,Y) + FP_3\nabla_X Y$, which implies $h^s(X,\overline{J}P_2Y) - h^s(Y,\overline{J}P_2X) = FP_3[X,Y]$. In view of (3.7), we have $h^l(X,\overline{J}P_2Y) = \overline{J}P_1\nabla_X Y$, which gives $h^l(X,\overline{J}P_2Y) - h^l(Y,\overline{J}P_2X) = \overline{J}P_1[X,Y]$. From (3.6), we obtain $P_3(\nabla_X\overline{J}P_2Y) = fP_3\nabla_X Y + Bh^s(X,Y)$, which implies $P_3(\nabla_X\overline{J}P_2Y) - P_3(\nabla_Y\overline{J}P_2X) = fP_3[X,Y]$. This proves the theorem. ∎

Theorem 3.5. *Let M be a radical transversal screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then D_2 is integrable if and only if*

(i) $h^l(X, fP_3Y) + D^l(X, FP_3Y) = h^l(Y, fP_3X) + D^l(Y, FP_3X)$,

(ii) $P_2(\nabla_X fP_3Y - \nabla_Y fP_3X) = P_2(A_{FP_3Y}X - A_{FP_3X}Y)$,

for all $X, Y \in \Gamma(D_2)$.

Proof. Let M be a radical transversal screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Let $X, Y \in \Gamma(D_2)$. From (3.7), we have $h^l(X, fP_3Y) + D^l(X, FP_3Y) = \overline{J}P_1\nabla_X Y$, which gives $h^l(X, fP_3Y) + D^l(X, FP_3Y) - h^l(Y, fP_3X) - D^l(Y, FP_3X) = \overline{J}P_1[X,Y]$. Also from (35), we obtain $P_2(\nabla_X fP_3Y) = P_2(A_{FP_3Y}X) + \overline{J}P_2\nabla_X Y$, which implies $P_2(\nabla_X fP_3Y - \nabla_Y fP_3X) = P_2(A_{FP_3Y}X - A_{FP_3X}Y) + \overline{J}P_2[X,Y]$. Thus, we obtain the required results. ∎

Theorem 3.6. *Let M be a radical transversal screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then the induced connection ∇ is a metric connection if and only if*

(i) $BD^s(X,N) = fP_3A_NX$,

(ii) $\overline{J}P_2A_NX = 0$, *for all $X \in \Gamma(TM)$ and $N \in \Gamma(ltr(TM))$.*

Proof. Let M be a radical transversal screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then the induced connection ∇ on M is a metric connection if and only if $RadTM$ is parallel distribution with respect to ∇ [2]. From (2.3), (2.4) and (2.7), we obtain $\nabla_X\overline{J}N + h^l(X,\overline{J}N) + h^s(X,\overline{J}N) = -\overline{J}A_NX + \overline{J}\nabla^l_X N + \overline{J}D^s(X,N)$. Now, on comparing tangential components of both sides of above equation, we get $\nabla_X\overline{J}N = -\overline{J}P_2A_NX - fP_3A_NX + \overline{J}\nabla^l_X N + BD^s(X,N)$, which completes the proof. ∎

4. Foliations determined by distributions

In this section, we obtain necessary and sufficient conditions for foliations determined by distributions on a radical transversal screen semi-slant lightlike submanifold of an indefinite Kaehler manifold to be totally geodesic.

Definition 4.1. An equivalence relation on an n-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ in which the equivalence classes are connected, immersed submanifolds (called the leaves of the foliation) of a common dimension k, $0 < k \leq n$ is called a foliation on $\overline{M}$. If each leaf of a foliation F on a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is totally geodesic submanifold of $\overline{M}$, we say that F is a totally geodesic foliation.

THEOREM 4.1. *Let M be a radical transversal screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then $RadTM$ defines a totally geodesic foliation if and only if $\overline{g}(A_{FP_3Z}X, \overline{J}Y) = \overline{g}(\nabla_X \overline{J}P_2Z + \nabla_X fP_3Z, \overline{J}Y)$, for all $X, Y \in \Gamma(RadTM)$ and $Z \in \Gamma(S(TM))$.*

Proof. Let M be a radical transversal screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. To prove $RadTM$ defines a totally geodesic foliation, it is sufficient to show that $\nabla_X Y \in \Gamma(RadTM)$, for all $X, Y \in \Gamma(RadTM)$. Since $\overline{\nabla}$ is metric connection, using (2.3), (2.6), (2.7) and (3.2), for any $X, Y \in \Gamma(RadTM)$ and $Z \in \Gamma(S(TM))$, we obtain $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\overline{\nabla}_X(\overline{J}P_2Z + fP_3Z + FP_3Z), \overline{J}Y)$, which implies $\overline{g}(\nabla_X Y, Z) = \overline{g}(P_1 A_{FP_3Z}X - P_1\nabla_X \overline{J}P_2Z - P_1\nabla_X fP_3Z, \overline{J}Y)$. This proves the theorem. ∎

THEOREM 4.2. *Let M be a radical transversal screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then D_1 defines a totally geodesic foliation if and only if*

(i) $\overline{g}(A_{FZ}X, \overline{J}Y) = \overline{g}(\nabla_X fZ, \overline{J}Y)$, *for all $X, Y \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$,*

(ii) $A^*_{\overline{J}N}$ *vanishes on D_1, for all $N \in \Gamma(ltr(TM))$.*

Proof. Let M be a radical transversal screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. The distribution D_1 defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(D_1)$, for all $X, Y \in \Gamma(D_1)$. Since $\overline{\nabla}$ is metric connection, from (2.3), (2.6) and (2.7), for any $X, Y \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$, we obtain $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\overline{\nabla}_X \overline{J}Z, \overline{J}Y)$, which implies $\overline{g}(\nabla_X Y, Z) = \overline{g}(A_{FZ}X - \nabla_X fZ, \overline{J}Y)$. Also, from (2.3), (2.6) and (2.7), for any $X, Y \in \Gamma(D_1)$ and $N \in \Gamma(ltr(TM))$, we have $\overline{g}(\nabla_X Y, N) = -\overline{g}(\overline{J}Y, \overline{\nabla}_X \overline{J}N)$, which gives $\overline{g}(\nabla_X Y, N) = -\overline{g}(\overline{J}Y, \nabla_X \overline{J}N) = \overline{g}(\overline{J}Y, A^*_{\overline{J}N}X)$. This concludes the theorem. ∎

THEOREM 4.3. *Let M be a radical transversal screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then D_2 defines a totally geodesic foliation if and only if*

(i) $\overline{g}(fY, \nabla_X \overline{J}Z) = -\overline{g}(FY, h^s(X, \overline{J}Z))$,

(ii) $\overline{g}(fY, \nabla_X \overline{J}N) = -\overline{g}(FY, h^s(X, \overline{J}N))$,

for all $X, Y \in \Gamma(D_2)$, $Z \in \Gamma(D_1)$ and $N \in \Gamma(ltr(TM))$.

Proof. Let M be a radical transversal screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. The distribution D_2 defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(D_2)$, for all $X, Y \in \Gamma(D_2)$. Since $\overline{\nabla}$ is metric connection, using (2.3), (2.6) and (2.7), for any $X, Y \in \Gamma(D_2)$ and $Z \in \Gamma(D_1)$, we get $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\overline{J}Y, \overline{\nabla}_X \overline{J}Z)$, which implies $\overline{g}(\nabla_X Y, Z) = -\overline{g}(fY, \nabla_X \overline{J}Z) - \overline{g}(FY, h^s(X, \overline{J}Z))$. Now, from (2.3), (2.6) and (2.7), for any $X, Y \in \Gamma(D_2)$ and $N \in$

$\Gamma(ltr(TM))$, we have $\overline{g}(\nabla_X Y, N) = -\overline{g}(\overline{J}Y, \overline{\nabla}_X \overline{J}N)$, which gives $\overline{g}(\nabla_X Y, N) = -\overline{g}(fY, \nabla_X \overline{J}N) - \overline{g}(FY, h^s(X, \overline{J}N))$. Thus, we obtain the required results. ∎

ACKNOWLEDGEMENT: Akhilesh Yadav gratefully acknowledges the financial support provided by the Council of Scientific and Industrial Research (C.S.I.R.), India.

REFERENCES

[**1**] B.Y. Chen, *Geometry of Slant Submanifolds*, Katholieke Universiteit, Leuven, 1990.

[**2**] K.L. Duggal and A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Vol. 364 of Mathematics and its applications, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.

[**3**] K.L. Duggal and B. Sahin, *Differential Geomety of Lightlike Submanifolds*, Birkhäuser Verlag AG, Basel, Boston, Berlin, 2010.

[**4**] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press New York, 1983.

[**5**] N. Papaghiuc, *Semi-slant submanifolds of a Kaehlerian manifold*, An. Stiint. Al.I.Cuza. Univ. Iasi, **40**, (1994), 55–61.

[**6**] B. Sahin, B., *Screen slant lightlike submanifolds*, Int. Electronic J. Geometry, **2** (2009), 41–54.

[**7**] B. Sahin,*Slant lightlike submanifolds of indefinite Hermitian manifolds*, Balkan J. Geometry Appl., **13** (1) (2008), 107–119.

[**8**] B. Sahin, *Transversal lightlike submanifolds of indefinite Kaehler manifolds*, Analele. Univ. Timisoara, **44** (1), (2006), 119–145.

[**9**] S.S. Shukla and A. Yadav, *Lightlike submanifolds of indefinite para-Sasakian manifolds*, Mat. Vesnik, **66** (4) (2014), 371–386.

[**10**] S.S. Shukla and A. Yadav, *Radical transversal lightlike submanifolds of indefinite para-Sasakian manifolds*, Demonstratio Math., **47** (4) (2014), 994–1011.

[**11**] S.S. Shukla and A. Yadav, *Radical transversal screen semi-slant lightlike submanifolds of indefinite Sasakian manifolds*, Lobachevskii J. Math. **36** (2) (2015), 160–168.

(received 13.05.2015; in revised form 17.01.2016; available online 03.02.2016)

Department of Mathematics, University of Allahabad, Allahabad-211002, India

E-mail: ssshukla_au@rediffmail.com, akhilesh_mathau@rediffmail.com