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COMMON FIXED POINTS IN b-METRIC SPACES ENDOWED WITH A GRAPH

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Abstract. We discuss the existence and uniqueness of points of coincidence and common fixed points for a pair of self-mappings defined on a b-metric space endowed with a graph. Our results improve and supplement several recent results of metric fixed point theory.

1. Introduction

Fixed point theory plays a major role in mathematics and applied sciences such as variational and linear inequalities, mathematical models, optimization, mathematical economics and the like. Different generalizations of the usual notion of a metric space were proposed by several mathematicians. In 1989, Bakhtin [5] introduced b-metric spaces as a generalization of metric spaces and generalized the famous Banach contraction principle in metric spaces to b-metric spaces. Since then, a series of articles have been dedicated to the improvement of fixed point theory in b-metric spaces.

In [17], Jungck introduced the concept of weak compatibility. Several authors have obtained coincidence points and common fixed points for various classes of mappings on a metric space by using this concept.

In recent investigations, the study of fixed point theory endowed with a graph occupies a prominent place in many aspects. In 2005, Echenique [13] studied fixed point theory by using graphs. Espinola and Kirk [14] applied fixed point results in graph theory. Recently, Jachymski [16] proved a sufficient condition for a selfmap f of a metric space (X, d) to be a Picard operator and applied it to the Kelisky-Rivlin theorem on iterates of the Bernstein operators on the space $C[0, 1]$.

Motivated by the idea given in some recent work on metric spaces with a graph (see [3,4,6–8]), we reformulate some important fixed point results in metric spaces to b-metric spaces endowed with a graph. As some consequences of our results, we obtain Banach contraction principle, Kannan fixed point theorem and Fisher fixed

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point theorem in metric spaces. Finally, some examples are provided to illustrate our results.

2. Some basic concepts

We begin with some basic notations, definitions, and necessary results in b-metric spaces.

DEFINITION 2.1. [12] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric on X if the following conditions hold:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq s\,(d(x, z) + d(z, y))$ for all $x, y, z \in X$.

The pair (X, d) is called a b-metric space.

It seems important to note that if $s = 1$, then the triangle inequality in a metric space is satisfied, however it does not hold true when $s > 1$. Thus the class of b-metric spaces is effectively larger than that of the ordinary metric spaces. The following example illustrates the above remarks.

EXAMPLE 2.2. [18] Let $X = \{-1, 0, 1\}$. Define $d : X \times X \to \mathbb{R}^+$ by $d(x, y) = d(y, x)$ for all $x, y \in X$, $d(x, x) = 0$, $x \in X$ and $d(-1, 0) = 3$, $d(-1, 1) = d(0, 1) = 1$. Then (X, d) is a b-metric space, but not a metric space since the triangle inequality is not satisfied. Indeed, we have that

$$d(-1, 1) + d(1, 0) = 1 + 1 = 2 < 3 = d(-1, 0).$$

It is easy to verify that $s = \frac{3}{2}$.

EXAMPLE 2.3. [19] Let (X, d) be a metric space and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. Then ρ is a b-metric with $s = 2^{p-1}$.

DEFINITION 2.4. [10] Let (X, d) be a b-metric space, $x \in X$ and (x_n) be a sequence in X. Then

(i) (x_n) converges to x if and only if $\lim_{n\to\infty} d(x_n, x) = 0$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ $(n \to \infty)$.

(ii) (x_n) is a Cauchy sequence if and only if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.

(iii) (X, d) is complete if and only if every Cauchy sequence in X is convergent.

REMARK 2.5. [10] In a b-metric space (X, d), the following assertions hold:

(i) A convergent sequence has a unique limit.

(ii) Each convergent sequence is Cauchy.

(iii) In general, a b-metric is not continuous.

Theorem 2.6. [2] *Let (X,d) be a b-metric space and suppose that (x_n) and (y_n) converge to $x, y \in X$, respectively. Then, we have*

$$\frac{1}{s^2}d(x,y) \leq \liminf_{n\to\infty} d(x_n,y_n) \leq \limsup_{n\to\infty} d(x_n,y_n) \leq s^2 d(x,y).$$

In particular, if $x = y$, then $\lim_{n\to\infty} d(x_n,y_n) = 0$. Moreover, for each $z \in X$, we have

$$\frac{1}{s}d(x,z) \leq \liminf_{n\to\infty} d(x_n,z) \leq \limsup_{n\to\infty} d(x_n,z) \leq sd(x,z).$$

Let T and S be self mappings of a set X. Recall that, if $y = Tx = Sx$ for some x in X, then x is called a coincidence point of T and S and y is called a point of coincidence of T and S. The mappings T, S are weakly compatible [17], if for every $x \in X$, the following holds:

$$T(Sx) = S(Tx) \text{ whenever } Sx = Tx.$$

Proposition 2.7. [1] *Let S and T be weakly compatible selfmaps of a nonempty set X. If S and T have a unique point of coincidence $y = Sx = Tx$, then y is the unique common fixed point of S and T.*

Definition 2.8. Let (X,d) be a b-metric space with the coefficient $s \geq 1$. A mapping $f : X \to X$ is called expansive if there exists a positive number $k > s$ such that

$$d(fx,fy) \geq k\, d(x,y)$$

for all $x, y \in X$.

We next review some basic notions in graph theory.

Let (X,d) be a b-metric space. We assume that G is a reflexive digraph where the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains no parallel edges. So we can identify G with the pair $(V(G), E(G))$. G may be considered as a weighted graph by assigning to each edge the distance between its vertices. By G^{-1} we denote the graph obtained from G by reversing the direction of edges, i.e., $E(G^{-1}) = \{(x,y) \in X \times X : (y,x) \in E(G)\}$. Let $\tilde{G}$ denote the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\tilde{G}$ as a digraph for which the set of its edges is symmetric. Under this convention,

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

Our graph theory notations and terminology are standard and can be found in all graph theory books, like [9,11,15]. If x, y are vertices of the digraph G, then a path in G from x to y of length n ($n \in \mathbb{N}$) is a sequence $(x_i)_{i=0}^n$ of $n+1$ vertices such that $x_0 = x$, $x_n = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, \ldots, n$. A graph G is connected if there is a path between any two vertices of G. G is weakly connected if $\tilde{G}$ is connected.

Definition 2.9. Let (X,d) be a b-metric space with the coefficient $s \geq 1$ and let $G=(V(G),E(G))$ be a graph. A mapping $f: X \to X$ is called a Banach G-contraction or simply G-contraction if there exists $\alpha \in (0, \frac{1}{s})$ such that

$$d(fx,fy) \leq \alpha\, d(x,y)$$

for all $x,y \in X$ with $(x,y) \in E(G)$.

Any Banach contraction is a G_0-contraction, where the graph G_0 is defined by $E(G_0) = X \times X$. But it is worth mentioning that a Banach G-contraction need not be a Banach contraction (see Remark 3.22).

Definition 2.10. Let (X,d) be a b-metric space with the coefficient $s \geq 1$ and let $G=(V(G),E(G))$ be a graph. A mapping $f: X \to X$ is called G-Kannan if there exists $k \in (0, \frac{1}{2s})$ such that

$$d(fx,fy) \leq k\,[d(fx,x)+d(fy,y)]$$

for all $x,y \in X$ with $(x,y) \in E(G)$.

Note that any Kannan operator is G_0-Kannan. However, a G-Kannan operator need not be a Kannan operator (see Remark 3.25).

Definition 2.11. Let (X,d) be a b-metric space with the coefficient $s \geq 1$ and let $G=(V(G),E(G))$ be a graph. A mapping $f: X \to X$ is called a Fisher G-contraction if there exists $k \in (0, \frac{1}{s(1+s)})$ such that

$$d(fx,fy) \leq k\,[d(fx,y)+d(fy,x)] \tag{2.1}$$

for all $x,y \in X$ with $(x,y) \in E(G)$.

If we take $G = G_0$, then condition (2.1) holds for all $x,y \in X$ and f is called a Fisher contraction. The following example shows that a Fisher G-contraction need not be a Fisher contraction.

Example 2.12. Let $X=[0,\infty)$ and define $d: X \times X \to \mathbb{R}^+$ by $d(x,y) = |x-y|^2$ for all $x,y \in X$. Then (X,d) is a b-metric space with the coefficient $s=2$. Let G be a digraph such that $V(G)=X$ and $E(G) = \Delta \cup \{(4^t x, 4^t(x+1)) : x \in X \text{ with } x \geq 2,\ t=0,1,2,\dots\}$, where $\Delta = \{(x,x) : x \in X\}$. Let $f: X \to X$ be defined by $fx = 4x$ for all $x \in X$.

For $x = 4^t z$, $y = 4^t(z+1)$, $z \geq 2$ with $k = \frac{16}{125}$, we have

$$\begin{aligned} d(fx,fy) &= d\left(4^{t+1}z, 4^{t+1}(z+1)\right) = 4^{2t+2} \\ &\leq \frac{16}{125}\, 4^{2t}(18z^2+18z+17) \\ &= \frac{16}{125}\, \left[d\left(4^{t+1}z, 4^t(z+1)\right) + d\left(4^{t+1}(z+1), 4^t z\right)\right] \\ &= k\,[d(fx,y)+d(fy,x)]. \end{aligned}$$

Thus, f is a Fisher G-contraction. But f is not a Fisher contraction because, if $x=4$, $y=0$, then for any arbitrary positive number $k<\frac{1}{s(1+s)}$, we have

$$\begin{aligned} k\,[d(fx,y)+d(fy,x)] &= k\,[d(f4,0)+d(f0,4)] = k\,[d(16,0))+d(0,4)] \\ &= 272k < 256 = d(fx,fy). \end{aligned}$$

Remark 2.13. If f is a G-contraction (resp., G-Kannan or Fisher G-contraction), then f is both a G^{-1}-contraction (resp., G^{-1}-Kannan or Fisher G^{-1}-contraction) and a $\tilde{G}$-contraction (resp., $\tilde{G}$-Kannan or Fisher $\tilde{G}$-contraction).

3. Main results

In this section, we assume that (X,d) is a b-metric space with the coefficient $s\geq 1$, and G is a reflexive digraph such that $V(G)=X$ and G has no parallel edges. Let $f,g\colon X\to X$ be such that $f(X)\subseteq g(X)$. If $x_0\in X$ is arbitrary, then there exists an element $x_1\in X$ such that $fx_0=gx_1$, since $f(X)\subseteq g(X)$. Proceeding in this way, we can construct a sequence (gx_n) such that $gx_n=fx_{n-1}$, $n=1,2,3,\dots$. By C_{gf} we denote the set of all elements x_0 of X such that $(gx_n,gx_m)\in E(\tilde{G})$ for $m,n=0,1,2,\dots$. If $g=I$, the identity map on X, then obviously C_{gf} becomes C_f which is the collection of all elements x of X such that $(f^n x, f^m x)\in E(\tilde{G})$ for $m,n=0,1,2,\dots$.

Theorem 3.1. *Let (X,d) be a b-metric space endowed with a graph G and the mappings $f,g\colon X\to X$ satisfy*

$$d(fx,fy)\leq k\,d(gx,gy) \tag{3.1}$$

for all $x,y\in X$ with $(gx,gy)\in E(\tilde{G})$, where $k\in(0,\frac{1}{s})$ is a constant. Suppose $f(X)\subseteq g(X)$ and $g(X)$ is a complete subspace of X with the following property:

$(*)$ If (gx_n) is a sequence in X such that $gx_n\to x$ and $(gx_n,gx_{n+1})\in E(\tilde{G})$ for all $n\geq 1$, then there exists a subsequence (gx_{n_i}) of (gx_n) such that $(gx_{n_i},x)\in E(\tilde{G})$ for all $i\geq 1$.

Then f and g have a point of coincidence in X if $C_{gf}\neq\emptyset$. Moreover, f and g have a unique point of coincidence in X if the graph G has the following property:

$(**)$ If x,y are points of coincidence of f and g in X, then $(x,y)\in E(\tilde{G})$.

Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. Suppose that $C_{gf}\neq\emptyset$. We choose an $x_0\in C_{gf}$ and keep it fixed. Since $f(X)\subseteq g(X)$, there exists a sequence (gx_n) such that $gx_n=fx_{n-1}$, $n=1,2,3,\dots$ and $(gx_n,gx_m)\in E(\tilde{G})$ for $m,n=0,1,2,\dots$.

We now show that (gx_n) is a Cauchy sequence in $g(X)$.

For any natural number n, we have by using condition (3.1) that

$$d(gx_n,gx_{n+1})=d(fx_{n-1},fx_n)\leq kd(gx_{n-1},gx_n). \tag{3.2}$$

By repeated use of condition (3.2), we get

$$d(gx_n, gx_{n+1}) \leq k^n d(gx_0, gx_1) \tag{3.3}$$

for all $n \in \mathbb{N}$. For $m, n \in \mathbb{N}$ with $m > n$, using condition (3.3), we have

$$\begin{aligned}
d(gx_n, gx_m) &\leq sd(gx_n, gx_{n+1}) + s^2 d(gx_{n+1}, gx_{n+2}) \\
&\quad + \cdots + s^{m-n-1} d(gx_{m-2}, gx_{m-1}) + s^{m-n-1} d(gx_{m-1}, gx_m) \\
&\leq \left[sk^n + s^2 k^{n+1} + \cdots + s^{m-n-1} k^{m-2} + s^{m-n-1} k^{m-1}\right] d(gx_0, gx_1) \\
&\leq sk^n \left[1 + sk + (sk)^2 + \cdots + (sk)^{m-n-2} + (sk)^{m-n-1}\right] d(gx_0, gx_1) \\
&\leq \frac{sk^n}{1 - sk} d(gx_0, gx_1) \to 0 \ \text{ as } \ n \to \infty.
\end{aligned}$$

Therefore, (gx_n) is a Cauchy sequence in $g(X)$. As $g(X)$ is complete, there exists an $u \in g(X)$ such that $gx_n \to u = gv$ for some $v \in X$.

As $x_0 \in C_{gf}$, it follows that $(gx_n, gx_{n+1}) \in E(\tilde{G})$ for all $n \geq 0$, and so by property $(*)$, there exists a subsequence (gx_{n_i}) of (gx_n) such that $(gx_{n_i}, gv) \in E(\tilde{G})$ for all $i \geq 1$. Again, using condition (3.1), we have

$$\begin{aligned}
d(fv, gv) &\leq sd(fv, fx_{n_i}) + sd(fx_{n_i}, gv) \\
&\leq skd(gv, gx_{n_i}) + sd(gx_{n_i+1}, gv) \\
&\to 0 \ \text{ as } \ i \to \infty.
\end{aligned}$$

This gives that $d(fv, gv) = 0$ and hence, $fv = gv = u$. Therefore, u is a point of coincidence of f and g.

The next is to show that the point of coincidence is unique. Assume that there is another point of coincidence u^* in X such that $fx = gx = u^*$ for some $x \in X$. By property $(**)$, we have $(u, u^*) \in E(\tilde{G})$. Then,

$$d(u, u^*) = d(fv, fx) \leq kd(gv, gx) = kd(u, u^*),$$

which gives that $u = u^*$. Therefore, f and g have a unique point of coincidence in X.

If f and g are weakly compatible, then by Proposition 2.7, f and g have a unique common fixed point in X. ■

The following corollary gives fixed point of Banach G-contraction in b-metric spaces.

Corollary 3.2. *Let (X, d) be a complete b-metric space endowed with a graph G and the mapping $f : X \to X$ be such that*

$$d(fx, fy) \leq kd(x, y) \tag{3.4}$$

for all $x, y \in X$ with $(x, y) \in E(\tilde{G})$, where $k \in (0, \frac{1}{s})$ is a constant. Suppose the triple (X, d, G) have the following property:

($*'$) *If* (x_n) *is a sequence in* X *such that* $x_n \to x$ *and* $(x_n, x_{n+1}) \in E(\tilde{G})$ *for all* $n \geq 1$, *then there exists a subsequence* (x_{n_i}) *of* (x_n) *such that* $(x_{n_i}, x) \in E(\tilde{G})$ *for all* $i \geq 1$.

Then f *has a fixed point in* X *if* $C_f \neq \emptyset$. *Moreover,* f *has a unique fixed point in* X *if the graph* G *has the following property:*

($**'$) *If* x, y *are fixed points of* f *in* X, *then* $(x, y) \in E(\tilde{G})$.

Proof. The proof can be obtained from Theorem 3.1 by considering $g = I$, the identity map on X. ∎

Corollary 3.3. *Let* (X, d) *be a* b*-metric space and mappings* $f, g : X \to X$ *satisfy* (3.1) *for all* $x, y \in X$, *where* $k \in (0, \frac{1}{s})$ *is a constant. If* $f(X) \subseteq g(X)$ *and* $g(X)$ *is a complete subspace of* X, *then* f *and* g *have a unique point of coincidence in* X. *Moreover, if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point in* X.

Proof. The proof follows from Theorem 3.1 by taking $G = G_0$, where G_0 is the complete graph $(X, X \times X)$. ∎

The following corollary is the b-metric version of Banach contraction principle.

Corollary 3.4. *Let* (X, d) *be a complete* b*-metric space and a mapping* $f : X \to X$ *be such that* (3.4) *holds for all* $x, y \in X$, *where* $k \in (0, \frac{1}{s})$ *is a constant. Then* f *has a unique fixed point* u *in* X *and* $f^n x \to u$ *for all* $x \in X$.

Proof. It follows from Theorem 3.1 by putting $G = G_0$ and $g = I$. ∎

From Theorem 3.1, we obtain the following corollary concerning the fixed point of expansive mapping in b-metric spaces.

Corollary 3.5. *Let* (X, d) *be a complete* b*-metric space and let* $g : X \to X$ *be an onto expansive mapping. Then* g *has a unique fixed point in* X.

Proof. The conclusion of the corollary follows from Theorem 3.1 by taking $G = G_0$ and $f = I$. ∎

Corollary 3.6. *Let* (X, d) *be a complete* b*-metric space endowed with a partial ordering* $\preceq$ *and the mapping* $f : X \to X$ *be such that* (3.4) *holds for all* $x, y \in X$ *with* $x \preceq y$ *or* $y \preceq x$, *where* $k \in (0, \frac{1}{s})$ *is a constant. Suppose the triple* $(X, d, \preceq)$ *has the following property:*

(†) *If* (x_n) *is a sequence in* X *such that* $x_n \to x$ *and* x_n, x_{n+1} *are comparable for all* $n \geq 1$, *then there exists a subsequence* (x_{n_i}) *of* (x_n) *such that* x_{n_i}, x *are comparable for all* $i \geq 1$.

If there exists $x_0 \in X$ *such that* $f^n x_0, f^m x_0$ *are comparable for* $m, n = 0, 1, 2, \ldots$, *then* f *has a fixed point in* X. *Moreover,* f *has a unique fixed point in* X *if the following property holds:*

(††) *If* x, y *are fixed points of* f *in* X, *then* x, y *are comparable.*

Proof. The proof can be obtained from Theorem 3.1 by taking $g = I$ and $G = G_2$, where the graph G_2 is defined by $E(G_2) = \{(x, y) \in X \times X : x \preceq y \text{ or } y \preceq x\}$. ■

THEOREM 3.7. *Let (X, d) be a b-metric space endowed with a graph G and the mappings $f, g : X \to X$ satisfy*

$$d(fx, fy) \leq kd(fx, gx) + ld(fy, gy) \tag{3.5}$$

for all $x, y \in X$ with $(gx, gy) \in E(\tilde{G})$, where k, l are positive numbers with $k+l < \frac{1}{s}$. Suppose $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X with the property $(*)$. Then f and g have a point of coincidence in X if $C_{gf} \neq \emptyset$. Moreover, f and g have a unique point of coincidence in X if the graph G has the property $(**)$. Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. As in the proof of Theorem 3.1, we can construct a sequence (gx_n) such that $gx_n = fx_{n-1}, n = 1, 2, 3, \ldots$ and $(gx_n, gx_m) \in E(\tilde{G})$ for $m, n = 0, 1, 2, \ldots$. We shall show that (gx_n) is a Cauchy sequence in $g(X)$.

For any natural number n, we have by using condition (3.5) that

$$\begin{aligned} d(gx_{n+1}, gx_n) &= d(fx_n, fx_{n-1}) \leq kd(fx_n, gx_n) + ld(fx_{n-1}, gx_{n-1}) \\ &= kd(gx_{n+1}, gx_n) + ld(gx_n, gx_{n-1}), \end{aligned}$$

which gives that

$$d(gx_{n+1}, gx_n) \leq \alpha d(gx_n, gx_{n-1}) \tag{3.6}$$

where $\alpha = \frac{l}{1-k} \in (0, \frac{1}{s})$. By repeated use of condition (3.6), we obtain

$$d(gx_{n+1}, gx_n) \leq \alpha^n d(gx_1, gx_0), \tag{3.7}$$

for all $n \in \mathbb{N}$. For $m, n \in \mathbb{N}$, using conditions (3.5) and (3.7), we have

$$\begin{aligned} d(gx_m, gx_n) &= d(fx_{m-1}, fx_{n-1}) \\ &\leq kd(fx_{m-1}, gx_{m-1}) + ld(fx_{n-1}, gx_{n-1}) \\ &= kd(gx_m, gx_{m-1}) + ld(gx_n, gx_{n-1}) \\ &\leq k\alpha^{m-1} d(gx_1, gx_0) + l\alpha^{n-1} d(gx_1, gx_0) \\ &\to 0 \text{ as } m, n \to \infty. \end{aligned}$$

Therefore, (gx_n) is a Cauchy sequence in $g(X)$. As $g(X)$ is complete, there exists an $u \in g(X)$ such that $gx_n \to u = gv$ for some $v \in X$. As $x_0 \in C_{gf}$, it follows that $(gx_n, gx_{n+1}) \in E(\tilde{G})$ for all $n \geq 0$, and so by property $(*)$, there exists a subsequence (gx_{n_i}) of (gx_n) such that $(gx_{n_i}, gv) \in E(\tilde{G})$ for all $i \geq 1$.

Now using conditions (3.5) and (3.7), we find

$$\begin{aligned} d(fv, gv) &\leq sd(fv, fx_{n_i}) + sd(fx_{n_i}, gv) \\ &\leq [skd(fv, gv) + sld(fx_{n_i}, gx_{n_i})] + sd(gx_{n_i+1}, gv) \\ &= skd(fv, gv) + sld(gx_{n_i+1}, gx_{n_i}) + sd(gx_{n_i+1}, gv), \end{aligned}$$

which yields

$$\begin{aligned} d(fv, gv) &\leq \frac{sl}{1-sk} d(gx_{n_i+1}, gx_{n_i}) + \frac{s}{1-sk} d(gx_{n_i+1}, gv) \\ &\leq \frac{sl\alpha^{n_i}}{1-sk} d(gx_1, gx_0) + \frac{s}{1-sk} d(gx_{n_i+1}, gv) \\ &\to 0 \text{ as } i \to \infty. \end{aligned}$$

This gives that, $fv = gv = u$. Therefore, u is a point of coincidence of f and g.

Finally, to prove the uniqueness of the point of coincidence, suppose that there is another point of coincidence u^* in X such that $fx = gx = u^*$ for some $x \in X$. By property $(**)$, we have $(u, u^*) \in E(\tilde{G})$. Then,

$$d(u, u^*) = d(fv, fx) \leq kd(fv, gv) + ld(fx, gx) = 0,$$

which gives that $u = u^*$. Therefore, f and g have a unique point of coincidence in X.

If f and g are weakly compatible, then by Proposition 2.7, f and g have a unique common fixed point in X. ■

Corollary 3.8. *Let (X, d) be a complete b-metric space endowed with a graph G and the mapping $f : X \to X$ be such that*

$$d(fx, fy) \leq kd(fx, x) + ld(fy, y) \tag{3.8}$$

for all $x, y \in X$ with $(x, y) \in E(\tilde{G})$, where k, l are positive numbers with $k + l < \frac{1}{s}$. Suppose the triple (X, d, G) has the property $(*')$. Then f has a fixed point in X if $C_f \neq \emptyset$. Moreover, f has a unique fixed point in X if the graph G has the property $(**')$.

Proof. The proof can be obtained from Theorem 3.7 by putting $g = I$. ■

Remark 3.9. In particular (i.e., taking $k = l$), the above corollary gives fixed points of G-Kannan operators in b-metric spaces.

Corollary 3.10. *Let (X, d) be a b-metric space and the mappings $f, g : X \to X$ satisfy (3.5) for all $x, y \in X$, where k, l are positive numbers with $k + l < \frac{1}{s}$. If $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.*

Proof. It can be obtained from Theorem 3.7 by taking $G = G_0$. ■

Corollary 3.11. *Let (X, d) be a complete b-metric space and $f : X \to X$ be a mapping such that (3.8) holds for all $x, y \in X$, where k, l are positive numbers with $k+l < \frac{1}{s}$. Then f has a unique fixed point u in X and $f^n x \to u$ for all $x \in X$.*

Proof. The proof follows from Theorem 3.7 by putting $G = G_0$ and $g = I$. ■

REMARK 3.12. In particular (i.e., taking $k = l$), the above corollary is the b-metric version of Kannan fixed point theorem.

COROLLARY 3.13. *Let (X, d) be a complete b-metric space endowed with a partial ordering $\preceq$ and the mapping $f : X \to X$ be such that (3.8) holds for all $x, y \in X$ with $x \preceq y$ or $y \preceq x$, where k, l are positive numbers with $k + l < \frac{1}{s}$. Suppose the triple $(X, d, \preceq)$ has the property (†). If there exists $x_0 \in X$ such that $f^n x_0, f^m x_0$ are comparable for $m, n = 0, 1, 2, \ldots$, then f has a fixed point in X. Moreover, f has a unique fixed point in X if the property (††) holds.*

Proof. The proof can be obtained from Theorem 3.7 by taking $g = I$ and $G = G_2$. ∎

THEOREM 3.14. *Let (X, d) be a b-metric space endowed with a graph G and the mappings $f, g : X \to X$ satisfy*

$$d(fx, fy) \leq kd(fx, gy) + ld(fy, gx) \tag{3.9}$$

for all $x, y \in X$ with $(gx, gy) \in E(\tilde{G})$, where k, l are positive numbers with $sk < \frac{1}{1+s}$ or $sl < \frac{1}{1+s}$. Suppose $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X with the property $(*)$. Then f and g have a point of coincidence in X if $C_{gf} \neq \emptyset$. Moreover, f and g have a unique point of coincidence in X if the graph G has the property $(**)$ and $k + l < 1$. Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. As in the proof of Theorem 3.1, we can construct a sequence (gx_n) such that $gx_n = fx_{n-1}, n = 1, 2, \ldots$ and $(gx_n, gx_m) \in E(\tilde{G})$ for $m, n = 0, 1, 2, \ldots$. We shall show that (gx_n) is Cauchy in $g(X)$. We assume that $sk < \frac{1}{1+s}$.

For any natural number n, we have by using condition (3.9) that

$$\begin{aligned} d(gx_{n+1}, gx_n) &= d(fx_n, fx_{n-1}) \\ &\leq kd(fx_n, gx_{n-1}) + ld(fx_{n-1}, gx_n) \\ &= kd(gx_{n+1}, gx_{n-1}) \\ &\leq skd(gx_{n+1}, gx_n) + skd(gx_n, gx_{n-1}), \end{aligned}$$

which gives that,

$$d(gx_{n+1}, gx_n)) \leq \alpha d(gx_n, gx_{n-1}) \tag{3.10}$$

where $\alpha = \frac{sk}{1-sk} \in (0, \frac{1}{s})$, since $sk < \frac{1}{1+s}$. By repeated use of condition (3.10), we obtain

$$d(gx_{n+1}, gx_n) \leq \alpha^n d(gx_1, gx_0), \quad \text{for all } n \in \mathbb{N}.$$

By an argument similar to that used in Theorem 3.1, it follows that (gx_n) is a Cauchy sequence in $g(X)$. As $g(X)$ is complete, there exists an $u \in g(X)$ such that $gx_n \to u = gv$ for some $v \in X$. Since $x_0 \in C_{gf}$, it follows that $(gx_n, gx_{n+1}) \in E(\tilde{G})$ for all $n \geq 0$, and so by property $(*)$, there exists a subsequence (gx_{n_i}) of (gx_n) such that $(gx_{n_i}, gv) \in E(\tilde{G})$ for all $i \geq 1$.

Now using condition (3.9), we find

$$\begin{aligned} d(fv, gv) &\leq sd(fv, fx_{n_i}) + sd(fx_{n_i}, gv) \\ &\leq s[kd(fv, gx_{n_i}) + ld(fx_{n_i}, gv)] + sd(gx_{n_i+1}, gv) \\ &\leq s^2kd(fv, gv) + s^2kd(gv, gx_{n_i}) + s(l+1)d(gx_{n_i+1}, gv) \end{aligned}$$

which gives that

$$d(fv, gv) \leq \frac{s^2k}{1-s^2k}d(gx_{n_i}, gv) + \frac{s(l+1)}{1-s^2k}d(gx_{n_i+1}, gv) \to 0 \text{ as } i \to \infty.$$

This proves that $fv = gv = u$. Therefore, u is a point of coincidence of f and g.

Finally, to prove the uniqueness of point of coincidence, suppose that there is another point of coincidence u^* in X such that $fx = gx = u^*$ for some $x \in X$. By property $(**)$, we have $(u, u^*) \in E(\tilde{G})$. Then,

$$\begin{aligned} d(u, u^*) &= d(fv, fx) \leq kd(fv, gx) + ld(fx, gv) \\ &= (k+l)d(u^*, u). \end{aligned}$$

If $k + l < 1$, then it must be the case that $d(u, u^*) = 0$ i.e., $u = u^*$. Therefore, f and g have a unique point of coincidence in X.

If f and g are weakly compatible, then by Proposition 2.7, f and g have a unique common fixed point in X. ■

COROLLARY 3.15. *Let (X, d) be a complete b-metric space endowed with a graph G and the mapping $f : X \to X$ be such that*

$$d(fx, fy) \leq kd(fx, y) + ld(fy, x) \tag{3.11}$$

for all $x, y \in X$ with $(x, y) \in E(\tilde{G})$, where k, l are positive numbers with $sk < \frac{1}{1+s}$ or $sl < \frac{1}{1+s}$. Suppose the triple (X, d, G) has the property $(*')$. Then f has a fixed point in X if $C_f \neq \emptyset$. Moreover, f has a unique fixed point in X if the graph G has the property $(**')$ and $k + l < 1$.

Proof. The proof can be obtained from Theorem 3.14 by putting $g = I$. ■

REMARK 3.16. In particular (i.e., taking $k = l$), the above corollary gives fixed points of Fisher G-contraction in b-metric spaces.

COROLLARY 3.17. *Let (X, d) be a b-metric space and the mappings $f, g : X \to X$ satisfy (3.9) for all $x, y \in X$, where k, l are positive numbers with $sk < \frac{1}{1+s}$ or $sl < \frac{1}{1+s}$. If $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a point of coincidence in X. Moreover, if $k + l < 1$, then f and g have a unique point of coincidence in X. Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point in X.*

Proof. The proof can be obtained from Theorem 3.14 by taking $G = G_0$. ■

The following corollary is [18, Theorem 5]. In particular (when $k = l$), it is the b-metric version of Fisher's theorem.

Corollary 3.18. *Let (X, d) be a complete b-metric space and let $f : X \to X$ be a mapping such that* (3.11) *holds for all $x, y \in X$, where k, l are positive numbers with $sk < \frac{1}{1+s}$ or $sl < \frac{1}{1+s}$. Then f has a fixed point in X. Moreover, if $k + l < 1$, then f has a unique fixed point u in X and $f^n x \to u$ for all $x \in X$.*

Proof. The proof can be obtained from Theorem 3.14 by considering $G = G_0$ and $g = I$. ■

Corollary 3.19. *Let (X, d) be a complete b-metric space endowed with a partial ordering $\preceq$ and the mapping $f : X \to X$ be such that* (3.11) *holds for all $x, y \in X$ with $x \preceq y$ or, $y \preceq x$, where k, l are positive numbers with $sk < \frac{1}{1+s}$ or $sl < \frac{1}{1+s}$. Suppose the triple $(X, d, \preceq)$ has the property (†). If there exists $x_0 \in X$ such that $f^n x_0, f^m x_0$ are comparable for $m, n = 0, 1, 2, \ldots$, then f has a fixed point in X. Moreover, f has a unique fixed point in X if the property (††) holds and $k + l < 1$.*

Proof. The proof can be obtained from Theorem 3.14 by taking $g = I$ and $G = G_2$. ■

We furnish some examples in favour of our results.

Example 3.20. Let $X = \mathbb{R}$ and define $d : X \times X \to \mathbb{R}^+$ by $d(x, y) = |x - y|^2$ for all $x, y \in X$. Then (X, d) is a complete b-metric space with the coefficient $s = 2$. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(0, \frac{1}{5^n}) : n = 0, 1, 2, \ldots\}$. Let $f, g : X \to X$ be defined by

$$fx = \begin{cases} \frac{x}{5}, & \text{if } x \neq \frac{2}{5}, \\ 1, & \text{if } x = \frac{2}{5}, \end{cases}$$

and $gx = 3x$ for all $x \in X$. Obviously, $f(X) \subseteq g(X) = X$.

If $x = 0, y = \frac{1}{3.5^n}$, then $gx = 0, gy = \frac{1}{5^n}$ and so $(gx, gy) \in E(\tilde{G})$.

For $x = 0, y = \frac{1}{3.5^n}$, we have

$$\begin{aligned} d(fx, fy) = d\left(0, \frac{1}{3.5^{n+1}}\right) &= \frac{1}{9.5^{2n+2}} \\ &< \frac{1}{9}.\frac{1}{5^{2n}} = kd(gx, gy), \text{ where } k = \frac{1}{9}. \end{aligned}$$

Therefore, $d(fx, fy) \leq kd(gx, gy)$ holds for all $x, y \in X$ with $(gx, gy) \in E(\tilde{G})$, where $k = \frac{1}{9} \in (0, \frac{1}{s})$ is a constant. We can verify that $0 \in C_{gf}$. In fact, $gx_n = fx_{n-1}, n = 1, 2, 3, \ldots$ gives that $gx_1 = f0 = 0 \Rightarrow x_1 = 0$ and so $gx_2 = fx_1 = 0 \Rightarrow x_2 = 0$. Proceeding in this way, we get $gx_n = 0$ for $n = 0, 1, 2, \ldots$ and hence $(gx_n, gx_m) = (0, 0) \in E(\tilde{G})$ for $m, n = 0, 1, 2, \ldots$.

Also, any sequence (gx_n) with the property $(gx_n, gx_{n+1}) \in E(\tilde{G})$ must be either a constant sequence or a sequence of the following form

$$gx_n = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ \frac{1}{5^n}, & \text{if } n \text{ is even,} \end{cases}$$

where the words 'odd' and 'even' are interchangeable. Consequently it follows that property $(*)$ holds. Furthermore, f and g are weakly compatible. Thus, we have all the conditions of Theorem 3.1 and 0 is the unique common fixed point of f and g in X.

We now show that the weak compatibility condition in Theorem 3.1 cannot be relaxed.

Remark 3.21. In Example 3.20, if we take $gx = 3x - 14$ for all $x \in X$ instead of $gx = 3x$, then $5 \in C_{gf}$ and $f(5) = g(5) = 1$ but $g(f(5)) \neq f(g(5))$ i.e., f and g are not weakly compatible. However, all other conditions of Theorem 3.1 are satisfied. We observe that 1 is the unique point of coincidence of f and g without being a common fixed point.

Remark 3.22. In Example 3.20, f is a Banach G-contraction with constant $k = \frac{1}{25}$ but it is not a Banach contraction. In fact, for $x = \frac{2}{5}, y = 1$, we have

$$d(fx, fy) = d\left(1, \frac{1}{5}\right) = \frac{16}{25} = \frac{16}{9} \cdot \frac{9}{25} > kd(x, y),$$

for any $k \in (0, \frac{1}{s})$. This implies that f is not a Banach contraction.

The next example shows that the property $(*)$ in Theorem 3.1 is necessary.

Example 3.23. Let $X = [0, 1]$ and define $d : X \times X \to \mathbb{R}^+$ by $d(x, y) = |x - y|^2$ for all $x, y \in X$. Then (X, d) is a complete b-metric space with the coefficient $s = 2$. Let G be a digraph such that $V(G) = X$ and $E(G) = \{(0, 0)\} \cup \{(x, y) : (x, y) \in (0, 1] \times (0, 1], x \geq y\}$. Let $f, g : X \to X$ be defined by

$$fx = \begin{cases} \frac{x}{3}, & \text{if } x \in (0, 1], \\ 1, & \text{if } x = 0, \end{cases}$$

and $gx = x$ for all $x \in X$. Obviously, $f(X) \subseteq g(X) = X$.

For $x, y \in X$ with $(gx, gy) \in E(\tilde{G})$, we have $d(fx, fy) = \frac{1}{9} d(gx, gy)$, where $\alpha = \frac{1}{9} \in (0, \frac{1}{s})$ is a constant. We see that f and g have no point of coincidence in X. We now verify that the property $(*)$ does not hold. In fact, (gx_n) is a sequence in X with $gx_n \to 0$ and $(gx_n, gx_{n+1}) \in E(\tilde{G})$ for all $n \in \mathbb{N}$ where $x_n = \frac{1}{n}$. But there exists no subsequence (gx_{n_i}) of (gx_n) such that $(gx_{n_i}, 0) \in E(\tilde{G})$.

The following example supports our Theorem 3.7.

Example 3.24. Let $X = [0, \infty)$ and define $d : X \times X \to \mathbb{R}^+$ by $d(x, y) = |x - y|^2$ for all $x, y \in X$. Then (X, d) is a complete b-metric space with the coefficient $s = 2$. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(3^t x, 3^t(x + 1)) : x \in X \text{ with } x \geq 2, t = 0, 1, 2, \ldots\}$.

Let $f, g : X \to X$ be defined by $fx = 3x$ and $gx = 9x$ for all $x \in X$. Clearly, $f(X) = g(X) = X$.

If $x = 3^{t-2}z, y = 3^{t-2}(z+1)$, then $gx = 3^t z, gy = 3^t(z+1)$ and so $(gx, gy) \in E(\tilde{G})$ for all $z \geq 2$.

For $x = 3^{t-2}z, y = 3^{t-2}(z+1), z \geq 2$ with $k = l = \frac{1}{52}$, we have

$$\begin{aligned} d(fx, fy) &= d\left(3^{t-1}z, 3^{t-1}(z+1)\right) = 3^{2t-2} \\ &\leq \frac{1}{52}3^{2t-2}(8z^2 + 8z + 4) \\ &= \frac{1}{52}\left[d\left(3^{t-1}z, 3^t z\right) + d\left(3^{t-1}(z+1), 3^t(z+1)\right)\right] \\ &= kd(fx, gx) + ld(fy, gy). \end{aligned}$$

Thus, condition (3.5) is satisfied. It is easy to verify that $0 \in C_{gf}$.

Also, any sequence (gx_n) with $gx_n \to x$ and $(gx_n, gx_{n+1}) \in E(\tilde{G})$ must be a constant sequence and hence property $(*)$ holds. Furthermore, f and g are weakly compatible. Thus, we have all the conditions of Theorem 3.7 and 0 is the unique common fixed point of f and g in X.

Remark 3.25. In Example 3.23, f is a G-Kannan operator with constant $k = \frac{9}{52}$. But f is not a Kannan operator because, if $x = 3, y = 0$, then for any arbitrary positive number $k < \frac{1}{2s}$, we have

$$k[d(fx, x) + d(fy, y)] = k[d(f3, 3) + d(f0, 0)] = 36k < 81 = d(fx, fy).$$

Example 3.26. Let $X = \mathbb{R}$ and define $d : X \times X \to \mathbb{R}^+$ by $d(x, y) = |x - y|^p$ for all $x, y \in X$, where $p > 1$ is a real number. Then (X, d) is a complete b-metric space with the coefficient $s = 2^{p-1}$. Let $f, g : X \to X$ be defined by

$$fx = \begin{cases} 1, & \text{if } x \neq 4, \\ 2, & \text{if } x = 4, \end{cases}$$

and $gx = 2x - 1$ for all $x \in X$. Obviously, $f(X) \subseteq g(X) = X$.

Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(1, 2), (2, 4)\}$. If $x = 1, y = \frac{3}{2}$, then $gx = 1, gy = 2$ and so $(gx, gy) \in E(\tilde{G})$. Again, if $x = \frac{3}{2}, y = \frac{5}{2}$, then $gx = 2, gy = 4$ and so $(gx, gy) \in E(\tilde{G})$.

It is easy to verify that condition (3.9) of Theorem 3.14 holds for all $x, y \in X$ with $(gx, gy) \in E(\tilde{G})$. Furthermore, $1 \in C_{gf}$, i.e., $C_{gf} \neq \emptyset$, f and g are weakly compatible, and the triple (X, d, G) have property $(*)$. Thus, all the conditions of Theorem 3.14 are satisfied and 1 is the unique common fixed point of f and g in X.

Remark 3.27. In Example 3.26, f is not a Fisher G-contraction for $p = 5$. In fact, for $x = 2, y = 4$ and $p = 5$, we have

$$\begin{aligned} k[d(fx, y) + d(fy, x)] = k[d(1, 4) + d(2, 2)] = 3^p k &< \frac{243}{272} \\ &< 1 = d(fx, fy), \end{aligned}$$

for arbitrary positive number k with $k < \frac{1}{s(1+s)}$. This implies that f is not a Fisher G-contraction for $p = 5$. However, we can verify that f is a Fisher G-contraction for $p = 4$.

REFERENCES

[**1**] M. Abbas and G. Jungck, *Common fixed point results for noncommuting mappings without continuity in cone metric spaces*, J. Math. Anal. Appl., **341** (2008), 416–420.

[**2**] A. Aghajani, M. Abbas and J. R. Roshan, *Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces*, Math. Slovaca, **4**, 2014, 941–960.

[**3**] S. Aleomraninejad, S. Rezapour and N. Shahzad, *Fixed point results on subgraphs of directed graphs*, Math. Sciences, **7**:41 (2013).

[**4**] M. R. Alfuraidan and M. A. Khamsi, *Caristi fixed point theorem in metric spaces with a graph*, Abstract Appl. Anal., **2014**, Art. ID 303484, 5 pages.

[**5**] I. A. Bakhtin, *The contraction mapping principle in almost metric spaces*, Funct. Anal., Gos. Ped. Inst. Unianowsk, **30** (1989), 26–37.

[**6**] I. Beg, A. R. Butt and S. Radojević, *The contraction principle for set valued mappings on a metric space with a graph*, Comput. Math. Appl., **60** (2010), 1214–1219.

[**7**] F. Bojor, *Fixed point of φ-contraction in metric spaces endowed with a graph*, Ann. Univ. Cralova, Mathematics and Computer Science Series, **37** (2010), 85–92.

[**8**] F. Bojor, *Fixed points of Kannan mappings in metric spaces endowed with a graph*, An. St. Univ. Ovidius Constanta, **20** (2012), 31–40.

[**9**] J. A. Bondy and U. S. R. Murty, *Graph theory with applications*, American Elsevier Publishing Co., Inc., New York, 1976.

[**10**] M. Boriceanu, *Strict fixed point theorems for multivalued operators in b-metric spaces*, Int. J. Mod. Math., **4** (2009), 285–301.

[**11**] G. Chartrand, L. Lesniak and P. Zhang, *Graph and Digraph*, CRC Press, New York, NY, USA, 2011.

[**12**] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostrav, **1** (1993), 5–11.

[**13**] F. Echenique, *A short and constructive proof of Tarski's fixed point theorem*, Internat. J. Game Theory, **33** (2005), 215–218.

[**14**] R. Espinola and W. A. Kirk, *Fixed point theorems in R-trees with applications to graph theory*, Topology Appl., **153** (2006), 1046–1055.

[**15**] J. I. Gross and J. Yellen, *Graph Theory and its Applications*, CRC Press, New York, NY, USA, 1999.

[**16**] J. Jachymski, *The contraction principle for mappings on a metric space with a graph*, Proc. Amer. Math. Soc., **136** (2008), 1359–1373.

[**17**] G. Jungck, *Common fixed points for noncontinuous nonself maps on non-metric spaces*, Far East J. Math. Sci., **4** (1996), 199–215.

[**18**] S. K. Mohanta, *Some fixed point theorems using wt-distance in b-metric spaces*, Fasciculi Math., **54** (2015), 125–140.

[**19**] J. R. Roshan, V. Parvaneh, S. Sedghi, N. Shobkolaei and W. Shatanawi, *Common fixed points of almost generalized $(\psi, \varphi)_s$-contractive mappings in ordered b-metric spaces*, Fixed Point Theory Appl., **2013**:159 (2013), 23 pages.

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