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CHARACTERIZATION OF $(\eta, \gamma, k, 2)$-DINI-LIPSCHITZ FUNCTIONS IN TERMS OF THEIR HELGASON FOURIER TRANSFORM

Radouan Daher and Salah El Ouadih

Abstract. In this paper, using a generalized translation operator, we obtain an analog of Younis Theorem 5.2 in [M. S. Younis, Fourier transforms of Dini-Lipschitz functions, Int. J. Math. Math. Sci. 9 (2),(1986), 301–312.] for the Helgason Fourier transform of a set of functions satisfying the $(\eta, \gamma, k, 2)$-Dini-Lipschitz condition in the space L^2 for functions on noncompact rank one Riemannian symmetric spaces.

1. Introduction

Younis Theorem 5.2 [10] characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Dini-Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transforms, namely we have

THEOREM 1.1 [10] *Let $f \in L^2(\mathbb{R})$. Then the following are equivalent*

(i) $\quad \|f(x+t) - f(x)\| = O\left(\frac{t^\eta}{(\log \frac{1}{t})^\gamma}\right)$, *as $t \to 0, 0 < \eta < 1, \gamma \geq 0$,*

(ii) $\quad \displaystyle\int_{|\lambda| \geq r} |\widehat{f}(\lambda)|^2 d\lambda = O\left(\frac{r^{-2\eta}}{(\log r)^{2\gamma}}\right)$, *as $r \to \infty$, where $\widehat{f}$ stands for the Fourier transform of f.*

In this paper, for rank one symmetric spaces, we prove the generalization of Theorem 1.1 for the Helgason Fourier transform of a class of functions satisfying the $(\eta, \gamma, k, 2)$-Dini-Lipschitz condition in the space L^2. For this purpose, we use the generalized translation operator. We point out that similar results have been established in the context of non compact rank one Riemannian symetric spaces [9].

2. Helgason Fourier transformation on symmetric spaces

Riemannian symmetric spaces constitute a remarkable class of Riemannian manifolds on which various problems of geometry, function theory, and mathematical physics are actively studied (e.g., see [2–6]). For example, the Fourier series

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expansion (more exactly, its analog) is defined on compact symmetric spaces and the Fourier transform is defined on noncompact symmetric spaces; moreover, many problems of the classical harmonic analysis have their natural analogs for symmetric spaces. Among all Riemannian symmetric spaces we especially distinguish the class of rank 1 Riemannian symmetric spaces. These manifolds possess nice geometric properties; in particular, they are two-point homogeneous spaces (see [9, Chapter 8]), while all geodesics on compact rank 1 symmetric spaces are closed and have the same length (see [1]). The class of rank 1 Riemannian symmetric spaces includes the n-dimensional sphere S^n and the n-dimensional Lobachevskiĭ space. Henceforth by a rank 1 symmetric space we mean a noncompact rank 1 Riemannian symmetric space.

Here we collect the necessary facts about the Fourier transformation on symmetric spaces and the spherical Fourier transformation (see [2, 3]). For the required properties of semisimple Lie groups and symmetric spaces, we refer the reader, e.g., to [4, 5]. An arbitrary Riemannian symmetric space X of noncompact type can be represented as the factor space G/K, where G is a connected noncompact semisimple Lie group with finite center, and K is a maximal Compact subgroup of G. ON $X = G/K$ the group G acts transitively by left shifts, and K coincides with the stabilizer of the point $o = eK$ (e is the unity of G). Let $G = NAK$ be an Iwasawa decomposition for G, and let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}$ be the Lie algebras of the groups G, K, A, N, respectively. We denote by M we mean the centralizer of the subgroup A in K and put $B = K/M$. Let dx be a G-invariant measure on X; the symbols db and dk will denote the normalized K-invariant measures on B and K, respectively.

We denote by $\mathfrak{a}^*$ the real space dual to $\mathfrak{a}$, and by W the finite Weyl group acting on $\mathfrak{a}^*$. Let $\sum$ be the set of restricted roots ($\sum \subset \mathfrak{a}^*$), let $\sum^+$ be the set of restricted positive roots, and let

$$\mathfrak{a}^+ = \left\{ h \in \mathfrak{a} : \alpha(h) > 0, \alpha \in \sum\nolimits^+ \right\}$$

be the positive Weyl chamber. If ρ is the half-sum of the positive roots (with multiplicity), then $\rho \in \mathfrak{a}^*$. Let $\langle , \rangle$ be the Killing form on the Lie algebra $\mathfrak{g}$. This form is positive definite on $\mathfrak{a}$. For $\lambda \in \mathfrak{a}^*$, let H_λ denote a vector in $\mathfrak{a}$ such that $\lambda(H) = \langle H_\lambda, H \rangle$ for all $H \in \mathfrak{a}$. For $\lambda, \mu \in \mathfrak{a}^*$ we put $\langle \lambda, \mu \rangle := \langle H_\lambda, H_\mu \rangle$. The correspondence $\lambda \mapsto H_\lambda$ enables us to identify $\mathfrak{a}^*$ and $\mathfrak{a}$. Via this identification, the action of the Weyl group W can be transferred to $\mathfrak{a}$. Let

$$\mathfrak{a}^*_+ = \left\{ \lambda \in \mathfrak{a}^* : H_\lambda \in \mathfrak{a}^+ \right\}.$$

If X is a symmetric space of rank 1, then $\dim \mathfrak{a}^* = 1$, and the set $\sum^+$ consists of the roots α and 2α with some multiplicities a and b depending on X (see [2]). In this case we identify the set $\mathfrak{a}^*$ with $\mathbb{R}$ via the correspondence $\lambda \leftrightarrow \lambda\alpha$, $\lambda \in \mathbb{R}$. Upon this identification positive numbers correspond to the set $\mathfrak{a}^*_+$. The numbers m_α and $m_{2\alpha}$ are frequent in various formulas for rank 1 symmetric spaces. For example, the area of a sphere of radius t on X is equal to

$$S(t) = c(\sinh t)^{m_\alpha} (\sinh 2t)^{m_{2\alpha}},$$

where c is some constant; the dimension of X is equal to

$$\dim X = m_\alpha + m_{2\alpha} + 1.$$

We return to the case in which $X = G/K$ is an arbitrary symmetric space. Given $g \in G$, denote by $A(g) \in \mathfrak{a}$ the unique element satisfying

$$g = n \cdot \exp A(g) \cdot u,$$

where $u \in K$ and $n \in N$. For $x = gK \in X$ and $b = kM \in B = K/M$, we put

$$A(x,b) := A(k^{-1}g).$$

We denote by $\mathcal{D}(X)$ and $\mathcal{D}(G)$ the sets of infinitely differentiable compactly-supported functions on X and G. Let dg be the element of the Haar measure on G. We assume that the Haar measure on G is normed so that

$$\int_X f(x)\,dx = \int_G f(go)\,dg, \quad f \in \mathcal{D}(X).$$

For a function $f \in \mathcal{D}(X)$, the Helgason Fourier transform on X was introduced by S. Helgason (see [3] or [6]) and is defined by the formula

$$\widehat{f}(\lambda,b) := \int_X f(x) e^{(i\lambda+\rho)(A(x,b))}\,dx, \quad \lambda \in \mathfrak{a}^*, \ \ b \in B = K/M.$$

We can norm the measure on X so that the inverse Fourier transform on X would have the form

$$f(x) = \frac{1}{|W|}\int_{\mathfrak{a}^*\times B} \widehat{f}(\lambda,b) e^{(i\lambda+\rho)(A(x,b))}|c(\lambda)|^{-2}d\lambda\,db,$$

where $|W|$ is the order of the Weyl group, $d\lambda$ is the element of the Euclidean measure on $\mathfrak{a}^*$, and $c(\lambda)$ is the Harish-Chandra function. Henceforth, for brevity, we use the notation

$$d\mu(\lambda) := |c(\lambda)|^{-2}d\lambda.$$

Also, the Plancherel formula is valid:

$$\|f\|_2^2 := \int_X |f(x)|^2\,dx = \frac{1}{|W|}\int_{\mathfrak{a}^*\times B} |\widehat{f}(\lambda,b)|^2\,d\mu(\lambda)\,db = \int_{\mathfrak{a}^*_+\times B} |\widehat{f}(\lambda,b)|^2\,d\mu(\lambda)\,db.$$

By continuity, the mapping $f \mapsto \widehat{f}(\lambda,b)$ extends from $\mathcal{D}(X)$ to an isomorphism of the Hilbert space $L^2(X) = L^2(X,dx)$ onto the Hilbert space $L^2(\mathfrak{a}^*_+ \times B, d\mu(\lambda)db)$.

Introduce the translation operator on X. Let $n = \dim X$. Denote by $d(x,y)$ the distance between points $x, y \in X$ and let

$$\sigma(x;t) = \{y \in X : d(x,y) = t\},$$

be the sphere of radius $t > 0$ on X centered at x. Let $d\sigma_x(y)$ be the $(n-1)$-dimensional area element of the sphere $\sigma(x;t)$ and let $|\sigma(t)|$ be the area of the whole sphere $\sigma(x;t)$ (it is independent of the point x). We denote by $C_0(X)$ the set of all continuous compactly-supported functions on X. Given $f \in C_0(X)$, define the generalized translation operator S^h by the formula

$$(S^t f)(x) = \frac{1}{|\sigma(t)|} \int_{\sigma(x;t)} f(y)\, d\sigma_x(y), \quad t > 0;$$

i.e., $(S^t f)(x)$ is the average of f over $\sigma(x;t)$. Observe that the operator S^t can also be called the spherical mean operator (this is the usual term if X coincides with the Euclidean space $\mathbb{R}^n$ when we have the natural translation operator $f(x) \to f(x+a)$).

LEMMA 2.1. [8] *The following inequality is valid for every function $f \in L^2(X)$ and every $t \in \mathbb{R}_+ = [0; +\infty)$:*

$$\|S^t f\|_2 \leq \|f\|_2.$$

An important role in harmonic analysis on symmetric spaces is played by spherical functions (see [2]). For $\lambda \in \mathfrak{a}^*$, let $\varphi_\lambda(t)$ denote the zonal spherical function on G defined by the Harish-Chandra formula

$$\varphi_\lambda(g) = \int_K e^{(i\lambda+\rho)(A(kg))}\, dk, \quad g \in G.$$

We list some properties of the spherical functions to be used later on

$$\varphi_\lambda(u_1 g u_2) = \varphi_\lambda(g), \quad u_1, u_2 \in K,$$
$$\varphi_\lambda(e) = 1,$$
$$\Lambda\varphi_\lambda = -(\lambda^2 + \rho^2)\varphi_\lambda,$$

where Λ is the Laplace operator on X, and

$$\int_K \varphi_\lambda(gkh)\, dk = \varphi_\lambda(g)\varphi_\lambda(h), \quad g, h \in G.$$

LEMMA 2.2. [8] *If $f \in L^2(X)$, then*

$$\widehat{S^t f}(\lambda, b) = \varphi_\lambda(t)\widehat{f}(\lambda, b), \quad \lambda, t \in \mathbb{R}_+ = [0; +\infty).$$

LEMMA 2.3. [7] *The following inequalities are valid for a spherical function $\varphi_\lambda(t)$ $(\lambda, t \in \mathbb{R}_+)$:*

(*i*) $|\varphi_\lambda(t)| \leq 1$,

(*ii*) $1 - \varphi_\lambda(t) \leq t^2(\lambda^2 + \rho^2)$,

(*iii*) *there is a constant $c > 0$ such that $1 - \varphi_\lambda(t) \geq c$, for $\lambda t \geq 1$.*

For $f \in L^2(X)$, we define the finite differences of first and higher order as follows:

$$\Delta_t^1 f = \Delta_t f = (I - S^t)f,$$
$$\Delta_t^k f = \Delta_t(\Delta_t^{k-1} f) = (I - S^t)^k f, \quad k = 2, 3, \ldots,$$

where I is the unit operator in the space $L^2(X)$.

3. Main result

In this section we give the main result of this paper. We need first to define the $(\eta,\gamma,k,2)$-Dini-Lipschitz class.

Definition 3.1. Let $\eta\in(0,1)$ and $\gamma\geq 0$. A function $f\in L^2(X)$ is said to be in the $(\eta,\gamma,k,2)$-Dini-Lipschitz class, denoted by $Lip(\eta,\gamma,k,2)$, if

$$\|\Delta_t^k f\|_2 = O\left(\frac{t^\eta}{(\log\frac{1}{t})^\gamma}\right) \quad \text{as} \quad t\to 0.$$

Lemma 3.2. *For $f\in L^2(X)$,*

$$\|\Delta_t^k f\|_2^2 = \int_0^{+\infty}\int_B |1-\varphi_\lambda(t)|^{2k}|\widehat{f}(\lambda,b)|^2\,d\mu(\lambda)\,db.$$

Proof. From Lemma 2.2, we have

$$\widehat{\Delta_t^k} f(\lambda,b)) = (1-\varphi_\lambda(t))^k\widehat{f}(\lambda,b), \quad \lambda,t\in\mathbb{R}_+=[0;+\infty).$$

Now by Plancherel formula, we have the result. ■

Theorem 3.3. *Let $f\in L^2(X)$. Then the following are equivalent:*
(a) *$f\in Lip(\eta,\gamma,k,2)$, $\eta\in(0,1)$,*
(b) $\displaystyle\int_r^{+\infty}\int_B|\widehat{f}(\lambda,b)|^2\,d\lambda\,db = O\left(\frac{r^{-2\eta-n+1}}{(\log r)^{2\gamma}}\right)$, *as* $r\to\infty$.

Proof. (a) ⇒ (b) Let $f\in Lip(\eta,\gamma,k,2)$. Then we have

$$\|\Delta_t^k f\|_2^2 = O\left(\frac{t^\eta}{(\log\frac{1}{t})^\gamma}\right) \quad \text{as} \quad t\to 0.$$

From Lemma 3.2, we have

$$\|\Delta_t^k f\|_2^2 = \int_0^{+\infty}\int_B |1-\varphi_\lambda(t)|^{2k}|\widehat{f}(\lambda,b)|^2\,d\mu(\lambda)\,db.$$

If $\lambda\in[\frac{1}{t},\frac{2}{t}]$, then $\lambda t\geq 1$ and (iii) of Lemma 2.3 implies that

$$1\leq\frac{1}{c^{2k}}|1-\varphi_\lambda(t)|^{2k}.$$

Then

$$\begin{aligned}\int_{\frac{1}{t}}^{\frac{2}{t}}\int_B|\widehat{f}(\lambda,b)|^2\,d\mu(\lambda)db &\leq \frac{1}{c^{2k}}\int_{\frac{1}{t}}^{\frac{2}{t}}\int_B|1-\varphi_\lambda(t)|^{2k}|\widehat{f}(\lambda,b)|^2\,d\mu(\lambda)\,db\\ &\leq\frac{1}{c^{2k}}\int_0^{+\infty}\int_B|1-\varphi_\lambda(t)|^{2k}|\widehat{f}(\lambda,b)|^2\,d\mu(\lambda)\,db\\ &\leq\frac{1}{c^{2k}}\|\Delta_t^k f\|_2^2 = O\left(\frac{t^{2\eta}}{(\log\frac{1}{t})^{2\gamma}}\right).\end{aligned}$$

From [8], we have $|c(\lambda)|^{-2} \asymp \lambda^{n-1}$, $n = \dim X$. Hence,

$$\int_{\frac{1}{t}}^{\frac{2}{h}} \int_B |\widehat{f}(\lambda, b)|^2 \lambda^{n-1}\, d\lambda\, db = O\left(\frac{t^{2\eta}}{(\log \frac{1}{t})^{2\gamma}}\right),$$

or, equivalently,

$$\int_r^{2r} \int_B |\widehat{f}(\lambda, b)|^2\, d\lambda\, db \le C \frac{r^{-2\eta-n+1}}{(\log r)^{2\gamma}}, \quad r \to \infty,$$

where C is a positive constant. Now,

$$\begin{aligned}\int_r^{+\infty} \int_B |\widehat{f}(\lambda, b)|^2\, d\lambda\, db &= \sum_{i=0}^{\infty} \int_{2^i r}^{2^{i+1} r} \int_B |\widehat{f}(\lambda, b)|^2\, d\lambda\, db \\ &\le C\left(\frac{r^{-2\eta-n+1}}{(\log r)^{2\gamma}} + \frac{(2r)^{-2\eta-n+1}}{(\log 2r)^{2\gamma}} + \frac{(4r)^{-2\eta-n+1}}{(\log 4r)^{2\gamma}} + \cdots\right) \\ &\le C \frac{r^{-2\eta-n+1}}{(\log r)^{2\gamma}} \left(1 + 2^{-2\eta-n+1} + (2^{-2\eta-n+1})^2 + (2^{-2\eta-n+1})^3 + \cdots\right) \\ &\le K_{\eta,n} \frac{r^{-2\eta-n+1}}{(\log r)^{2\gamma}},\end{aligned}$$

where $K_{\eta,n} = C(1 - 2^{-2\eta-n+1})^{-1}$ since $2^{-2\eta-n+1} < 1$. Consequently

$$\int_r^{+\infty} \int_B |\widehat{f}(\lambda, b)|^2\, d\lambda\, db = O\left(\frac{r^{-2\eta-n+1}}{(\log r)^{2\gamma}}\right), \quad \text{as} \quad r \to \infty.$$

(b) $\Rightarrow$ (a). Suppose now that

$$\int_r^{+\infty} \int_B |\widehat{f}(\lambda, b)|^2\, d\lambda\, db = O\left(\frac{r^{-2\eta-n+1}}{(\log r)^{2\gamma}}\right), \quad \text{as} \quad r \to \infty. \tag{1}$$

Then

$$\int_r^{2r} \int_B |\widehat{f}(\lambda, b)|^2\, d\lambda\, db = O\left(\frac{r^{-2\eta-n+1}}{(\log r)^{2\gamma}}\right),$$

whence

$$\begin{aligned}\int_r^{2r} \int_B |\widehat{f}(\lambda, b)|^2 \lambda^{n-1}\, d\lambda\, db &\le 2^{n-1} r^{n-1} \int_r^{2r} \int_B |\widehat{f}(\lambda, b)|^2\, d\lambda\, db \\ &\le C' \frac{r^{-2\eta}}{(\log r)^{2\gamma}}.\end{aligned}$$

Now,

$$\begin{aligned}\int_r^{+\infty} \int_B |\widehat{f}(\lambda, b)|^2 \lambda^{n-1}\, d\lambda\, db &\le \sum_{k=0}^{\infty} \int_{2^k r}^{2^{k+1} r} \int_B |\widehat{f}(\lambda, b)|^2 \lambda^{n-1}\, d\lambda \\ &\le C' \sum_{k=0}^{\infty} 2^{-2k\eta} \frac{r^{-2\eta}}{(\log r)^{2\gamma}}.\end{aligned}$$

Consequently,

$$\int_r^{+\infty}\int_B |\widehat{f}(\lambda,b)|^2\lambda^{n-1}\,d\lambda\,db = O\left(\frac{r^{-2\eta}}{(\log r)^{2\gamma}}\right),$$

and, by $|c(\lambda)|^{-2} \asymp \lambda^{n-1}$,

$$\int_r^{+\infty}\int_B |\widehat{f}(\lambda,b)|^2\,d\mu(\lambda)\,db = O\left(\frac{r^{-2\eta}}{(\log r)^{2\gamma}}\right). \tag{2}$$

Write $\|\Delta_t^k f\|_2^2 = I_1 + I_2$, where

$$I_1 = \int_0^{\frac{1}{t}}\int_B |1-\varphi_\lambda(t)|^{2k}|\widehat{f}(\lambda,b)|^2\,d\mu(\lambda)\,db,$$

and

$$I_2 = \int_{\frac{1}{t}}^{+\infty}\int_B |1-\varphi_\lambda(t)|^{2k}|\widehat{f}(\lambda,b)|^2\,d\mu(\lambda)\,db.$$

Firstly, it follows from the inequality $|\varphi_\lambda(t)| \le 1$ that

$$I_2 \le 2^{2k}\int_{\frac{1}{t}}^{+\infty}\int_B |\widehat{f}(\lambda,b)|^2\,d\mu(\lambda)\,db = O\left(\frac{t^{2\eta}}{(\log\frac{1}{t})^{2\gamma}}\right), \quad \text{as} \quad t\to 0.$$

In order to estimate I_1, we use the inequalities (i) and (ii) of Lemma 2.3:

$$\begin{aligned}
I_1 &= \int_0^{\frac{1}{t}}\int_B |1-\varphi_\lambda(t)|^{2k-1}|1-\varphi_\lambda(t)||\widehat{f}(\lambda,b)|^2\,d\mu(\lambda)db\\
&\le 2^{2k-1}\int_0^{\frac{1}{t}}\int_B |1-\varphi_\lambda(t)||\widehat{f}(\lambda,b)|^2\,d\mu(\lambda)\,db\\
&\le 2^{2k-1}t^2\int_0^{\frac{1}{t}}\int_B (\lambda^2+\rho^2)|\widehat{f}(\lambda,b)|^2\,d\mu(\lambda)\,db.
\end{aligned}$$

Now, we apply integration by parts for the function

$$\phi(r) = \int_r^{+\infty}\int_B |\widehat{f}(\lambda,b)|^2\,d\mu(\lambda)\,db,$$

to get

$$\begin{aligned}
I_1 &\le 2^{2k-1}t^2\int_0^{1/t} -(r^2+\rho^2)\phi'(r)\,dr\\
&\le 2^{2k-1}t^2\int_0^{1/t} -r^2\phi'(r)\,dr\\
&\le 2^{2k-1}t^2\left(-\frac{1}{t^2}\phi(\frac{1}{t}) + 2\int_0^{1/t} r\phi(r)\,dr\right)\\
&\le -2^{2k-1}\phi(\frac{1}{t}) + 2^{2k}t^2\int_0^{1/t} r\phi(r)\,dr\\
&\le 2^{2k}t^2\int_0^{1/t} r\phi(r)\,dr.
\end{aligned}$$

Since $\phi(r) = O\left(\frac{r^{-2\eta}}{(\log r)^{2\gamma}}\right)$, we have $r\phi(r) = O\left(\frac{r^{1-2\eta}}{(\log r)^{2\gamma}}\right)$ and

$$\int_0^{1/t} r\phi(r)\,dr = O\biggl(\int_0^{1/t} \frac{r^{1-2\eta}}{(\log r)^{2\gamma}}\,dr\biggr) = O\left(\frac{t^{2\eta-2}}{(\log\frac{1}{t})^{2\gamma}}\right),$$

so that

$$I_1 = O\left(\frac{t^{2\eta}}{(\log\frac{1}{t})^{2\gamma}}\right).$$

Combining the estimates for I_1 and I_2 gives

$$\|\Delta_t^k f\|_2 = O\left(\frac{t^{\eta}}{(\log\frac{1}{t})^{\gamma}}\right) \quad \text{as} \quad t \to 0,$$

and this ends the proof of the theorem. ■

4. Remarks

Noncompact rank 1 Riemannian symmetric spaces together with Euclidean spaces constitute the class of noncompact two-point homogeneous Riemannian spaces (see [9]), and many theorems of analysis on rank 1 symmetric spaces have natural analogs for Euclidean spaces. We now consider analogs of the Younis Theorem 5.2 [10] for the Euclidean space $\mathbb{R}^n$, $n \geq 1$, which can be obtained from Theorems 1.1 and 3.3.

Let $x, y \in \mathbb{R}^n$, $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$. By definition we put

$$\langle x, y\rangle := x_1y_1 + \cdots + x_ny_n, \quad |x| := \sqrt{\langle x, y\rangle},$$

where dx is the element of the Lebesgue measure on $\mathbb{R}^n$. For every function $f \in C_0(\mathbb{R}^n)$, the Fourier transform $\widehat{f}(\lambda)$ is defined by

$$\widehat{f}(\lambda) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x)e^{-i\langle\lambda, x\rangle}\,dx, \quad \lambda \in \mathbb{R}^n,$$

and the Fourier transform extends by continuity to the Hilbert space $L^2(\mathbb{R}^n)$.

Let $f \in L^2(\mathbb{R}^n)$. We say that the function f belongs to the Dini-Lipschitz class $Lip_{\mathbb{R}^n}(\eta, \gamma, k, 2)$, $0 < \eta < 1$, $\gamma \geq 0$, if

$$\|\Delta_y^k f\|_{L^2(\mathbb{R}^n)} = O\left(\frac{|y|^{\eta}}{(\log\frac{1}{|y|})^{\gamma}}\right), \quad \text{as} \quad |y| \to 0,$$

where

$$\Delta_y^1 f(x) = \Delta_y f(x) = f(x+y) - f(x), \quad \Delta_t^k f(x) = \Delta_y(\Delta_y^{k-1} f(x)), \quad k = 2, 3, \dots .$$

By analogy with the proof of Theorem 1.1 (see [10, Theorem 5.2]), we can establish the following

THEOREM 4.1. *If $f \in L^2(\mathbb{R}^n)$ and $\widehat{f}(\lambda)$ is its Fourier transform then the conditions*

$$f \in Lip_{\mathbb{R}^n}(\eta,\gamma,k,2), \quad 0<\eta<1, \quad \gamma \geq 0$$

and

$$\int_{|\lambda|\geq r} |\widehat{f}(\lambda)|^2 \, d\lambda = O\left(\frac{r^{-2\eta}}{(\log r)^{2\gamma}}\right), \quad \textit{as} \quad r\to\infty, \tag{3}$$

are equivalent.

Suppose that $\sigma = \sigma^{n-1} := \{w \in \mathbb{R}^n : |w| = 1\}$ is the unit sphere in $\mathbb{R}^n$, dw is the $(n-1)$-dimensional area element of the sphere σ, and $|\sigma|$ is the area of the whole sphere σ. Given $f \in C_0(\mathbb{R}^n)$, define the operator S^t by the following formula (if it is given on the space $\mathbb{R}^n$ then we call it the spherical mean operator):

$$(S^t f)(x) := \frac{1}{|\sigma|}\int_\sigma f(x+tw)\,dw, \quad t \geq 0.$$

In particular, for $n=1$ the operator S^t has the form $(S^t f)(x) = \frac{1}{2}(f(x+t)+f(x-t))$. The operator S^t extends by continuity to the Hilbert space $L^2(\mathbb{R}^n)$.

We say that a function f belongs to the spherical Dini-Lipschitz class $Lip^s_{\mathbb{R}^n}(\eta,\gamma,k,2)$, $0<\eta<1$, $\gamma \geq 0$ if $f \in L^2(\mathbb{R}^n)$ and

$$\|\Delta_t^k f\|_{L^2(\mathbb{R}^n)} = O\left(\frac{t^\eta}{(\log\frac{1}{t})^\gamma}\right), \quad \text{as} \quad t\to 0,$$

where

$$\Delta_t^1 f = \Delta_t f = (I-S^t)f, \quad \Delta_t^k f = \Delta_t(\Delta_t^{k-1} f) = (I-S^t)^k f, \quad k=2,3,\ldots.$$

By analogy with the proof of Theorem 3.3, we can establish the following

THEOREM 4.2. *If $f \in L^2(\mathbb{R}^n)$ and $\widehat{f}(\lambda) = \widehat{f}(tw)$ ($\lambda \in \mathbb{R}^n$, $t \geq 0$, and $w \in \sigma^{n-1}$) is its Fourier transform then the conditions*

$$f \in Lip^s_{\mathbb{R}^n}(\eta,\gamma,k,2), \tag{4}$$

and

$$\int_r^\infty \int_{\sigma^{n-1}} |\widehat{f}(tw)|^2 \, dt\, dw = O\left(\frac{r^{-2\eta-n+1}}{(\log r)^{2\gamma}}\right), \quad \textit{as} \quad r\to\infty, \tag{5}$$

are equivalent.

Suppose that the function $\widehat{f}(\lambda)$ satisfies (3). We pass to the polar coordinates $\lambda = tw$, $t\geq 0$, $w \in \sigma^{n-1}$. Then (3) takes the form

$$\int_r^\infty \int_{\sigma^{n-1}} |\widehat{f}(tw)|^2 t^{n-1} \, dt\, dw = O\left(\frac{r^{-2\eta}}{(\log r)^{2\gamma}}\right), \quad \text{as} \quad r\to\infty. \tag{6}$$

It is easy to see that (6) is equivalent to (5) (the corresponding arguments can be carried out by analogy with the proof of equivalence of (1) and (2) in Theorem 3.3); therefore, (3) and (4) are equivalent, and we obtain the following

Corollary 4.3. *The function classes* $Lip_{\mathbb{R}^n}(\eta, \gamma, k, 2)$ *and* $Lip^s_{\mathbb{R}^n}(\eta, \gamma, k, 2)$ *coincide.*

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Department of Mathematics, Faculty of Sciences Aïn Chock, University of Hassan II, Casablanca, Morocco

E-mail: rjdaher024@gmail.com, salahwadih@gmail.com