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SOME CONSTRUCTIONS OF GRAPHS WITH INTEGRAL SPECTRUM

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Abstract. A graph G is said to be an integral graph if all the eigenvalues of the adjacency matrix of G are integers. A natural question to ask is which graphs are integral. In general, characterizing integral graphs seems to be a difficult task. In this paper, we define some graph operations on ordered triple of graphs. We compute their spectrum and, as an application, we give some new methods to construct infinite families of integral graphs starting with either an arbitrary integral graph or integral regular graph. Also, we present some new infinite families of integral graphs by applying our graph operations to some standard graphs like complete graphs, complete bipartite graphs etc.

1. Introduction

Throughout the paper we consider only graphs with no loops and no multiple edges. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. The adjacency matrix $A(G)$ of G is the $n \times n$ real symmetric matrix $[a_{ij}]$, where $a_{ij} = 1$ if the vertices v_i and v_j are adjacent in G, otherwise $a_{ij} = 0$. The spectrum of the adjacency matrix of a graph G is known as adjacency spectrum of G or simply, spectrum of G. For studies on spectrum of graphs, we refer to a classical book by Cvetković, Doob and Sachs [5]. If all the eigenvalues of the adjacency matrix of a graph G are integers, then the graph G is said to be an integral graph. The graphs K_n, $K_{m,n}$ (mn a perfect square), C_6, the cocktail parity graph $CP(n) = \overline{nK_2}$, are some examples of integral graphs. Integral graphs finds its applications in perfect state transfers in graphs [7]. The notion of integral graph was first introduced by Harary and Schwenk in 1974 [10]. In general, the problem of characterizing integral graphs is a difficult task. A result of Ahmadi et al. [3] shows that for a sufficiently large n, the number of integral graphs on n vertices can be at most $2^{\frac{n(n-1)}{2} - \frac{n}{400}}$. In literature, researchers mainly focussed on classifying integral graphs among some interesting families of graphs such as trees, regular graphs, complete r-partite graphs etc. Some works on integral trees and complete r-partite integral

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graphs can be found in a PhD thesis of Wang [18] and also in [8]. In [9], Hansen et al. characterized integral graphs in the families of complete split graphs and multiple complete split-like graphs. So [16] considered circulant graphs and characterized integral graphs among them. In [1], Abdollahi and Vatandoost determined all connected cubic integral Cayley graphs. Some studies on integral regular graphs can be found in [6, 17, 21].

We now recall some well-known graph products [11]. Let H be a graph with vertex set $V(H) = \{u_1, u_2, \ldots, u_m\}$. The cartesian product $G \,\Box\, H$ of two graphs G and H is the graph with vertex set $V(G \,\Box\, H) = V(G) \times V(H)$ and in which two vertices (v_i, u_k) and (v_j, u_l) are adjacent if either $v_i = v_j$ and $u_k u_l$ is an edge in H or $u_k = u_l$ and $v_i v_j$ is an edge in G. The Kronecker product $G \otimes H$ of two graphs G and H is the graph with vertex set $V(G \otimes H) = V(G) \times V(H)$ and in which two vertices (v_i, u_k) and (v_j, u_l) are adjacent if and only if $v_i v_j$ is an edge in G and $u_k u_l$ is an edge in H. The strong product $G \boxtimes H$ of two graphs G and H is the union of cartesian and Kronecker product of graphs G and H. It is worth to note that the cartesian and strong product of graphs G and H consists of $|V(G)|$ copies of H and $|V(H)|$ copies of G. Also the Kronecker product consists of $|V(G)|$ copies of $\overline{K_m}$. Interestingly, these graph products when applied on integral graphs produces again integral graphs.

The problem of constructing infinite families of integral graphs has attracted many researchers. In [12,19–21], several families of integral graphs are constructed by employing some known graphs (integral graphs). Mohammadian and Tayfeh-Rezaie [15] investigated various forms of $(0, 1)$-matrices (obtained using Kronecker product) for integer eigenvalues. More information about integral graphs can be found in [4]. Most of the graph constructions demonstrated in literature are applied either on complete graphs or complete bipartite graphs to produce infinite families of integral graphs, for example, see [2,12–15,20]. Our aim in this paper is to construct infinite families of integral graphs starting with an arbitrary integral graph. We define some graph operations on ordered triple of graphs using some well-known graph products. We compute their spectrum and as an application, we give some new methods to construct infinite families of integral graphs starting with either an arbitrary integral graph or integral regular graph. Also, we present some new infinite families of integral graphs by applying our graph operations on some standard graphs like complete graphs, complete bipartite graphs etc. In the sequel, we denote n copies of a graph G by nG.

2. Spectrum of $\psi_\alpha(G_1, G_2, G_3)$

Let G_i $(i = 1, 2, 3)$ be a graph on n_i vertices. Let $V(G_1) = \{u_1, u_2 \ldots, u_{n_1}\}$, $V(G_2) = \{v_1, v_2 \ldots, v_{n_2}\}$ and $V(G_3) = \{w_1, w_2, \ldots, w_{n_3}\}$ be the vertex sets of G_1, G_2 and G_3, respectively. Denote by $\psi_\alpha(G_1, G_2, G_3)$ $(\alpha = 1, 2, 3)$, the graph obtained from G_i $(i = 1, 2, 3)$ as follows:

Definition 2.1. $\psi_1(G_1, G_2, G_3)$ is the graph obtained from $G_1 \,\Box\, G_3$ and $G_2 \,\Box\, G_3$, by joining each vertex in the i-th copy of G_1 in $G_1 \,\Box\, G_3$ to every vertex in the j-th copy of G_2 in $G_2 \square G_3$, whenever the vertices w_i and w_j are adjacent in G_3.

Definition 2.2. $\psi_2(G_1, G_2, G_3)$ is the graph with vertex set $V(\psi_2(G_1, G_2, G_3)) = (V(G_1) \times V(G_3)) \cup (V(G_2) \times V(G_3))$ and edge set defined as follows:

a. (u_i, w_k) and (u_j, w_l) are adjacent in $\psi_2(G_1, G_2, G_3)$ if either u_iu_j is an edge in G_1 and $w_k = w_l$.
b. (v_i, w_k) and (v_j, w_l) are adjacent in $\psi_2(G_1, G_2, G_3)$ if either v_iv_j is an edge in G_2 and $w_k = w_l$.
c. (u_i, w_k) and (u_j, w_l) are adjacent in $\psi_2(G_1, G_2, G_3)$ if u_iu_j is an edge in G_1 and w_kw_l is an edge in G_3.
d. (v_i, w_k) and (v_j, w_l) are adjacent in $\psi_2(G_1, G_2, G_3)$ if v_iv_j is an edge in G_2 and w_kw_l is an edge in G_3.
e. (v_i, w_k) and (u_j, w_l) are adjacent in $\psi_2(G_1, G_2, G_3)$ if w_kw_l is an edge in G_3.

Definition 2.3. $\psi_3(G_1, G_2, G_3)$ is the graph obtained from $G_1 \boxtimes G_3$ and $G_2 \boxtimes G_3$, by joining each vertex in the i-th copy of G_1 in $G_1 \boxtimes G_3$ to every vertex in the j-th copy of G_2 in $G_2 \boxtimes G_3$, whenever the vertices w_i and w_j are adjacent in G_3.

Let $A = (a_{ij})$ be a $n \times m$ matrix and $B = (b_{ij})$ be a $p \times q$ matrix. Then the Kronecker product $A \otimes B$ of A and B is the $np \times mq$ matrix obtained by replacing each entry a_{ij} of A by $a_{ij}B$. It is well-known that $(A \otimes B)(C \otimes D) = AC \otimes BD$, whenever the products AC, BD are defined and $\lambda\mu$ is the eigenvalue of $A \otimes B$, whenever λ and μ are the eigenvalues of A and B, respectively.

In this section, we compute the spectrum of $\psi_\alpha(G_1, G_2, G_3)$, $\alpha = 1, 2, 3$, when G_1 and G_2 are regular graphs.

Theorem 2.4. *Let G_i $(i = 1, 2)$ be an r_i-regular graph on n_i vertices and let G_3 be an arbitrary graph on n_3 vertices. Suppose $Spec(G_1) = \{\lambda_1 = r_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n_1}\}$, $Spec(G_2) = \{\mu_1 = r_2 \geq \mu_2 \geq \cdots \geq \mu_{n_2}\}$ and $Spec(G_3) = \{\nu_1 \geq \nu_2 \geq \cdots \geq \nu_{n_3}\}$, then the spectrum of $\psi_1(G_1, G_2, G_3)$ consists of*

a. $\lambda_i + \nu_j$, $i = 2, 3, \ldots, n_1$ *and* $j = 1, 2, 3, \ldots, n_3$.
b. $\mu_i + \nu_j$, $i = 2, 3, \ldots, n_2$ *and* $j = 1, 2, 3, \ldots, n_3$.
c. $\left(2\nu_i + r_1 + r_2 \pm \sqrt{4\nu_i^2 n_1 n_2 + (r_1 - r_2)^2}\right)/2$, $i = 1, 2, \ldots, n_3$.

Proof. With suitable labelling of the vertices of $G := \psi_1(G_1, G_2, G_3)$, the adjacency matrix of G can be formulated as follows:

$$A(G) = \begin{bmatrix} I_{n_3} \otimes A(G_1) + A(G_3) \otimes I_{n_1} & A(G_3) \otimes J_{n_1 \times n_2} \\ A(G_3) \otimes J_{n_2 \times n_1} & I_{n_3} \otimes A(G_2) + A(G_3) \otimes I_{n_2} \end{bmatrix},$$

where $J_{n_1 \times n_2}$ is the $n_1 \times n_2$ matrix whose all entries are 1.

Since $A(G_3)$ is a real symmetric matrix of order n_3, it has n_3 orthonormal eigenvectors. Let $X_1, X_2, \ldots, X_{n_3}$ be a set of orthonormal eigenvectors of $A(G_3)$ corresponding to the eigenvalues $\nu_1, \nu_2, \ldots, \nu_{n_3}$, respectively.

Case 1: $\nu_i \neq 0$.

Let $\omega_i = \left(2\nu_i + r_1 + r_2 + \sqrt{4\nu_i^2 n_1 n_2 + (r_1 - r_2)^2}\right)/2$,

$\overline{\omega_i} = \left(2\nu_i + r_1 + r_2 - \sqrt{4\nu_i^2 n_1 n_2 + (r_1 - r_2)^2}\right)/2$,

$\Phi_i = \begin{bmatrix} \dfrac{X_i}{\omega_i - \nu_i - r_1} \otimes \mathbf{1}_{n_1 \times 1} \\ \dfrac{X_i}{\nu_i n_2} \otimes \mathbf{1}_{n_2 \times 1} \end{bmatrix}$ and $\overline{\Phi_i} = \begin{bmatrix} \dfrac{X_i}{\overline{\omega_i} - \nu_i - r_1} \otimes \mathbf{1}_{n_1 \times 1} \\ \dfrac{X_i}{\nu_i n_2} \otimes \mathbf{1}_{n_2 \times 1} \end{bmatrix}$,

where $i = 1, 2, \ldots, n_3$ and $\mathbf{1} = (1, 1, \ldots, 1)^T$. Then

$$A(G)\Phi_i = \begin{bmatrix} I_{n_3} \otimes A(G_1) + A(G_3) \otimes I_{n_1} & A(G_3) \otimes J_{n_1 \times n_2} \\ A(G_3) \otimes J_{n_2 \times n_1} & I_{n_3} \otimes A(G_2) + A(G_3) \otimes I_{n_2} \end{bmatrix} \times \begin{bmatrix} \dfrac{X_i}{\omega_i - \nu_i - r_1} \otimes \mathbf{1}_{n_1 \times 1} \\ \dfrac{X_i}{\nu_i n_2} \otimes \mathbf{1}_{n_2 \times 1} \end{bmatrix}$$

$$= \begin{bmatrix} (I_{n_3} \otimes A(G_1) + A(G_3) \otimes I_{n_1}) \left(\dfrac{X_i}{\omega_i - \nu_i - r_1} \otimes \mathbf{1}_{n_1 \times 1}\right) \\ +(A(G_3) \otimes J_{n_1 \times n_2}) \left(\dfrac{X_i}{\nu_i n_2} \otimes \mathbf{1}_{n_2 \times 1}\right) \\ (A(G_3) \otimes J_{n_2 \times n_1}) \left(\dfrac{X_i}{\omega_i - \nu_i - r_1} \otimes \mathbf{1}_{n_1 \times 1}\right) \\ +(I_{n_3} \otimes A(G_2) + A(G_3) \otimes I_{n_2}) \left(\dfrac{X_i}{\nu_i n_2} \otimes \mathbf{1}_{n_2 \times 1}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \left(\dfrac{X_i}{\omega_i - \nu_i - r_1}\right) \otimes r_1 \mathbf{1}_{n_1 \times 1} + \left(\dfrac{\nu_i X_i}{\omega_i - \nu_i - r_1}\right) \otimes \mathbf{1}_{n_1 \times 1} + (X_i \otimes \mathbf{1}_{n_1 \times 1}) \\ \left(\dfrac{\nu_i n_1 X_i}{\omega_i - \nu_i - r_1}\right) \otimes \mathbf{1}_{n_2 \times 1} + \left(\dfrac{X_i}{\nu_i n_2} \otimes r_2 \mathbf{1}_{n_2 \times 1}\right) + \left(\dfrac{X_i}{n_2} \otimes \mathbf{1}_{n_2 \times 1}\right) \end{bmatrix}$$

$$= \omega_i \begin{bmatrix} \dfrac{X_i}{\omega_i - \nu_i - r_1} \otimes \mathbf{1}_{n_1 \times 1} \\ \dfrac{X_i}{\nu_i n_2} \otimes \mathbf{1}_{n_2 \times 1} \end{bmatrix} = \omega_i \Phi_i.$$

Thus Φ_i $(i = 1, 2, \ldots, n_3)$ is an eigenvector of $A(G)$ corresponding to the eigenvalue ω_i. Similarly, it can be proved that $\overline{\Phi_i}$ $(i = 1, 2, \ldots, n_3)$ is an eigenvector of $A(G)$ corresponding to the eigenvalue $\overline{\omega_i}$.

Case 2: $\nu_i = 0$.

Let X_i be an eigenvector of $A(G_3)$ with corresponding eigenvalue $\nu_i = 0$. Then

$$A(G) \begin{bmatrix} X_i \otimes \mathbf{1}_{n_1 \times 1} \\ \mathbf{0} \end{bmatrix} = r_1 \begin{bmatrix} X_i \otimes \mathbf{1}_{n_1 \times 1} \\ \mathbf{0} \end{bmatrix}.$$

Hence, $\begin{bmatrix} X_i \otimes \mathbf{1}_{n_1 \times 1} \\ \mathbf{0} \end{bmatrix}$ is an eigenvector of $A(G)$ with corresponding eigenvalue r_1.

Similarly, it can be shown that $\begin{bmatrix} \mathbf{0} \\ X_i \otimes \mathbf{1}_{n_2\times 1} \end{bmatrix}$ is an eigenvector of $A(G)$ corresponding to the eigenvalue r_2.

Since $A(G_1)$ is a r_1-regular graph, it follows that $\mathbf{1}_{n_1\times 1}$ is an eigenvector of $A(G_1)$ corresponding to the eigenvalue r_1. Let $Z_1 = \mathbf{1}_{n_1\times 1}/\sqrt{n_1}, Z_2, \ldots, Z_{n_1}$ be a set of orthonormal eigenvectors of $A(G_1)$ corresponding to the eigenvalues $\lambda_1 = r_1, \lambda_2, \ldots, \lambda_{n_1}$, respectively. For $i = 1, 2, \ldots, n_3$ and $j = 2, 3, \ldots, n_1$, we have

$$\begin{aligned} A(G) &\begin{bmatrix} X_i \otimes Z_j \\ \mathbf{0} \end{bmatrix} \\ &= \begin{bmatrix} I_{n_3} \otimes A(G_1) + A(G_3) \otimes I_{n_2} & A(G_3) \otimes J_{n_1\times n_2} \\ A(G_3) \otimes J_{n_2\times n_1} & I_{n_3} \otimes A(G_2) + A(G_3) \otimes I_{n_2} \end{bmatrix} \begin{bmatrix} X_i \otimes Z_j \\ \mathbf{0} \end{bmatrix} \\ &= \begin{bmatrix} (X_i \otimes \lambda_j Z_j) + (\nu_i X_i \otimes Z_j) \\ \nu_i X_i \otimes 0 \end{bmatrix} = (\nu_i + \lambda_j) \begin{bmatrix} X_i \otimes Z_j \\ \mathbf{0} \end{bmatrix}. \end{aligned}$$

Thus, $\begin{bmatrix} X_i \otimes Z_j \\ \mathbf{0} \end{bmatrix}$ is an eigenvector of $A(G)$ corresponding to the eigenvalue $\nu_i + \lambda_j$, where $i = 1, 2, \ldots, n_3$ and $j = 2, 3, \ldots, n_1$.

Let $Y_1 = \mathbf{1}_{n_2\times 1}/\sqrt{n_2}, Y_2, \ldots, Y_{n_2}$ be an orthonormal set of eigenvectors of $A(G_2)$ corresponding to the eigenvalues $\mu_1 = r_2, \mu_2, \ldots, \mu_{n_2}$, respectively. Then it is easy to see that $\begin{bmatrix} \mathbf{0} \\ X_i \otimes Y_j \end{bmatrix}$ is an eigenvector with corresponding eigenvalue $\nu_i + \mu_j$ for $i = 1, 2, \ldots, n_3$ and $j = 2, 3, \ldots, n_2$. This completes the proof of the theorem. ■

The following corollary is an immediate consequence of the above theorem.

Corollary 2.5. *Let G_i $(i = 1, 2)$ be an integral r_i-regular graph and let G_3 be an integral graph. Then $\psi_1(G_1, G_2, G_3)$ is integral if and only if $4\nu_i^2 n_1 n_2 + (r_1 - r_2)^2$ is a perfect square for $i = 1, 2, \ldots, n_3$.*

The following theorems give the spectrum of $\psi_\alpha(G_1, G_2, G_3)$ $(\alpha = 2, 3)$ when G_1 and G_2 are regular graphs. As the proofs of these theorems are analogous to that of the above one, we omit the details.

Theorem 2.6. *For $i = 1, 2$, let G_i be a r_i-regular graph on n_i vertices and let G_3 be an arbitrary graph. Suppose $Spec(G_1) = \{\lambda_1 = r_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n_1}\}$, $Spec(G_2) = \{\mu_1 = r_2 \geq \mu_2 \geq \cdots \geq \mu_{n_2}\}$ and $Spec(G_3) = \{\nu_1 \geq \nu_2 \geq \cdots \geq \nu_{n_3}\}$. Then the spectrum of $\psi_2(G_1, G_2, G_3)$ consists of:*

a. $\lambda_i(1 + \nu_j)$, *$i = 2, 3, \ldots, n_1$ and $j = 1, 2, 3, \ldots, n_3$.*

b. $\mu_i(1 + \nu_j)$, *$i = 2, 3, \ldots, n_2$ and $j = 1, 2, 3, \ldots, n_3$.*

c. $\left((\nu_i + 1)(r_1 + r_2) \pm \sqrt{4\nu_i^2 n_1 n_2 + (\nu_i + 1)^2 (r_1 - r_2)^2}\right)/2$, *$i = 1, 2, \ldots, n_3$.*

Theorem 2.7. *Let G_i $(i = 1, 2)$ be a r_i-regular graph on n_i vertices and let G_3 be an arbitrary graph. Suppose $Spec(G_1) = \{\lambda_1 = r_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n_1}\}$,* *Spec*$(G_2) = \{\mu_1 = r_2 \geq \mu_2 \geq \cdots \geq \mu_{n_2}\}$ *and* *Spec*$(G_3) = \{\nu_1 \geq \nu_2 \geq \cdots \geq \nu_{n_3}\}$. *Then the spectrum of* $\psi_3(G_1, G_2, G_3)$ *consists of:*

a. $\lambda_i + \nu_j + \lambda_i\nu_j$, $i = 2, 3, \ldots, n_1$ *and* $j = 1, 2, 3, \ldots, n_3$.

b. $\mu_i + \nu_j + \mu_i\nu_j$, $i = 2, 3, \ldots, n_2$ *and* $j = 1, 2, 3, \ldots, n_3$.

c. $\left((\nu_i + 1)(r_1 + r_2) + 2\nu_i \pm \sqrt{4\nu_i^2 n_1 n_2 + (\nu_i + 1)^2(r_1 - r_2)^2}\right)/2$, $i = 1, 2, \ldots, n_3$.

Corollary 2.8. *Let* G_i $(i = 1, 2)$ *be an integral* r_i*-regular graph and let* G_3 *be an integral graph. Then* $\psi_\alpha(G_1, G_2, G_3)$ $(\alpha = 2, 3)$ *is integral if and only if* $4\nu_i^2 n_1 n_2 + (\nu_i + 1)^2(r_1 - r_2)^2$ *is a perfect square for* $i = 1, 2, \ldots, n_3$.

Using Corollaries 2.5 and 2.8, in the following propositions, we give some families of integral graphs.

Proposition 2.9. *Let* G_1, G_2 *be integral regular graphs of same degrees on* n_1, n_2 *vertices, respectively and let* G_3 *be an integral graph. Then* $\psi_\alpha(aG_1, bG_2, G_3)$ $(\alpha = 1, 2, 3)$ *is an integral graph for* $a, b \in \mathbb{N}$ *and* abn_1n_2 *a perfect square.*

Proposition 2.10. *Let* G_i $(i = 1, 2)$ *be an integral* r_i*-regular graph on* n_i *vertices. Let* $a, b \in \mathbb{N}$ *and* $ab = (r_1 - r_2)^2 p(pn^2 n_1 n_2 \pm 1)$ $(p = 1, 2, 3, \ldots)$. *Then* $\psi_1(aG_1, bG_2, K_{n,n})$ *is an integral graph.*

Proposition 2.11. *Let* G_1 *be an integral (n+k)-regular graph on* n_1 *vertices and* G_2 *be an integral k-regular graph on* n_2 *vertices. Then* $\psi_1(aG_1, bG_2, K_{n,n})$ *is an integral graph for* $a, b \in \mathbb{N}$ *and* $ab = p(pn_1n_2 \pm 1)$ $(p = 1, 2, 3, \ldots)$. *Also if* $k = 0$ *and* $n_2 = abn_1 \pm 1$, *then* $\psi_1(aG_1, bG_2, K_{n,n})$ *is an integral graph.*

Proposition 2.12. *Let a, b and j be arbitrary positive integers. Then:*

(1) *the graph* $\psi_1(aK_{2nj,2nj},\ bK_2,\ K_{n,n})$ *is integral for* $ab = \dfrac{j(nj-1)^2}{8n} \in \mathbb{N}$.

(2) *For* $ab = \dfrac{j(nj-1)^2}{2n} \in \mathbb{N}$, *the graph* $\psi_1(aK_{2nj},\ bK_1,\ K_{n,n})$ *is integral.*

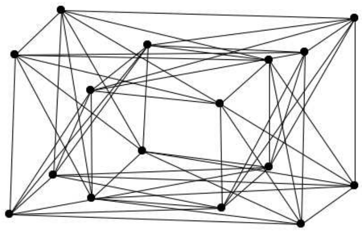
(3) *The graph* $\psi_1(aH,\ bC_4,\ K_{n,n})$, *where* $H = K_{2nj,2nj} \,\square\, K_2$ *is integral for* $ab = \dfrac{j(nj-1)^2}{32n} \in \mathbb{N}$.

(4) *for* $ab = \dfrac{(nj-1)^2}{2n^2} \in \mathbb{N}$, *the graph* $\psi_1(a(K_{nj,nj} \,\square\, K_{nj}),\ bK_1,\ K_{n,n})$ *is integral.*

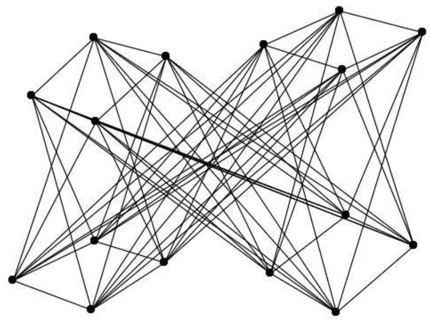
Proposition 2.13. *Let a, b and j be arbitrary positive integers and let* $\alpha = 2,\ 3$. *Then:*

(1) *For* $ab = (n+1)^2 \dfrac{j(nj-1)^2}{8n} \in \mathbb{N}$, *the graph* $\psi_\alpha(aK_{2nj,2nj},\ bK_2,\ K_{n+1})$ *is integral.*

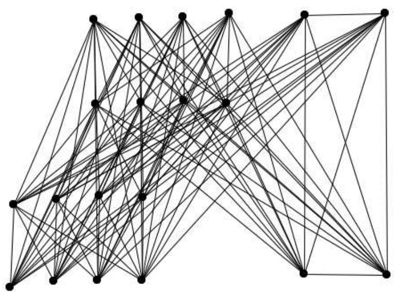
(2) *The graph* $\psi_\alpha(aK_{2nj},\ bK_1,\ K_{n+1})$ *is integral for* $ab = (n+1)^2 \dfrac{j(nj-1)^2}{2n} \in \mathbb{N}$.



a) $\psi_1(K_2, K_2, C_4)$ with spectrum $\{7^1, 3^1, 1^6, -1^5, -3^2, -5^1\}$



b) $\psi_2(C_4, C_4, K_2)$ with spectrum $\{8^1, 4^1, 0^{11}, -4^3\}$



c) $\psi_3(K_{4,4}, K_2, K_2)$ with spectrum $\{11^1, 3^1, 1^7, -1^9, -5^1, -7^1\}$

Fig. 1. Some integral graphs obtained from Corollaries 2.5 and 2.8

(3) *For* $ab = (n+1)^2 \dfrac{j(nj-1)^2}{32n} \in \mathbb{N}$, *the graph* $\psi_\alpha(aH,\ bC_4,\ K_{n+1})$ *is integral, where* $H = K_{2nj,2nj} \,\square\, K_2$.

(4) *The graph* $\psi_\alpha(a(K_{nj,nj} \,\square\, K_{nj}),\ bK_1,\ K_{n+1})$ *is integral for* $ab = (n+1)^2 \dfrac{(nj-1)^2}{2n^2} \in \mathbb{N}$.

3. Spectrum of $\phi_\alpha(G_1, G_2, G_3)$

Denote by $\phi_\alpha(G_1, G_2, G_3)$ $(\alpha = 1, 2, 3)$, the graph obtained from G_i $(i = 1, 2, 3)$ as follows:

Definition 3.1. $\phi_1(G_1, G_2, G_3)$ is the graph obtained from $G_1 \,\square\, G_3$ and $G_2 \otimes G_3$, by joining each vertex in the i-th copy of G_1 in $G_1 \,\square\, G_3$ to every vertex in the j-th copy of G_2 in $G_2 \otimes G_3$, whenever the vertices w_i and w_j are adjacent in G_3.

Definition 3.2. $\phi_2(G_1, G_2, G_3)$ is the graph obtained from $G_1 \,\square\, G_3$ and $G_2 \boxtimes G_3$, by joining each vertex in the i-th copy of G_1 in $G_1 \,\square\, G_3$ to every vertex in the j-th copy of G_2 in $G_2 \boxtimes G_3$, whenever the vertices w_i and w_j are adjacent in G_3.

Definition 3.3. $\phi_3(G_1, G_2, G_3)$ is the graph obtained from $G_1 \otimes G_3$ and $G_2 \boxtimes G_3$, by joining each vertex in the i-th copy of G_1 in $G_1 \otimes G_3$ to every vertex in the j-th copy of G_2 in $G_2 \boxtimes G_3$, whenever the vertices w_i and w_j are adjacent in G_3.

In this section, we give the spectrum of $\phi_\alpha(G_1, G_2, G_3)$ ($\alpha = 1, 2, 3$) when G_1 and G_2 are regular graphs. We use the following lemma to prove our main results.

Lemma 3.4. (see [5]) *If M, N, P, Q are matrices with M being non-singular then*

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = |M|\,|Q - PM^{-1}N|.$$

The proof of the following theorem can be given in an analogous way as that of Theorem 2.4, but here we give a different proof using the above lemma.

Theorem 3.5. *Let G_i ($i = 1, 2$) be an r_i-regular graph on n_i vertices and let G_3 be an arbitrary graph. Suppose $Spec(G_1) = \{\lambda_1 = r_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n_1}\}$, $Spec(G_2) = \{\mu_1 = r_2 \geq \mu_2 \geq \cdots \geq \mu_{n_2}\}$ and $Spec(G_3) = \{\nu_1 \geq \nu_2 \geq \cdots \geq \nu_{n_3}\}$. Then the spectrum of $\phi_1(G_1, G_2, G_3)$ consists of:*

a. $\lambda_i + \nu_j$, $i = 2, 3, \ldots, n_1$ *and* $j = 1, 2, 3, \ldots, n_3$.

b. $\mu_i \nu_j$, $i = 2, 3, \ldots, n_2$ *and* $j = 1, 2, 3, \ldots, n_3$.

c. $\left(r_1 + \nu_i(r_2 + 1) \pm \sqrt{4\nu_i^2 n_1 n_2 + (\nu_i(r_2 - 1) - r_1)^2}\right)/2$, $i = 1, 2, \ldots, n_3$.

Proof. Since $A(G_i)$ ($i = 1, 2, 3$) is a real symmetric matrix, it is orthogonally diagonalizable. Let P_i ($i = 1, 2, 3$) be an orthogonal matrix such that $P_i^T A(G_i) P_i = D_i$, where $D_1 = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{n_1})$, $D_2 = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_{n_2})$ and $D_3 = \mathrm{diag}(\nu_1, \nu_2, \ldots, \nu_{n_3})$. As G_i ($i = 1, 2$) is an r_i-regular graph, without loss of generality, we can assume that the first column of P_i is $\mathbf{1}_{n_i \times 1}/\sqrt{n_i}$, where $\mathbf{1} = (1, 1, \ldots, 1)^T$.

Upon labelling the vertices of $G := \phi_1(G_1, G_2, G_3)$ suitably, the adjacency matrix of G can be formulated as follows:

$$A(G) = \begin{bmatrix} I_{n_3} \otimes A(G_1) + A(G_3) \otimes I_{n_1} & A(G_3) \otimes J_{n_1 \times n_2} \\ A(G_3) \otimes J_{n_2 \times n_1} & A(G_3) \otimes A(G_2) \end{bmatrix},$$

where $J_{n_1 \times n_2}$ is the $n_1 \times n_2$ matrix whose all entries are 1. Now,

$$\begin{aligned} A(G) &= \begin{bmatrix} I_{n_3} \otimes P_1 D_1 P_1^T + P_3 D_3 P_3^T \otimes I_{n_1} & P_3 D_3 P_3^T \otimes J_{n_1 \times n_2} \\ P_3 D_3 P_3^T \otimes J_{n_2 \times n_1} & P_3 D_3 P_3^T \otimes P_2 D_2 P_2^T \end{bmatrix} \\ &= \begin{bmatrix} P_3 \otimes P_1 & \mathbf{0} \\ \mathbf{0} & P_3 \otimes P_2 \end{bmatrix} \begin{bmatrix} I_{n_3} \otimes D_1 + D_3 \otimes I_{n_1} & D_3 \otimes P_1^T J_{n_1 \times n_2} P_2 \\ D_3 \otimes P_2^T J_{n_2 \times n_1} P_1 & D_3 \otimes D_2 \end{bmatrix} \\ &\qquad \times \begin{bmatrix} P_3^T \otimes P_1^T & \mathbf{0} \\ \mathbf{0} & P_3^T \otimes P_2^T \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} P_3 \otimes P_1 & \mathbf{0} \\ \mathbf{0} & P_3 \otimes P_2 \end{bmatrix} \begin{bmatrix} I_{n_3} \otimes D_1 + D_3 \otimes I_{n_1} & D_3 \otimes \sqrt{n_1 n_2} J'_{n_1 \times n_2} \\ D_3 \otimes \sqrt{n_1 n_2} J'_{n_2 \times n_1} & D_3 \otimes D_2 \end{bmatrix}$$
$$\times \begin{bmatrix} P_3^T \otimes P_1^T & \mathbf{0} \\ \mathbf{0} & P_3^T \otimes P_2^T \end{bmatrix},$$

where $J'_{n_1 \times n_2}$ is the matrix obtained from $J'_{n_1 \times n_2}$ by replacing all its entry except the first diagonal entry by 0. Thus $A(G)$ is similar to

$$B := \begin{bmatrix} I_{n_3} \otimes D_1 + D_3 \otimes I_{n_1} & D_3 \otimes \sqrt{n_1 n_2} J'_{n_1 \times n_2} \\ D_3 \otimes \sqrt{n_1 n_2} J'_{n_2 \times n_1} & D_3 \otimes D_2 \end{bmatrix}.$$

The rest of the proof follows by applying Lemma 3.4 to the matrix B. ■

COROLLARY 3.6. *Let G_i $(i = 1, 2)$ be an integral r_i-regular graph and let G_3 be an integral graph. Then $\phi_1(G_1, G_2, G_3)$ is integral if and only if $4\nu_i^2 n_1 n_2 + (\nu_i(r_2 - 1) - r_1)^2$ is a perfect square for $i = 1, 2, \ldots, n_3$.*

The following theorems give the spectrum of $\phi_\alpha(G_1, G_2, G_3)$, when G_1 and G_2 are regular graphs. As the proofs are analogous to those of Theorems 2.4 and 3.5, we omit the details.

THEOREM 3.7. *For $i = 1, 2$, let G_i be an r_i-regular graph on n_i vertices and let G_3 be an arbitrary graph. Suppose $Spec(G_1) = \{\lambda_1 = r_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n_1}\}$, $Spec(G_2) = \{\mu_1 = r_2 \geq \mu_2 \geq \cdots \geq \mu_{n_2}\}$ and $Spec(G_3) = \{\nu_1 \geq \nu_2 \geq \cdots \geq \nu_{n_3}\}$. Then the spectrum of $\phi_2(G_1, G_2, G_3)$ consists of:*

a. $\lambda_i + \nu_j$, $i = 2, 3, \ldots, n_1$ *and* $j = 1, 2, 3, \ldots, n_3$.

b. $\mu_i + \nu_j + \mu_i \nu_j$, $i = 2, 3, \ldots, n_2$ *and* $j = 1, 2, 3, \ldots, n_3$.

c. $\left(r_1 + r_2 + \nu_i(r_2 + 2) \pm \sqrt{4\nu_i^2 n_1 n_2 + (r_2(\nu_i + 1) - r_1)^2}\right)/2$, $i = 1, 2, \ldots, n_3$.

COROLLARY 3.8. *Let G_i $(i = 1, 2)$ be an integral r_i-regular graph and let G_3 be an integral graph. Then $\phi_2(G_1, G_2, G_3)$ is integral if and only if $4\nu_i^2 n_1 n_2 + (r_2(\nu_i + 1) - r_1)^2$ is a perfect square for $i = 1, 2, \ldots, n_3$.*

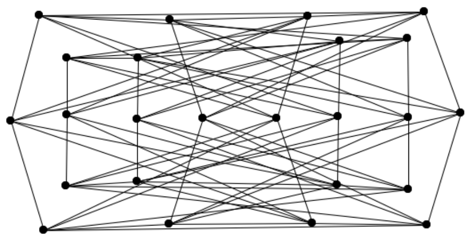
THEOREM 3.9. *Let G_i $(i = 1, 2)$ be a r_i-regular graph on n_i vertices and let G_3 be an arbitrary graph. Suppose $Spec(G_1) = \{\lambda_1 = r_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n_1}\}$, $Spec(G_2) = \{\mu_1 = r_2 \geq \mu_2 \geq \cdots \geq \mu_{n_2}\}$ and $Spec(G_3) = \{\nu_1 \geq \nu_2 \geq \cdots \geq \nu_{n_3}\}$. Then the spectrum of $\phi_3(G_1, G_2, G_3)$ consists of:*

a. $\lambda_i \nu_j$, $i = 2, 3, \ldots, n_1$ *and* $j = 1, 2, 3, \ldots, n_3$.

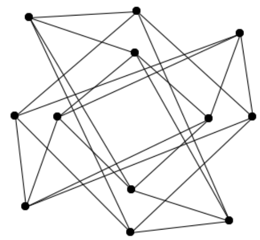
b. $\mu_i + \nu_j + \mu_i \nu_j$, $i = 2, 3, \ldots, n_2$ *and* $j = 1, 2, 3, \ldots, n_3$.

c. $\left(r_2(\nu_i + 1) + \nu_i(r_1 + 1) \pm \sqrt{4\nu_i^2 n_1 n_2 + (\nu_i(r_1 - 1) - r_2(\nu_i + 1))^2}\right)/2$, $i = 1, 2, \ldots, n_3$.

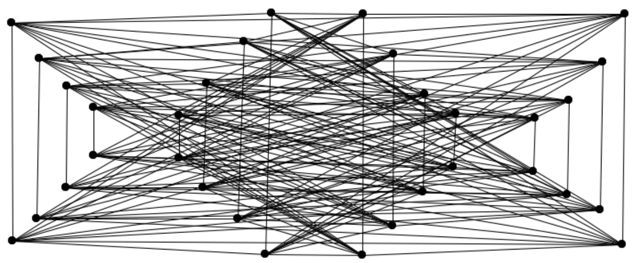
COROLLARY 3.10. *Let G_i $(i = 1, 2)$ be an integral r_i-regular graph and G_3 be an integral graph. Then $\phi_3(G_1, G_2, G_3)$ is integral if and only if $4\nu_i^2 n_1 n_2 + (\nu_i(r_1 - 1) - r_2(\nu_i + 1))^2$ is a perfect square for $i = 1, 2, \ldots, n_3$.*



a) $\phi_2(\overline{K_2}, \overline{K_2}, C_6)$ with spectrum $\{6^1, 3^2, 2^3, 1^6, -1^6, -2^3, -3^2, -6^1\}$



b) $\phi_3(K_1, \overline{K_2}, C_4)$ with spectrum $\{4^1, 2^2, 0^6, -2^2, -4^1\}$



c) $\phi_1(4K_1, 2K_2, C_4)$ with spectrum $\{10^1, 6^1, 2^6, 0^{16}, -2^6, -6^1, -10^1\}$

Fig. 2. Some integral graphs obtained from Corollaries 3.6, 3.8 and 3.10

Using Corollaries 3.6, 3.8 and 3.10, in the following propositions, we give some families of integral graphs.

Proposition 3.11. *Let G and H be integral graphs of order m and n with G being an r-regular graph. Let $a, b \in \mathbb{N}$ and $ab = (r-1)^2 p(pm \pm 1) (p = 1, 2, 3, \dots)$. Then $\phi_1(aK_1, bG, H)$ is an integral graph.*

Proposition 3.12. *Let G be an integral graph and let $a, b \in \mathbb{N}$ and $ab = \dfrac{j(j-1)^2}{4} \in \mathbb{N}, j = 1, 2, 3, \dots$. Then the graph $\phi_1(aK_1,\ bK_{2j,2j}, G)$ is integral.*

Proposition 3.13. *Let G be an integral graph and let $a, b \in \mathbb{N}$ and $ab = \dfrac{j(j+1)^2}{16}, j = 1, 2, 3, \dots$. Then the graph $\phi_1(aK_1,\ b(K_{2j,2j} \,\Box\, C_4), G)$ is integral.*

Proposition 3.14. *Let G be an integral graph and let $a, b \in \mathbb{N}$ and $ab = \dfrac{(j-1)^2}{4} \in \mathbb{N}, j = 1, 2, 3, \dots$. Then the graph $\phi_1(aK_1,\ b(K_{j,j} \,\Box\, K_{j,j}), G)$ is integral.*

Proposition 3.15. *Let G_i $(i = 1, 2)$ be an integral r-regular graph on n_i vertices and G_3 be an integral graph. Let $a, b \in \mathbb{N}$ and $ab = r^2 p(pn_1n_2 \pm 1)$ $(p = 1, 2, 3 \dots)$. Then $\phi_2(aG_1, bG_2, G_3)$ is an integral graph.*

Proposition 3.16. *Let G be an integral graph and let $a, b \in \mathbb{N}$ and $ab = \frac{(j-1)^2}{4} \in \mathbb{N}$, $j = 1, 2, 3, \dots$. Then the graph $\phi_2(aK_{2j},\ bK_{2j}, G)$ is integral.*

Proposition 3.17. *Let G be an integral graph and let $a, b \in \mathbb{N}$ and $ab = \left(\frac{j+1}{8}\right)^2 \in \mathbb{N}$, $j = 1, 2, 3, \dots$. Then the graph $\phi_2(aH,\ bH, G)$ is integral, where $H = K_{2j,2j} \,\square\, K_2$.*

Proposition 3.18. *Let G_i $(i = 1, 2)$ be an integral r_i-regular graph on n_i vertices. Let $r_1 = r_2+1$, $a, b \in \mathbb{N}$ and $ab = r_2^2 p(pn^2 n_1 n_2 \pm 1)$. Then $\phi_3(aG_1, bG_2, K_{n,n})$ is an integral graph.*

Proposition 3.19. *For $a, b \in \mathbb{N}$ and $ab = \frac{(j-1)^2}{8} \in \mathbb{N}$, $j = 1, 2, 3, \dots$, the graph $\phi_3(aK_{2j,2j},\ bK_{2j}, K_{n,n})$ is integral.*

Proposition 3.20. *For $a, b \in \mathbb{N}$ and $ab = \frac{j^2}{128} \in \mathbb{N}$, $j = 1, 2, 3, \dots$, the graph $\phi_3(aH, bG, K_{n,n})$ is integral, where $H = K_{2(j-1),2(j-1)} \,\square\, C_4$ and $G = K_{2(j-1),2(j-1)} \,\square\, K_2$.*

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