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CUBIC SYMMETRIC GRAPHS OF ORDER $6p^3$

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Abstract. A graph is called s-regular if its automorphism group acts regularly on the set of its s-arcs. In this paper, we classify all connected cubic s-regular graphs of order $6p^3$ for each $s \geq 1$ and all primes p.

1. Introduction

Throughout this paper, graphs are assumed to be finite, simple, undirected and connected. For a graph X, we use $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}(X)$ to denote its vertex set, the edge set, the arc set and the full automorphism group of X, respectively. For $u, v \in V(X)$, uv is the edge incident to u and v in X and the neighborhood $N_X(u)$ is the set of vertices adjacent to u in X. Denote by $\mathbb{Z}_n$ the cyclic group of order n as well as the ring of integers modulo n, and by $\mathbb{Z}_n^*$ the multiplicative group of $\mathbb{Z}_n$ consisting of numbers coprime to n. For two groups M and N, $N < M$, means that N is a proper subgroup of M.

Given a finite group G and an inverse closed subset $S \subseteq G \setminus \{1\}$, the Cayley graph $\mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. Given a $g \in G$, define the permutation $R(g)$ on G by $x \to xg$, $x \in G$. Then $R(G) = \{R(g) \mid g \in G\}$, called the right regular representation of G, is a permutation group isomorphic to G. It is well-known that $R(G)$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))$, acting regularly on the vertex set of $\mathrm{Cay}(G, S)$. A Cayley graph $\mathrm{Cay}(G, S)$ is said to be normal if $R(G)$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G, S))$.

Let X be a graph and N a subgroup of $\mathrm{Aut}(X)$. Denote by X_N *the quotient graph* corresponding to the orbits of N, that is the graph having the orbits of N as vertices with two orbits adjacent in X_N whenever there is an edge between those orbits in X.

A graph $\widetilde{X}$ is called a *covering* of a graph X with projection $p : \widetilde{X} \to X$ if there is a surjection $p : V(\widetilde{X}) \to V(X)$ such that $p|_{N_{\widetilde{X}}(\tilde{v})} : N_{\widetilde{X}}(\tilde{v}) \to N_X(v)$ is a bijection

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for any vertex $v \in V(X)$ and $\tilde{v} \in p^{-1}(v)$. A covering $\widetilde{X}$ of X with a projection p is said to be *regular* (or *K-covering*) if there is a semiregular subgroup K of the automorphism group $\mathrm{Aut}(\widetilde{X})$ such that graph X is isomorphic to the quotient graph $\widetilde{X}_K$, say by isomorphism h, and the quotient map $\widetilde{X} \to \widetilde{X}_K$ is the composition ph of p and h (for the purpose of this paper, all functions are composed from left to right). If $\widetilde{X}$, is connected, K becomes the covering transformation group.

An *s-arc* in a graph X is an ordered $(s+1)$-tuple $(v_0, v_1, \dots, v_{s-1}, v_s)$ of vertices of X such that v_{i-1} is adjacent to v_i for $1 \le i \le s$ and $v_{i-1} \ne v_{i+1}$ for $1 \le i < s$. A graph X is said to be *s-arc-transitive* if $\mathrm{Aut}(X)$ is transitive on the set of s-arcs in X. In particular, 0-arc-transitive means *vertex-transitive*, and 1-arc-transitive means *arc-transitive* or *symmetric*. A graph X is said to be *edge-transitive* if $\mathrm{Aut}(X)$ is transitive on $E(X)$ and *half-transitive* if X is vertex-transitive, edge-transitive, but not arc-transitive. A subgroup of the automorphism group of X is said to be *s-regular* if it is acts regularly on the set of s-arcs in X. In particular, if the subgroup is the full automorphism group $\mathrm{Aut}(X)$, then X is said to be *s-regular*. Tutte [21, 22] showed that every cubic symmetric graph is s-regular for s at most 5.

Many people have investigated the automorphism group of cubic symmetric graphs, for example, see [4, 5, 7, 18]. Djoković and Miller [7] constructed an infinite family of cubic 2-regular graphs, and Conder and Praeger [5] constructed two infinite families of cubic s-regular graphs for $s = 2$ or 4. Cheng and Oxley [2] classified symmetric graphs of order $2p$, where p is a prime. Marušić and Xu [17], showed a way to construct a cubic 1-regular graph from a tetravalent half-transitive graph with girth 3. Also, Marušić and Pisanski [16] classified s-regular cubic Cayley graphs on the dihedral groups for each $s \ge 1$ and Feng et al. [8, 9, 10] classified the s-regular cubic graphs of orders $2p^2, 2p^3, 4p, 4p^2, 6p$ and $6p^2$ for each prime p and each $s \ge 1$. In this paper we classify the s-regular cubic graphs of order $6p^3$ for each prime p and each $s \ge 1$.

2. Preliminaries

Let X be a graph and K a finite group. By a^{-1} we mean the reverse arc to an arc a. A *voltage assignment* (or, *K-voltage assignment*) of X is a function $\phi : A(X) \to K$ with the property that $\phi(a^{-1}) = \phi(a)^{-1}$ for each arc $a \in A(X)$ (a^{-1}) is a group inverse). The values of ϕ are called *voltages*, and K is the *voltage group*. The *graph* $X \times_\phi K$ derived from a voltage assignment $\phi : A(X) \to K$ has vertex set $V(X) \times K$ and edge set $E(X) \times K$, so that an edge (e, g) of $X \times_\phi K$ joins a vertex (u, g) to $(v, \phi(a)g)$ for $a = (u, v) \in A(X)$ and $g \in K$, where $e = uv$.

Clearly, the derived graph $X \times_\phi K$ is a covering of X with the first coordinate projection $p : X \times_\phi K \to X$, which is called the *natural projection*. By defining $(u, g^{'})^g := (u, g^{'}g)$ for any $g \in K$ and $(u, g^{'}) \in V(X \times_\phi K)$, K becomes a subgroup of $\mathrm{Aut}(X \times_\phi K)$ which acts semiregularly on $V(X \times_\phi K)$. Therefore, $X \times_\phi K$ can be viewed as a *K-covering*. Conversely, each regular covering $\widetilde{X}$ of X with a covering transformation group K can be derived from a K-voltage assignment. Giving a spanning tree T of the graph X, a voltage assignment ϕ is said to be *T-reduced* if the voltages on the tree arcs are the identity. Gross and Tucker [12] showed that every regular covering $\widetilde{X}$ of a graph X can be derived from a T-reduced voltage assignment ϕ with respect to an arbitrary fixed spanning tree T of X. It is clear that if ϕ is reduced, the derived graph $X \times_\phi K$ is connected if and only if the voltages on the cotree arcs generate the voltage group K.

Let $\widetilde{X}$ be a K-covering of X with a projection p. If $\alpha \in \mathrm{Aut}(X)$ and $\tilde{\alpha} \in \mathrm{Aut}(\widetilde{X})$ satisfy $\tilde{\alpha}p = p\alpha$, we call $\tilde{\alpha}$ a *lift* of α, and α the *projection* of $\tilde{\alpha}$. Concepts such as a lift of a subgroup of $\mathrm{Aut}(X)$ and the projection of a subgroup of $\mathrm{Aut}(\widetilde{X})$ are self-explanatory. The lifts and the projections of such subgroups are of course subgroups in $\mathrm{Aut}(\widetilde{X})$ and $\mathrm{Aut}(X)$ respectively. In particular, if the covering graph $\widetilde{X}$ is connected, then the covering transformation group K is the lift of the trivial group, that is $K=\{\tilde{\alpha} \in \mathrm{Aut}(\widetilde{X}) \colon p = \tilde{\alpha}p\}$. Clearly, if $\tilde{\alpha}$ is a lift of α, then $K\tilde{\alpha}$ is the set of all lifts of α.

Let $X \times_\phi K \to X$ be a connected K-covering derived from a T-reduced voltage assignment ϕ. The problem whether an automorphism α of X lifts or not can be grasped in terms of voltages as follows. Observe that a voltage assignment on arcs extends to a voltage assignment on walks in a natural way. Given $\alpha \in \mathrm{Aut}(X)$, we define a function $\bar{\alpha}$ from the set of voltages on fundamental closed walks based at a fixed vertex $v \in V(X)$ to the voltage group K by

$$(\phi(C))^{\bar{\alpha}} = \phi(C^\alpha),$$

where C ranges over all fundamental closed walks at v, and $\phi(C)$ and $\phi(C^\alpha)$ are the voltages on C and C^α, respectively. Note that if K is Abelian, $\bar{\alpha}$ does not depend on the choice of the base vertex, and the fundamental closed walks at v can be substituted by the fundamental cycles generated by the cotree arcs of X.

The next proposition is a special case of [14, Theorem 4.2].

Proposition 2.1. *Let $X \times_\phi K \to X$ be a connected K-covering derived from a T-reduced voltage assignment ϕ. Then, an automorphism α of X lifts if and only if $\bar{\alpha}$ from the set of voltages on fundamental closed walks based at a fixed vertex of $V(X)$ to the voltage group K extends to an automorphism of K.*

Proposition 2.2. [13, Theorem 9] *Let X be a connected symmetric graph of prime valency and G an s-arc-transitive subgroup of Aut(X) for some $s \geq 1$. If a normal subgroup N of G has more than two orbits, then it is semiregular and G/N is an s-arc-transitive subgroup of Aut(X_N). Furthermore, X is a regular covering of X_N with the covering transformation group N.*

Two coverings $\widetilde{X}_1$ and $\widetilde{X}_2$ of X with projections p_1 and p_2 respectively, are said to be *equivalent* if there exists a graph isomorphism $\tilde{\alpha} : \widetilde{X}_1 \to \widetilde{X}_2$ such that $\tilde{\alpha}p_2 = p_1$. We quote the following proposition.

Proposition 2.3. [20, Proposition 1.5] *Two connected regular coverings $X \times_\phi K$ and $X \times_\psi K$, where ϕ and ψ are T-reduced, are equivalent if and only if there* *exists an automorphism* $\sigma \in Aut(K)$ *such that* $\phi(u,v)^\sigma = \psi(u,v)$ *for any cotree arc* (u,v) *of* X.

Let $p \geq 5$ be a prime. By [10, Theorem 3.2], every cubic symmetric graph of order $2p^3$ is a normal Cayley graph on a group of order $2p^3$. Thus, we have the following result.

LEMMA 2.4. *Let* $p \geq 5$ *be a prime and* X *a cubic symmetric graph of order* $2p^3$. *Then* $Aut(X)$ *has a normal Sylow* p*-subgroup.*

3. Graph constructions and isomorphisms

In this section we construct some examples of cubic symmetric graphs to use later for classification of cubic symmetric graphs of order $6p^3$. In the following examples, let $V(K_{3,3}) = \{\mathbf{0},\ \mathbf{1},\ \mathbf{2},\ \mathbf{3},\ \mathbf{4},\ \mathbf{5},\ \mathbf{6}\}$ be the vertex set of $K_{3,3}$ as illustrated in Fig. 1.

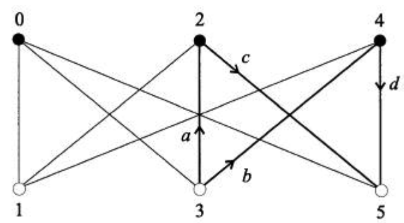

Fig. 1. The complete bipartite graph $K_{3,3}$ with voltage assignment ϕ.

EXAMPLE 3.1. Let p be a prime such that $p-1$ is divisible by 3 and let k be an element of order 3 in $\mathbb{Z}^*_{p^3}$. Let $P = \langle x \rangle$ with $o(x) = p^3$. The graphs AF_{6p^3} and $A'F_{6p^3}$ are defined to have the same vertex set $V(K_{3,3}) \times P$ and edge sets

$$E(AF_{6p^3}) = \{(\mathbf{0},t)(\mathbf{1},t),\ (\mathbf{0},t)(\mathbf{3},t),\ (\mathbf{0},t)(\mathbf{5},t),\ (\mathbf{2},t)(\mathbf{1},t),\ (\mathbf{2},t)(\mathbf{3},tx^{-1}),$$
$$(\mathbf{2},t)(\mathbf{5},tx^{k}),\ (\mathbf{4},t)((\mathbf{1},t),\ (\mathbf{4},t)(\mathbf{3},tx^{-k-1}),\ (\mathbf{4},t)(\mathbf{5},tx^{-1}) \mid t \in P\},$$
$$E(A'F_{6p^3}) = \{(\mathbf{0},t)(\mathbf{1},t),\ (\mathbf{0},t)(\mathbf{3},t),\ (\mathbf{0},t)(\mathbf{5},t),\ (\mathbf{2},t)(\mathbf{1},t),\ (\mathbf{2},t)(\mathbf{3},tx^{-1}),$$
$$(\mathbf{2},t)(\mathbf{5},tx^{k^2}),\ (\mathbf{4},t)((\mathbf{1},t),\ (\mathbf{4},t)(\mathbf{3},tx^{-k^2-1}),\ (\mathbf{4},t)(\mathbf{5},tx^{-1}) \mid t \in P\},$$

respectively.

EXAMPLE 3.2. Let p be a prime such that $p-1$ is divisible by 3 and let k be an element of order 3 in $\mathbb{Z}^*_p$. Also let $P = \langle x \rangle \times \langle y \rangle \times \langle z \rangle$ with $o(x) = o(y) = o(z) = p$. The graphs BF_{6p^3} and $B'F_{6p^3}$ are defined to have the same vertex set $V(K_{3,3}) \times P$ and edge sets

$$E(BF_{6p^3}) = \{(\mathbf{0},t)(\mathbf{1},t),\ (\mathbf{0},t)(\mathbf{3},t),\ (\mathbf{0},t)(\mathbf{5},t),\ (\mathbf{2},t)(\mathbf{1},t),\ (\mathbf{2},t)(\mathbf{3},tx^{-1}),$$
$$(\mathbf{2},t)(\mathbf{5},tz),\ (\mathbf{4},t)((\mathbf{1},t),\ (\mathbf{4},t)(\mathbf{3},ty^{-1}),\ (\mathbf{4},t)(\mathbf{5},txy^{k}z^{-k^2}) \mid t \in P\},$$
$$E(B'F_{6p^3}) = \{(\mathbf{0},t)(\mathbf{1},t),\ (\mathbf{0},t)(\mathbf{3},t),\ (\mathbf{0},t)(\mathbf{5},t),\ (\mathbf{2},t)(\mathbf{1},t),\ (\mathbf{2},t)(\mathbf{3},tx^{-1}),$$
$$(\mathbf{2},t)(\mathbf{5},tz),\ (\mathbf{4},t)((\mathbf{1},t),\ (\mathbf{4},t)(\mathbf{3},ty^{-1}),\ (\mathbf{4},t)(\mathbf{5},txy^{k^2}z^{-k}) \mid t \in P\},$$

respectively.

EXAMPLE 3.3. Let p be a prime such that $p-1$ is divisible by 3 and let k be an element of order 3 in $\mathbb{Z}_{p^2}^*$. Also let $P = \langle x\rangle \times \langle y\rangle$ with $o(x) = p^2$ and $o(y) = p$. The graphs CF_{6p^3} and $C'F_{6p^3}$ are defined to have the same vertex set $V(K_{3,3}) \times P$ and edge sets

$$E(CF_{6p^3}) = \{(\mathbf{0},t)(\mathbf{1},t), (\mathbf{0},t)(\mathbf{3},t), (\mathbf{0},t)(\mathbf{5},t), (\mathbf{2},t)(\mathbf{1},t), (\mathbf{2},t)(\mathbf{3},tx^{-1}),$$
$$(\mathbf{2},t)(\mathbf{5},tx^{-k-1}y), (\mathbf{4},t)((\mathbf{1},t), (\mathbf{4},t)(\mathbf{3},tx^{k}y^{-1}), (\mathbf{4},t)(\mathbf{5},tx^{-1}) \mid t \in P\},$$
$$E(C'F_{6p^3}) = \{(\mathbf{0},t)(\mathbf{1},t), (\mathbf{0},t)(\mathbf{3},t), (\mathbf{0},t)(\mathbf{5},t), (\mathbf{2},t)(\mathbf{1},t), (\mathbf{2},t)(\mathbf{3},tx^{-1}),$$
$$(\mathbf{2},t)(\mathbf{5},tx^{-k^2-1}), (\mathbf{4},t)((\mathbf{1},t), (\mathbf{4},t)(\mathbf{3},tx^{k^2}y^{-1}), (\mathbf{4},t)(\mathbf{5},tx^{-1}) \mid t \in P\},$$

respectively.

EXAMPLE 3.4. Let p be a prime and let $P = \langle x,\ y,\ z \mid x^p = y^p = z^p = 1,\ [x,y] = z,\ [z,x] = [z,y] = 1\rangle$. For any $k \in \mathbb{Z}_p^*$, denote by k^{-1} the inverse of k in $\mathbb{Z}_p^*$. The graphs DF_{6p^3} and EF_{6p^3} are defined to have the same vertex set $V(K_{3,3}) \times P$ and edge sets

$$E(DF_{6p^3}) = \{(\mathbf{0},t)(\mathbf{1},t), (\mathbf{0},t)(\mathbf{3},t), (\mathbf{0},t)(\mathbf{5},t), (\mathbf{2},t)(\mathbf{1},t), (\mathbf{2},t)(\mathbf{3},tx^{-1}),$$
$$(\mathbf{2},t)(\mathbf{5},ty), (\mathbf{4},t)((\mathbf{1},t), (\mathbf{4},t)(\mathbf{3},ty^{-1}x^{-1}z^{3^{-1}}), (\mathbf{4},t)(\mathbf{5},tx^{-1}z^{-(3^{-1})}) \mid t \in P\},$$
$$E(EF_{6p^3}) = \{(\mathbf{0},t)(\mathbf{1},t), (\mathbf{0},t)(\mathbf{3},t), (\mathbf{0},t)(\mathbf{5},t), (\mathbf{2},t)(\mathbf{1},t), (\mathbf{2},t)(\mathbf{3},tx^{-1}),$$
$$(\mathbf{2},t)(\mathbf{5},ty), (\mathbf{4},t)((\mathbf{1},t), (\mathbf{4},t)(\mathbf{3},tyz^{3^{-1}}), (\mathbf{4},t)(\mathbf{5},txyz^{-(3^{-1})}) \mid t \in P\},$$

respectively.

It is easy to see that all graphs in the above examples are bipartite and regular coverings of the complete bipartite graph $K_{3,3}$. Note that if k is an element of order 3 in $\mathbb{Z}_{p^n}^*$ for some positive integer n, then k and k^2 are the only elements of order 3 in $\mathbb{Z}_{p^n}^*$. The graphs $A'F_{6p^3}$, $B'F_{6p^3}$ and $C'F_{6p^3}$ are obtained by replacing k with k^2 in each edge of AF_{6p^3}, BF_{6p^3} and CF_{6p^3}, respectively. In Lemma 3.6, it will be shown that $AF_{6p^3} \cong A'F_{6p^3}$, $BF_{6p^3} \cong B'F_{6p^3}$ and $CF_{6p^3} \cong C'F_{6p^3}$. Thus the graphs AF_{6p^3}, BF_{6p^3} and CF_{6p^3} are independent of the choice of k. Later in Theorem 6.5, it will be shown that the graphs AF_{6p^3}, BF_{6p^3} and CF_{6p^3} are 1-regular and the graphs DF_{6p^3} and EF_{6p^3} are 2-regular.

LEMMA 3.5. *Let $p > 3$ be a prime and n a positive integer. Then k is an element of order 3 in $\mathbb{Z}_{p^n}^*$ if and only if $k^2 + k + 1 = 0$ in the ring $\mathbb{Z}_{p^n}$.*

Proof. Suppose first that $k^2 + k + 1 = 0$. If $k = 1$ then $3 = 0$, which implies that $n = 1$ and $p = 3$, a contradiction. Hence $k \neq 1$. On the other hand, since $k^3 - 1 = (k-1)(k^2 + k + 1)$, we have $k^3 = 1$. Thus k is an element of order 3 in $\mathbb{Z}_{p^n}^*$.

Now suppose that k is an element of order 3 in $\mathbb{Z}_{p^n}^*$. Then $(k-1)(k^2+k+1) = k^3 - 1 = 0$. To prove $k^2+k+1 = 0$, it suffices to show that $(k-1, p) = 1$. Suppose to the contrary that $k \equiv 1 \pmod{p}$. Then $k^2 + k + 1 = 3 \pmod{p}$ and since $p > 3$, $k^2 + k + 1$ is coprime with p. This forces $k - 1 = 0$, a contradiction. Thus $k^2 + k + 1 = 0$. This complete the proof of the lemma. ■

Lemma 3.6. *$AF_{6p^3} \cong A'F_{6p^3}, BF_{6p^3} \cong B'F_{6p^3}, CF_{6p^3} \cong C'F_{6p^3}$ and $DF_{6p^3} \cong EF_{6p^3}$.*

Proof. First we show that $BF_{6p^3} \cong B'F_{6p^3}$. To do this we define a map α from $V(BF_{6p^3})$ to $V(B'F_{6p^3})$ by

$$\begin{aligned}&(0,t)\longmapsto(0,g),\quad (2,t)\longmapsto(4,g),\quad (4,t)\longmapsto(2,g),\\ &(1,t)\longmapsto(1,g),\quad (3,t)\longmapsto(5,g),\quad (5,t)\longmapsto(3,g),\end{aligned}$$

where $t = x^iy^jz^l$ and $g = x^{-i}y^{-l-k^2i}z^{-j+ki}$ for some $i,j,l \in \mathbb{Z}_p$. Clearly, the neighborhood

$$\begin{aligned}N_{BF_{6p^3}}((4,t)) &= \{(1,t),\ (3,ty^{-1}),\ (5,txy^kz^{-k^2})\},\\ N_{B'F_{6p^3}}((4,t)^\alpha) = N_{B'F_{6p^3}}((2,g)) &= \{(1,g),\ (3,gx^{-1}),\ (5,gz)\}.\end{aligned}$$

Since k is an element of order 3 in $\mathbb{Z}_p^*$, by Lemma 3.5, $k^2+k+1=0$ in the ring $\mathbb{Z}_p$. With the aid of this equation, one can easily show that

$$[N_{BF_{6p^3}}((4,t))]^\alpha = N_{B'F_{6p^3}}((4,t)^\alpha).$$

Similarly,

$$[N_{BF_{6p^3}}((u,t))]^\alpha = N_{B'F_{6p^3}}((u,t)^\alpha),$$

for $u = 0, 2$. It follows that α is an isomorphism from BF_{6p^3} to $B'F_{6p^3}$, because the graphs are bipartite. Thus $BF_{6p^3} \cong B'F_{6p^3}$.

Also, by a similar method as above, one can show that the following three maps are isomorphisms from AF_{6p^3} to $A'F_{6p^3}$, CF_{6p^3} to $C'F_{6p^3}$ and DF_{6p^3} to EF_{6p^3}, respectively:

$$\begin{aligned}&(0,t)\longmapsto(0,t),\quad (2,t)\longmapsto(4,t),\quad (4,t)\longmapsto(2,t),\\ &(1,t)\longmapsto(1,t),\quad (3,t)\longmapsto(5,t),\quad (5,t)\longmapsto(3,t),\end{aligned}$$

where $t = x^i$ for some $i \in \mathbb{Z}_{p^3}$,

$$\begin{aligned}&(0,t)\longmapsto(0,g_1),\quad (2,t)\longmapsto(4,g_1),\quad (4,t)\longmapsto(2,g_1),\\ &(1,t)\longmapsto(1,g_1),\quad (3,t)\longmapsto(5,g_1),\quad (5,t)\longmapsto(3,g_1),\end{aligned}$$

where $t = x^iy^j$ and $g_1 = x^iy^{-j}$ for some $i \in \mathbb{Z}_{p^2}$ and $j \in \mathbb{Z}_p$,

$$\begin{aligned}&(0,t)\longmapsto(0,g_2),\quad (2,t)\longmapsto(2,g_2),\quad (4,t)\longmapsto(4,g_2),\\ &(1,t)\longmapsto(1,g_2),\quad (3,t)\longmapsto(3,g_2),\quad (5,t)\longmapsto(5,g_2),\end{aligned}$$

where $t = x^iy^jz^l$ and $g_2 = y^{-i}x^{-j}z^{-l}$. ∎

4. Cubic symmetric graphs of order $6p^3$

In this section, we shall determine all connected cubic symmetric graphs of order $6p^3$ for each prime p.

By [3], we have the following lemma.

LEMMA 4.1. *Let $p \geq 5$ be a prime, and let X be a connected cubic symmetric graph of order $6p^3$. Then X is one of the following:*

(i) *The 1-regular graph F_{162B},*

(ii) *The 2-regular graphs F_{48}, F_{162A} or F_{750},*

(iii) *The 3-regular graph F_{162C}.*

(the graphs are labeled in accordance with the Foster census.)

LEMMA 4.2. *Let $p \geq 7$ be a prime and let X be a connected cubic symmetric graph of order $6p^3$. Then Aut(X) has a normal Sylow p-subgroup.*

Proof. Let X be a cubic graph satisfying the assumptions and let $A :=$Aut(X). Since X is symmetric, by Tutte [21], X is s-regular for some $1 \leq s \leq 5$. Thus $|A| = 2^s.\ 3^2.\ p^3$. Let N be a minimal normal subgroup of A.

Suppose that N is unsolvable. Then $N \cong T \times T \times ... \times T = T^k$, where T is a non-abelian simple group. Since $p \geq 7$ and A is a $\{2,\ 3,\ p\}$-group, by [11, pp. 12-14] and [6], T is one of the following groups

$$PSL_2(7),\ PSL_2(8),\ PSL_2(17),\ PSL_3(3),\ PSU_3(3) \qquad (1)$$

with orders $2^4.\ 3.\ 7$, $2^4.\ 3^2.\ 7$, $2^4.\ 3^2.\ 17$, $2^4.\ 3^3.\ 13$, and $2^5.\ 3^3.\ 7$, respectively. Since 2^6 does not divide $|A|$, one has $k = 1$ and hence $p^2 \nmid |N|$. It follows that N has more than two orbits on $V(X)$. By Proposition 2.2, N is semiregular on V(X), which implies that $|N| \mid 6p^3$, a contradiction.

Table 1. Voltages on fundamental cycles and their images under $\alpha, \beta, \gamma, \tau$ and δ.

C	$\phi(C)$	C^α	$\phi(C^\alpha)$	C^β	$\phi(C^\beta)$
03210	a	23412	$a^{-1}b$	05230	$c^{-1}a^{-1}$
03410	b	23012	a^{-1}	05430	$d^{-1}b^{-1}$
01250	c	21452	dc^{-1}	03210	a
01450	d	21052	c^{-1}	03410	b
C^γ	$\phi(C^\gamma)$	C^τ	$\phi(C^\tau)$	C^δ	$\phi(C^\delta)$
14501	d	12301	a^{-1}	14301	b^{-1}
14301	b^{-1}	12501	c	14501	d
10521	c^{-1}	12301	b	10321	a
10321	a	12301	d^{-1}	10541	c^{-1}

Thus, N is solvable. Let $O_q(A)$ denote the maximal normal q-subgroup of A, $q \in \{2, 3, p\}$. Since X is of order $6p^3$, by Proposition 2.2, $O_q(A)$ is semiregular on $V(X)$. Moreover, the quotient graph $X_{O_q(A)}$ of X corresponding to the orbits of $O_q(A)$ is a cubic symmetric graph with $A/O_q(A)$ as an arc-transitive subgroup of Aut$(X_{O_q(A)})$. The semiregularity of $O_q(A)$ implies that $|O_q(A)| \mid 6p^3$. If $O_2(A) \neq 1$, then $O_2(A) \cong \mathbb{Z}_2$ and hence $X_{O_2(A)}$ has odd order and valency 3, a contradiction. By the solvability of N, either $O_3(A) \neq 1$ or $O_p(A) \neq 1$.

Let $O_3(A) \neq 1$. Then by the semiregularity of $O_3(A)$ on $V(X)$, $|O_3(A)| = 3$, so $X_{O_3(A)}$ is a cubic symmetric graph of order $2p^3$. Let P be a Sylow p-subgroup of A. By Proposition 2.4, $\mathrm{Aut}(X_{O_3(A)})$ has a normal Sylow p-subgroup and hence $PO_3(A)/O_3(A) \lhd A/O_3(A)$ because $A/O_3(A) \leq \mathrm{Aut}(X_{O_3(A)})$. Consequently, $PO_3(A) \lhd A$. Since $|PO_3(A)| = 3p^3$, P is characteristic in $PO_3(A)$, implying $P \lhd A$, as required. Thus, to complete the proof, one may assume that $O_3(A) = 1$. Hence $O_p(A) \neq 1$. Set $Q := O_p(A)$. To prove the lemma, we need to show that $|Q| = p^3$. Suppose to the contrary that $|Q| = p^t$ for $t = 1$ or 2. Then $Q \cong \mathbb{Z}_p$, $\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p^2$.

Suppose first that $Q \cong \mathbb{Z}_p$. Let $C := C_A(Q)$ be the centralizer of Q in A. Clearly $Q < C$. Let L/Q be a minimal normal subgroup of A/Q contained in C/Q. By the same argument as above we may prove that L/Q is solvable and hence elementary abelian. By Proposition 2.2, L/Q is semiregular on $V(X_Q)$, which implies that $L/Q \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$. Since $L \leq C$, Q has a normal complement, say M such that $M \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$. therefore $L = M \times Q$. Now M is characteristic in L and $L \lhd A$, so $M \lhd A$, contradicting $O_2(A) = O_3(A) = 1$.

Suppose now that $Q \cong \mathbb{Z}_{p^2}$. Set $C := C_A(Q)$. Clearly $Q \leq C$. Suppose that $Q = C$. Then by [19 Theorem 10.6.13], A/Q is isomorphic to a subgroup of $\mathrm{Aut}(Q) \cong \mathbb{Z}_{p(p-1)}$, which implies that A/Q is abelian. Since A/Q is transitive on $V(X_Q)$, by [23, Proposition 4.4], A/Q is regular on $V(X_Q)$. Consequently $|A| = 6p^3$, which contradicts the fact that X is symmetric. Hence $Q < C$. Let L/Q be a minimal normal subgroup of A/Q contained in C/Q. Assume that L/Q is unsolvable. Then $L/Q = T^k$ where T is a nonabelian simple group listed in (1). Clearly, $k = 1$. Let P be a Sylow p-subgroup of L. Then $Q \leq Z(P)$ and hence P is abelian. By [19, Theorems 10.1.5, 10.1.6], $L' \bigcap Q = 1$, where L' is the derived subgroup of L. The simplicity of L/Q implies that $L = L'Q$. It follows that $L/Q \cong L'$ and since $p^2 \nmid |L/Q|$, L' has more than two orbits on $V(X)$. By Proposition 2.2, L' is semiregular on $V(X)$, implying $|L'| \mid 6p^3$. This forces L' is solvable, a contradiction. Thus L/Q is solvable. In this case by the same argument as in the preceding paragraph a similar contradiction is obtained.

Suppose finally that $Q \cong \mathbb{Z}_p^2$. Then X_Q is a cubic symmetric graph of order $6p$. If $p \neq 17$, by [9, Theorem 5.2], X_Q is 1-regular, because $p \geq 7$. Thus $|\mathrm{Aut}(X_Q)| = 18p$. By Sylow's theorem $\mathrm{Aut}(X_Q)$ has a normal Sylow p-subgroup. Let P be a Sylow p-subgroup of A. Then P/Q is a Sylow p-subgroups of A/Q and since $A/Q \leq \mathrm{Aut}(X_Q)$, one has $P/Q \lhd A/Q$, implying $P \lhd A$, a contradiction. Thus $p = 17$. By [9, Theorem 2.5], X_Q is isomorphic to the 4-regular Smith-Biggs graph SB_{102} and by [1], $\mathrm{Aut}(X_Q) \cong PSL_2(17)$. Since $|A| = 2^s.\ 3^2.\ p^3$, we have $|\mathrm{Aut}(X_Q) : A/Q|$ is a 2-power. By [6], $PSL_2(17)$ has no subgroup of index 2^t for $t \geq 1$ and so $A/Q = \mathrm{Aut}(X_Q) \cong PSL_2(17)$. Set $C := C_A(Q)$. Then $Q = C$ or $Q \leq Z(A)$. If $Q = C$, then A/Q is isomorphic to a subgroup of $\mathrm{Aut}(Q) \cong GL_2(17)$. Therefore $PSL_2(17)$ is a subgroup of $GL_2(17)$. Since $GL_2(17) \cong SL_2(17) \rtimes Z_{16}$, it follows that $PSL_2(17) \leq SL_2(17)$. This is impossible because $SL_2(17)$ contains a unique involution, while $PSL_2(17)$ contains more. Thus $Q \leq Z(A)$, which implies that the Sylow p-subgroups of A are abelian. This leads to a contradiction similar to the one in preceding paragraph (replacing L by A). ■

Let $p \geq 7$ be a prime and X be a connected cubic symmetric graph of order $6p^3$. Also let P be a Sylow p-subgroup of $\mathrm{Aut}(X)$. By Lemma 4.2, $P \lhd \mathrm{Aut}(X)$. Then

X is a P-covering of the bipartite graph $K_{3,3}$ of order 6 such that $\mathrm{Aut}(X)$ projects to an arc-transitive subgroup of $\mathrm{Aut}(K_{3,3})$. Thus to classify the cubic symmetric graph of order $6p^3$ for $p \geq 7$, it suffices to determine all pairwise non-isomorphic P-coverings of the graph $K_{3,3}$ that admit a lift of an arc-transitive subgroup of $\mathrm{Aut}(K_{3,3})$, that is symmetric. Note that each P-covering of the graph $K_{3,3}$ with a lift of an arc-transitive subgroup of $\mathrm{Aut}(K_{3,3})$ is symmetric.

We now introduce some notations and terminology to be use the reminder the paper. From elementary group theory we know that up to isomorphism there are five groups of order p^3 for each odd prime p, of which three are abelian, that is,

$$\mathbb{Z}_{p^3}, \quad \mathbb{Z}_{p^2} \times \mathbb{Z}_p \quad \text{and} \quad \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p,$$

and two are nonabelian defined by

$$N(p^2,p)* = \langle x, y \mid x^{p^2} = y^p = 1, [x,y] = x^p \rangle,$$
$$N(p,p,p) = \langle x, y, z \mid x^p = y^p = z^p = 1, [x,y] = z, [z,x] = [z,y] = 1 \rangle.$$

Let $\{\mathbf{0,\ 2,\ 4}\}$ and $\{\mathbf{1,\ 3,\ 5}\}$ be the two partite sets of $K_{3,3}$ (see Fig. 1). Take a spanning tree of $K_{3,3}$, say T, with edge set $\{\{\mathbf{0,1}\}, \{\mathbf{0,3}\}, \{\mathbf{0,5}\}, \{\mathbf{2,1}\}, \{\mathbf{4,1}\}\}$ denoted by semi dark lines in Fig. 1.

Let P be a group of order p^3 for a prime $p \geq 7$ and let $X = K_{3,3} \times_\phi P$ be a connected P-covering of $K_{3,3}$ admitting a lift of an arc-transitive group of automorphisms of $K_{3,3}$, say L, where ϕ is a voltage assignment valued in the voltage group P. Assign voltage 1 to the tree arcs of T and voltages a, b, c and d in P to cotree arcs $(\mathbf{1\ 2}), (\mathbf{1\ 4}), (\mathbf{2\ 5})$ and $(\mathbf{4\ 5})$, respectively. By the connectivity of X, we have $P = \langle a, b, c, d \rangle$. Note that $\mathrm{Aut}(K_{3,3}) \cong (S_3 \times S_3) \rtimes Z_2$. Thus $\mathrm{Aut}(K_{3,3})$ has a normal Sylow 3-subgroup, say H, that is, $H = \langle \alpha, \beta \rangle$, where $\alpha = (\mathbf{0\ 2\ 4})$ and $\beta = (\mathbf{1\ 3\ 5})$. Clearly, each arc-transitive subgroup of $\mathrm{Aut}(K_{3,3})$ contains H as a subgroup. Furthermore, an arc-transitive subgroup of $\mathrm{Aut}(K_{3,3})$ must contain an automorphism which reverses the arc (0 1). It is easy to see that in this case at least one of the three automorphisms $\gamma = (\mathbf{0\ 1})(\mathbf{2\ 5})(\mathbf{3\ 4})$, $\tau = (\mathbf{0\ 1})(\mathbf{2\ 3})(\mathbf{4\ 5})$ and $\delta = (\mathbf{0\ 1})(\mathbf{2\ 3\ 4\ 5})$ belong to this subgroup. From these, it can be easily verified that $\mathrm{Aut}(K_{3,3}) = \langle \alpha, \beta, \gamma, \delta \rangle$ and each proper arc-transitive subgroup of $\mathrm{Aut}(K_{3,3})$ is conjugate in $\mathrm{Aut}(K_{3,3})$ to one of the three subgroups $L_1 = \langle \alpha, \beta, \gamma \rangle$, $L_2 = \langle \alpha, \beta, \gamma, \tau \rangle$ and $L_3 = \langle \alpha, \beta, \delta \rangle$. Furthermore, L_1 is 1-regular, L_2 and L_3 are 2-regular, $L_1 \leq L_2$ and L_3 does not contain a 1-regular subgroup. Thus we may assume that α, β and either γ or δ lift to automorphisms of X.

Denote by $i_1 i_2 \cdots i_s$ a directed cycle which has vertices $i_1,\ i_2,\ \cdots,\ i_s$ in a consecutive order. There are four fundamental cycles **03210**, **03410**, **01250** and **01450** in $K_{3,3}$, which are generated by the four cotree arcs **(3 2)**, **(3 4)**, **(2 5)** and **(4 5)**, respectively. Each cycle is mapped to a cycle of the same length under the actions of α, β, γ, τ and δ. We list all these cycles and their voltages in Table 1, in which C denotes a fundamental cycle of Q_3 and $\phi(C)$ denotes the voltage of C.

Consider the mapping $\bar{\alpha}$ from the set $\{a, b, c, d\}$ of voltages of the four fundamental cycles of $K_{3,3}$ to the group P, which is defined by $(\phi(C))^{\bar{\alpha}} = \phi(C^\alpha)$, where C ranges over the four fundamental cycles. Similarly, we can define $\bar{\beta}$, $\bar{\gamma}$, $\bar{\tau}$ and $\bar{\delta}$.

Since α, β and either γ or δ lift, by Proposition 2.1, either $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$ or $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\delta}$ can be extended to automorphisms of P. We denote by α^*, β^*, γ^* and δ^* these automorphisms, respectively. From Table 1, $b^{\alpha^*} = a^{-1}$, $d^{\alpha^*} = c^{-1}$ and $c^{\beta^*} = a$. It follows that a, b, c and d have the same order in P. For any $x \in P$, denote by $o(x)$ the order of x in P. Then $o(a) = o(b) = o(c) = o(d)$. We summarise the previous paragraph's observations as follows.

Observation. (1) *$P = \langle a, b, c, d\rangle$ and $o(a) = o(b) = o(c) = o(d)$.*
(2) *$\bar{\alpha}$, $\bar{\beta}$ and either $\bar{\gamma}$ or $\bar{\delta}$ can be extended to automorphisms of P.*

Lemma 4.3. *Let P be an abelian group of order p^3 for a prime $p \geq 7$ and X be a connected P-covering of the graph $K_{3,3}$ admiting a lift of an arc-transitive subgroup of Aut($K_{3,3}$). Then $p-1$ is divisible by 3 and X is isomorphic to the 1-regular graphs AF_{6p^3}, BF_{6p^3} or CF_{6p^3} .*

Proof. Let $X = K_{3,3} \times_\phi P$ be a P-covering of the graph $K_{3,3}$ satisfying the assumptions. Then all statements in the above observation are valid. Since P is abelian, $P = \mathbb{Z}_{p^3}$, $\mathbb{Z}_p^3$ or $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$.

Case I. $P = \mathbb{Z}_{p^3}$.

By observation (1), $P = \langle a\rangle$. Note that an automorphism of P is of the form $x \longmapsto x^t$, $x \in P$, where t is coprime to p^3. Hence we may assume that $\alpha^* : x \longmapsto x^k$ and $\beta^* : x \longmapsto x^l$, for each $x \in P$, where k and l are coprime to p^3. By Table 1, $a^{\alpha^*} = a^{-1}b$ and $a^{\beta^*} = c^{-1}a^{-1}$ imply that $b = a^{k+1}$ and $c = a^{-(l+1)}$. Considering the image of $b = a^{k+1}$ under β^*, one has $d^{-1}b^{-1} = a^{l(k+1)}$, which implies that $d = a^{-(k+1)(l+1)}$. Thus we obtain that

$$b = a^{k+1}, \quad c = a^{-(l+1)}, \quad d = a^{-(k+1)(l+1)}.$$

Furthermore, because $b^{\alpha^*} = a^{-1}$ and $c^{\beta^*} = a$, we have $k^2+k+1 = 0$ and $l^2+l+1 = 0$ in $\mathbb{Z}_{p^3}$ and since $p \geq 7$, k and l are of order 3 in $\mathbb{Z}_{p^3}^*$. Then $p-1$ is divisible by 3. Since there are exactly two elements of order 3 in $\mathbb{Z}_{p^3}^*$, $l = k$ or k^2. Assume that $l = k$. By using $k^2+k+1 = 0$, we have $b = a^{k+1}, c = a^{-k-1}$ and $d = a^{-k}$. Suppose $\bar{\gamma}$ can be extended to an automorphism of P, say γ^*. By Table 1, $b^{\gamma^*} = b^{-1}$, one has $-(k+1) = -k(k+1)$, implying $k = -2$ and hence $p = 3$ because $k^2+k+1 = 0$, a contradiction. With the same argument one can prove that $\bar{\delta}$ cannot be extended to an automorphism of P, contrary to Observation (2).

Thus $l = k^2$. Similarly, in this case one can get $b = a^{k+1}$, $c = a^k$ and $d = a^{-1}$. By Table 1, it is easy to check that $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$ can be extended to automorphisms of P induced by $a \longmapsto a^k$, $a \longmapsto a^{k^2}$ and $a \longmapsto a^{-1}$, respectively, but $\bar{\tau}$ and $\bar{\delta}$ cannot. By Proposition 2.1, α, β and γ lift but τ and δ cannot. Since $\langle \alpha, \beta, \gamma\rangle$ is 1-regular, Proposition 2.1, Lemma 4.1 and Proposition 2.2 imply that X is 1-regular. Set $\lambda = k$. By Example 3.1 and Lemma 3.5, $X \cong AF_{6p^3}$.

Case II. $P = \mathbb{Z}_p^3$.

By Observation (1), $P = \langle a, b, c, d\rangle$ and $o(a) = o(b) = o(c) = o(d) = p$. Suppose $\langle a\rangle = \langle c\rangle$. Then $c = a^k$ for some $k \in \mathbb{Z}_p^*$ and hence $c^{\alpha^*} = (a^{\alpha^*})^k$. By

Table 1, $dc^{-1} = a^{-k}b^k$. It follows $d = b^k$. Thus $P = \langle a, b\rangle$, which contradicts the hypothesis $P = \mathbb{Z}_p^3$. Suppose $c \in \langle a, b\rangle$. Considering the image of a, b and c under α^*, one has $dc^{-1} \in \langle a^{-1}b, a^{-1}\rangle$ and so $P = \langle a, c\rangle$, a contradiction. This implies that $P = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$.

One may assume that $d = a^i b^j c^k$ for some $i, j, k \in \mathbb{Z}_p$. By considering the image of d under α^* and β^*, we have $c^{-1} = (a^{-1}b)^i a^{-j}(dc^{-1})^k$ and $b = (c^{-1}a^{-1})^i(d^{-1}b^{-1})^j a^k$. Since $P = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \cong \mathbb{Z}_p^3$, by considering the powers of b and c in the first equation and the power of b in the second equation we have the following equations in $\mathbb{Z}_p$:

$$i + jk = 0, \quad k^2 - k + 1 = 0, \quad j^2 + j + 1 = 0.$$

By the last two equations, j and $-k$ are of order 3 in $\mathbb{Z}_p^*$. It follows that $k = -j$ or $-j^2$ and $p - 1$ is divisible by 3. Assume that $k = -j$. Since $i + jk = 0$, one has $i = j^2$. Then $d = a^{j^2} b^j c^{-j}$. By Table 1, it is easy to show that $\bar{\gamma}$ and $\bar{\delta}$ cannot be extended to automorphisms of P, contradicting Observation (2).

Thus $k = -j^2$. By $i+jk = 0$, $i = 1$ because $j^3 = 1$. It follows that $d = ab^j c^{-j^2}$. Set $\lambda = j$. By Proposition 2.1, Example 3.2 and Lemma 3.5, $X \cong BF_{6p^3}$. From Table 1, one can check that $\bar{\alpha}, \bar{\delta}$ and $\bar{\gamma}$ can be extended to automorphisms of P induced by

$$\begin{aligned} &a \longmapsto a^{-1}b, && b \longmapsto a^{-1}, && c \longmapsto ab^j c^j,\\ &a \longmapsto c^{-1}a^{-1}, && b \longmapsto a^{-1}b^{j^2}c^{j^2}, && c \longmapsto a,\\ &a \longmapsto ab^j c^{-j^2}, && b \longmapsto b^{-1}, && c \longmapsto c^{-1}, \end{aligned}$$

respectively, but $\bar{\tau}$ and $\bar{\delta}$ cannot. Then, the some reason as in Case I implies that X is 1-regular.

Case III: $P = \mathbb{Z}_{p^2} \times \mathbb{Z}_p$

Let $P = \langle x\rangle \times \langle y\rangle$ with $o(x) = p^2$ and $o(y) = p$. By Observation (1), $o(a) = o(b) = o(c) = o(t) = p^2$.

First assume that $P = \langle a, b\rangle$. Then $\langle a\rangle = \langle b\rangle = \langle a^p\rangle = \langle b^p\rangle$, implying $b = a^r$ for some $r \in \mathbb{Z}_p^*$. Thus $o(a^{-r}b) = p$ and so $P = \langle a, a^{-r}b\rangle$. One may assume that $a = x$ and $a^{-r}b = y$. Hence

$$a = x, \quad b = x^r y, \quad c = x^i y^j, \quad d = x^k y^l, \tag{2}$$

where $i, k \in \mathbb{Z}_{p^2}$ and $j, l \in \mathbb{Z}_p$. Clearly $c = a^{i-rj}b^j$ and $d = a^{k-rl}b^l$. By considering the image of $c = a^{i-rj}b^j$ under α^* and β^*, we conclude that $dc^{-1} = a^{rj-i}b^{i-rj}a^{-j}$ and $a = c^{rj-i}a^{rj-i}d^{-j}b^{-j}$ from Table 1, which, together with (2), implies the following equations:

$$rj + ri - r^2 j - j - k = 0 \pmod{p^2}, \tag{3}$$

$$i - rj - l + j = 0, \tag{4}$$

$$rij - i^2 - i - kj - 1 = 0 \pmod{p^2}, \tag{5}$$

$$rj^2 - ij - lj - j = 0. \tag{6}$$

In the above equations we have adopted the convention to suppress the modulus when the equation is to be taken modulus p. We will continue in this way with all the forthcoming equations that are to be taken mod p; unless specified otherwise. Similarly, by considering the image of $d = a^{k-rl}b^l$ under α^* and β^* one gets the following:

$$rl - k + rk - r^2 l - l + i = 0 \pmod{p^2}, \tag{7}$$

$$k - rl + j = 0, \tag{8}$$

$$ril - ik - kl - r - k = 0 \pmod{p^2}, \tag{9}$$

$$rjl - jk - l^2 - l - 1 = 0. \tag{10}$$

By (6), $j = 0$ or $rj - i - l - 1 = 0$. First assume that $j = 0$. From Eqs. (4) and (5), $i = l$ and $i^2 + i + 1 = 0 \pmod{p^2}$. It follows that $a = x,\ b = x^r y,\ c = x^i$ and $d = x^k y^i$. Suppose $\bar{\gamma}$ can be extended to an automorphism of P, say γ^*. By considering the image of $c = a^i$ under γ^*, we have $c^{-1} = d^i$, which implies that $x^{-i} = x^{ki} y^{i^2}$ from Table 1. Consequently $i^2 = 0$, implying $i = 0$ and by $i^2 + i + 1 = 0$, one has $1 = 0$, a contradiction. Similarly, one can show that $\bar{\delta}$ cannot be extended to an automorphism of P, contrary to Observation (2).

Thus $rj - i - l - 1 = 0$. Eq. (4) implies that $j = 2l + 1$ and multiplying Eq. (8) by j and adding to (10), we conclude that $j^2 = l^2 + l + 1$. Consequently $3l(l + 1) = 0$, one has $l = 0$ or -1 because $p \geq 7$.

Assume that $l = -1$. Then $j = 2l + 1 = -1$, and by (4) and (8), $i = -r$ and $k = 1 - r$. One may assume $l = -1 + l_1 p \pmod{p^2}$, $j = -1 + j_1 p \pmod{p^2}$, $i = -r + i_1 p \pmod{p^2}$ and $k = 1 - r + k_1 p \pmod{p^2}$. By (3), (5), (7) and (9), we have the following equations:

$$rj_1 + ri_1 - r^2 j_1 - j_1 - k_1 = 0,$$

$$-r^2 j_1 + rj_1 + ri_1 - i_1 + k_1 - j_1 = 0,$$

$$rl_1 - k_1 + rk_1 - r^2 l_1 - l_1 + i_1 = 0,$$

$$-r^2 l_1 + rk_1 + rl_1 - i_1 - l_1 = 0.$$

By the first two equations, $i_1 = 2k_1$ and by last two equations, $k_1 = 2i_1$. Consequently $i_1 = k_1 = 0$. It follows that $a = x,\ b = x^r y,\ c = x^{-r}y^{-1}$ and $d = x^{1-r}y^{-1}$. By Table 1, $y^{\alpha^*} = (a^{-r}b)^{\alpha^*} = a^{r-1}b^{-r} = x^{-r^2-1}y^{-r}$. This implies that $r^2 - r + 1 = 0$, because $o(y) = p$. Suppose $\bar{\gamma}$ can be extended to an automorphism of P, say γ^*. By Table 1, $y^{\gamma^*} = (a^{-r}b)^{\gamma^*} = d^{-r}b^{-1} = x^{-r-1}y^{r-1}$, which is impossible because $o(x^{-r-1}y^{r-1}) = p^2$. One may obtain a similar contradiction if $\bar{\delta}$ can be extended to an automorphism of P, contradicting Observation (2).

Suppose that $l = 0$. Then $j = 2l + 1 = 1$. From Eqs. (4) and (8), one has $i = r - 1$ and $k = -1$. One may assume that $l = l_1 p \pmod{p^2}$, $j = 1 + j_1 p \pmod{p^2}$, $i = r - 1 + i_1 p \pmod{p^2}$ and $k = -1 + k_1 p \pmod{p^2}$. By (3), (5), (7) and (9), we have the following:

$$\begin{aligned} rj_1 + ri_1 - r^2 j_1 - j_1 - k_1 &= 0, \\ -r^2 j_1 + rj_1 + ri_1 - i_1 + k_1 - j_1 &= 0, \\ rl_1 - k_1 + rk_1 - r^2 l_1 - l_1 + i_1 &= 0, \\ -r^2 l_1 + rk_1 + rl_1 - i_1 - l_1 &= 0. \end{aligned}$$

By the first two equations, $i_1 = 2k_1$ and by the last two equations, $k_1 = 2i_1$. Hence $i_1 = k_1 = 0$, implying

$$a = x, \quad b = x^r y, \quad c = x^{r-1} y, \quad d = x^{-1}.$$

By Table 1, $y^{\alpha^*} = (a^{-r}b)^{\alpha^*} = a^r b^{-r} a^{-1} = x^{-r^2+r-1} y^{-r}$. Since b has order p, one has $r^2 - r + 1 = 0$, which implies that $-r$ has order 3 in $\mathbb{Z}_p^*$ and hence $p-1$ is divisible by 3. Thus, there is an integer m such that $-(r+mp)$ is of order 3 in $\mathbb{Z}_{p^2}^*$, implying $(r+mp)^2 - (r+mp) + 1 = 0 \pmod{p^2}$. Set $\lambda = -(r+mp)$. Since $x \longmapsto x$ and $y \longmapsto x^{mp} y$ extended to an automorphism of P, by Proposition 2.1, one may assume that

$$a = x, \quad b = x^{-\lambda} y, \quad c = x^{-\lambda-1} y, \quad d = x^{-1}.$$

By Example 3.3 and Lemma 3.5, $X \cong CF_{6p^3}$. By Table 1, it is easy to check that $\bar{\alpha}, \bar{\beta}$ and $\bar{\gamma}$ can be extends to automorphisms of P induced by

$$\begin{aligned} &x \longmapsto x^{-\lambda-1} y, \quad y \longmapsto y^{\lambda}, \\ &x \longmapsto x^{\lambda} y^{-1}, \quad\ \ y \longmapsto y^{-\lambda-1}, \\ &x \longmapsto x^{-1}, \qquad\ \ y \longmapsto y^{-1}, \end{aligned}$$

respectively. By the same argument as above one can show that $\bar{\tau}$ and $\bar{\delta}$ cannot be extended to automorphisms of P. Then, using same kind reasoning as in Case I, X is 1-regular.

Now assume that $P \neq \langle a, b \rangle$. Then $b = a^s$ where $s \in \mathbb{Z}_{p^2}^*$. Since $b^{\beta^*} = (a^{\beta^*})^s$, one has $d^{-1} b^{-1} = c^{-s} a^{-s}$ from Table 1, which implies that $d = c^s$ because b. Thus $P = \langle a, c \rangle$ and one may assume that

$$a = x, \quad b = x^s, \quad c = x^r y, \quad d = x^{rs} y^s,$$

where $r \in \mathbb{Z}_{p^2}^*$. Considering the image of b under α^* and using Table 1, $a^{-1} = a^{-s} b^s$, implying $x^{-1} = x^{s^2 - s}$. It follows that $s^2 - s + 1 = 0 \pmod{p^2}$. Suppose $\bar{\gamma}$ can be extended to an automorphism of P, say γ^*. By considering the image of $b = a^s$ under γ^*, one has $b^{-1} = d^s$ and hence $x^{-s} = x^{rs^2} y^{s^2}$. It follows that $s^2 = 0$, implying $s = 0$ and by $s^2 - s + 1 = 0$, one has $1 = 0$, a contradiction. Similarly, one can show that $\bar{\delta}$ cannot be extended to an automorphism of P, contradicting Observation (2). ■

LEMMA 4.4. *Let P be a nonabelian group of order p^3 for a prime $p \geq 7$, and X is a connected P-covering of the graph $K_{3,3}$ admitting a lift of an arc-transitive subgroup of Aut($K_{3,3}$). Then X is isomorphic to the 2-regular graph DF_{6p^3}.*

Proof. Let $X = K_{3,3} \times_\phi P$ be a P-covering of the graph $K_{3,3}$ satisfying the assumptions and let $A :=\mathrm{Aut}(X)$. Then all statements in the observation preceding Lemma 4.3 are valid. Since P is nonabelian, $P = N(p^2, p)$ or $N(p, p, p)$.

Case I. $P = N(p^2, p) = \langle x, y \mid x^{p^2} = y^p = 1, [x, y] = x^p\rangle$.

Set $C := C_A(P)$. By Lemma 4.1, P is normal in A. Then by [19, Theorem 1.6.13], A/C is isomorphic to a subgroup of $\mathrm{Aut}(P)$. Note that each arc-transitive subgroup of $\mathrm{Aut}(K_{3,3})$ contains the Sylow 3-subgroup H of $\mathrm{Aut}(K_{3,3})$. By the hypothesis H lifts to a subgroup of $\mathrm{Aut}(X)$, say B. Then $B = P \rtimes H$, because $P \lhd B$. Since $H \cong Z_3 \times Z_3$, the Sylow 3-subgroups of B as well as A are isomorphic to $Z_3 \times Z_3$. By [24, Lemma 2.3], $\mathrm{Aut}(P)$ has a cyclic Sylow 3-subgroup. Thus $3 \mid |C|$. Note that $Z(P)$ is a normal Sylow p-subgroup of C so by [19, Theorem 9.1.2], there is a subgroup L of C such that $C = Z(P) \times L$, implying that L is characteristic in C and hence is normal in A. Since $(|L|, |Z(P)|) = 1$, L has more than two orbits on $V(X)$. By Proposition 2.2, L is semiregular on $V(X)$ and the quotient graph X_L of X corresponding to the orbits of L is a cubic symmetric graph. Since L is semiregular and $p^2 \nmid |L|$, one has $|L| \mid 6$. If L was even order, then X_L would be a cubic graph of odd order, a contradiction. Thus $|L| = 3$ and so X_L is a cubic symmetric graph of order $2p^3$. But by [10, Theorem 3.2], there is no cubic symmetric graph of order $2p^3$ for $p \geq 7$ whose automorphism group has a Sylow p-subgroup isomorphic to $N(p^2, p)$.

Case II. $N(p, p, p) = \langle x, y, z \mid x^p = y^p = z^p = 1, [x, y] = z, [z, x] = [z, y] = 1\rangle$.

It is easy to see that $P' = Z(P) = \langle z\rangle$. Then for any $s, w \in P$ and any integers i, j, one has $[s^i, w^j] = [s, w]^{ij}$ and $s^i w^j = w^j s^i [s^i, w^j]$. Furthermore, by [19, Theorem 5.3.5], $(sw)^i = s^i w^i [w, s]^{\binom{i}{2}}$. First assume that $P = \langle a, c\rangle$. One may show that $\mathrm{Aut}(P)$ acts transitively on the set of ordered pairs of generators of P and so by proposition 2.3, one may let $a = x,\ c = y$ and $b = x^i y^j z^k$ for some $i,\ j,\ k \in \mathbb{Z}_p$. Thus $b = a^i c^j [a, c]^k$. Considering the image of $b = a^i c^j [a, c]^k$ under β^*, one has $d^{-1}b^{-1} = (c^{-1}a^{-1})^i a^j [c^{-1}a^{-1}, a]^k = c^{-i} a^{j-i} [a, c]^{k+\binom{i}{2}} = y^{-i} x^{j-i} z^{k+\binom{i}{2}}$ and hence $d^{-1} = y^{-i} x^{j-i} z^{k+\binom{i}{2}} b = y^{j-i} x^j z^{j^2+2k+\binom{i}{2}}$. Set $t = j^2 + 2k + \binom{i}{2}$. Then $d^{-1} = y^{j-i} x^j z^t = c^{j-i} a^j [a, c]^t$ and by considering its image under α^*, we have that

$$\begin{aligned} y = c &= (dc^{-1})^{j-i} (a^{-1}b)^j [dc^{-1}, a^{-1}b]^t \\ &= (x^{-j} y^{i-j-1})^{j-i} (x^{i-1} y^j)^j z^{t(j-i)+kj} [x^{-j} y^{i-j-1}, x^{i-1} y^j]^t. \end{aligned}$$

Since $P/P' = \langle xP'\rangle \times \langle yP'\rangle$, one has $yP' = x^{j(i-j)+j(i-1)} y^{(j-i)(i-j-1)+j^2} P'$, which implies the following equations:

$$j(2i - j - 1) = 0, \quad 2ij - j - i^2 + i = 1.$$

By the first equation, either $j = 0$ or $j = 2i - 1$. Suppose that $j = 0$. By the second equation, $i^2 - i + 1 = 0$. Considering the image of $d^{-1} = c^{-i}[a, c]^t$ under β^*, we have $x^{-i} z^{-k} = b^{-1} = a^{-i} [c^{-1}a^{-1}, a]^t = x^{-i} z^t$, implying $z^{-k} = z^t$ and so $t + k = 0$. It follows that $6k + (i^2 - i) = 0$, because $t = 2k + \binom{i}{2}$. Since $i^2 - i + 1 = 0$, we have $k = -6^{-1}$, where 6^{-1} denotes the inverse of 6 in $\in \mathbb{Z}_p^*$. Also $t = -6^{-1}$. Thus, $a = x,\ b = x^i z^{6^{-1}},\ c = y$ and $d = y^i z^{6^{-1}}$. Suppose $\bar{\gamma}$ can be extended to an automorphism of P, say γ^*. By Table 1, $z^{\gamma^*} = [x,y]^{\gamma^*} = [a,c]^{\gamma^*} = [d,c^{-1}] = [y^i,y^{-1}] = 1$, which impossible. One may obtain a similar contradiction if $\bar{\delta}$ can be extended to an automorphism of P, contradicting Observation (2).

Thus $j = 2i-1$. By $2ij - i^2 + i - j = 1$, one has $3i^2 - 3i = 0$, and since $p \geq 7$, $i = 0$ or 1. Suppose that $i = 1$. Then $j = 2i-1 = 1$. Considering the image of $d^{-1} = a[a,c]^t$ under β^*, we have $y^{-1}x^{-1}z^{-k} = b^{-1} = c^{-1}a^{-1}[c^{-1}a^{-1},a]^t = y^{-1}x^{-1}z^t$. It follows that $k+t=0$, which implies that $3k+1=0$. Hence $k = -3^{-1}$ and $t = 3^{-1}$. Thus $a = x$, $b = xyz^{-3^{-1}}$, $c = y$, $d = x^{-1}z^{-3^{-1}}$. By Example 3.4, $X \cong DF_{6p^3}$. Based on Table 1, $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ and $\bar{\tau}$ can be extended to automorphisms of P induced by

$$\begin{array}{lll} x \longmapsto yz^{-3^{-1}}, & y \longmapsto x^{-1}y^{-1}z^{-3^{-1}}, & z \longmapsto z, \\ x \longmapsto y^{-1}x^{-1}, & y \longmapsto x, & z \longmapsto z, \\ x \longmapsto x^{-1}z^{-3^{-1}}, & y \longmapsto y^{-1}, & z \longmapsto z, \\ x \longmapsto x^{-1}, & y \longmapsto xyz^{-3^{-1}}, & z \longmapsto z^{-1}, \end{array}$$

respectively. Suppose $\bar{\delta}$ can be extended to an automorphism of P, say δ^*. Since $d^{\delta^*} = (a^{\delta^*})^{-1}[a^{\delta^*},c^{\delta^*}]^{-3^{-1}}$, one has $c^{-1} = b[b^{-1},a]^{-3^{-1}}$, implying $y^{-1} = xyz^{(-2)3^{-1}}$, which is impossible. By Proposition 2.1, α, β, γ and τ lift but δ cannot. Since $\langle\alpha,\beta,\gamma,\tau\rangle$ is 2-regular, by Proposition 2.1, Lemma 4.1 and Proposition 2.2, X is 2-regular.

Assume that $i = 0$. Then $j = 2i-1 = -1$. In this case by a similar argument as in the preceding paragraph one can show that $k = -3^{-1}$ and $t = 3^{-1}$. It follows that

$$a = x,\ b = y^{-1}z^{-3^{-1}},\ c = y,\ d = xyz^{-3^{-1}}.$$

By Example 3.4 and Lemma 3.5 $X \cong DF_{6p^3}$. From Table 1, one can check that $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ and $\bar{\tau}$ can be extended to automorphisms of P induced by

$$\begin{array}{lll} x \longmapsto x^{-1}y^{-1}z^{-3^{-1}}, & y \longmapsto xz^{-3^{-1}}, & z \longmapsto z, \\ x \longmapsto y^{-1}x^{-1}, & y \longmapsto x, & z \longmapsto z, \\ x \longmapsto xyz^{-3^{-1}}, & y \longmapsto y^{-1}, & z \longmapsto z^{-1}, \\ x \longmapsto x^{-1}, & y \longmapsto y^{-1}z^{-3^{-1}}, & z \longmapsto z, \end{array}$$

respectively, but $\bar{\delta}$ cannot. Then, with the same reason as in the preceding paragraph X is 2-regular.

Now assume that $P \neq \langle a,c\rangle$. Thus $|\langle a,c\rangle| = p$ or p^2. Assume that $|\langle a,c\rangle| = p$. Then $c = a^r$ where $r \in \mathbb{Z}_p^*$. By considering the image of c under β^*, one has $a = (c^{-1}a^{-1})^r$, which implies that $a = a^{-r^2-r}$. Consequently $r^2+r+1 = 0$. Since $\langle a\rangle = \langle c\rangle$, we have $\langle a^{\alpha^*}\rangle = \langle c^{\alpha^*}\rangle$, implying $\langle a^{-1}b\rangle = \langle dc^{-1}\rangle$. Hence $P = \langle a,b,c,d\rangle = \langle a,c,a^{-1}b,dc^{-1}\rangle = \langle a,b\rangle$. One may assume that $a = x$, $b = y$ and $c = x^r$. Since $c^{\alpha^*} = (a^{\alpha^*})^r$, by Table 1, $dc^{-1} = (a^{-1}b)^r$. This implies that $d = (a^{-1}b)^r c = (x^{-1}y)^r x^r = y^r z^{-r^2+\binom{i}{2}}$, implying $d = y^r z^{2^{-1}}$ because $r^2+r+1 = 0$.

Then $d^{\alpha^*} = (b^{\alpha^*})^r(z^{\alpha^*})^{2^{-1}}$ and hence $c^{-1} = a^{-r}(z^{\alpha^*})$. Consequently $z^{\alpha^*} = 1$, a contradiction. Hence $|\langle a, c\rangle| = p^2$ and $\langle a, c\rangle = \langle a\rangle \times \langle c\rangle$. Suppose $b \in \langle a, c\rangle$. By considering the image of a, b and c under β^*, we have $d \in \langle a, c\rangle$ and hence $P = \langle a, c\rangle$, a contradiction. Thus $b \notin \langle a, c\rangle$, forcing $P = \langle a, b, c\rangle$. Assume that $P \neq \langle a, b\rangle$. Then $|\langle a, b\rangle| = p^2$ and so $\langle a, b\rangle \cap \langle a, c\rangle = Z(P)$. It follows that $a \in Z(P)$. Therefore $a^{\alpha^*}, a^{\beta^*} \in Z(P)$ implying $b, c \in Z(P)$ and consequently P is abelian, a contradiction. Thus $P = \langle a, b\rangle$ and one may assume that $a = x$ and $b = y$. Since $|\langle a, c\rangle| = p^2$, one has $z \in \langle a, c\rangle$ and hence $c = x^i z^j$ for some $i, j \in \mathbb{Z}_p$. Then $c = a^i[a, b]^j$ and since $c^{\beta^*} = (a^{\beta^*})^i[a^{\beta^*}, b^{\beta^*}]^j$, we have that $a^{i^2+i+1} \in P'$ which implies that $i^2 + i + 1 = 0$. Similarly, by considering the image of $c = a^i[a, b]^j$ under α^*, one has $dc^{-1} = (a^{-1}b)^i[a^{-1}b, a^{-1}]^j$, implying $d = b^i[a, b]^{2j+2^{-1}}$ because $i^2 + i + 1 = 0$. Then $d^{\alpha^*} = (b^{\alpha^*})^i[a^{\alpha^*}, b^{\alpha^*}]^{2j+2^{-1}}$. It follows that $c^{-1} = a^i[a^{-1}b, a^{-1}]^{2j+2^{-1}}$. Therefore $x^{-i}z^{-j} = x^{-i}z^{2j+2^{-1}}$, that is $z^{-j} = z^{2j+2^{-1}}$ and consequently $j = -6^{-1}$. Hence $a = x,\ b = y,\ c = x^i z^{-6^{-1}}$ and $d = y^i z^{6^{-1}}$ where $i^2 + i + 1 = 0$. Suppose $\overline{\gamma}$ can be extended to an automorphism of P, say γ^*. By Table 1, $z^{\gamma^*} = [a^{\gamma^*}, b^{\gamma^*}] = [d, b^{-1}] = [y^i, y^{-1}] = 1$, which is impossible. One may obtain a similar contradiction if $\overline{\delta}$ can be extended to an automorphism of P, contrary to Observation (2). ■

Now by Lemmas 4.1, 4.2, 4.3 and 4.4 we have the following classification theorem which is the main result of this paper.

THEOREM 4.5. *Let X be a connected cubic symmetric graph of order $6p^3$, where p is a prime. Then X is 1-, 2- or 3-regular. Furthermore,*

(1) *X is 1-regular if and only if X is isomorphic to one of the graphs F_{162B}, AF_{6p^3}, BF_{6p^3} and CF_{6p^3}, where $p-1$ is divisible by 3.*

(2) *X is 2-regular if and only if X is isomorphic to one of the graphs F_{48}, F_{750}, F_{162A} and DF_{6p^3} where $p \geq 7$.*

(3) *X is 3-regular if and only if X is isomorphic to the graph F_{162C}.*

REFERENCES

[1] N. Biggs, *Three remarkable graphs*, Canad. J. Math. **25** (1973,) 397–411.

[2] Y. Cheng, J. Oxley, *On weakly symmetric graphs of order twice a prime*, J. Combin. Theory, Ser. B **42** (1987), 196–211.

[3] M. Conder, P. Dobcsányi, *Trivalent symmetric graphs on up to* 768 *vertices*, J. Combin. Math. Combin. Comput. **40** (2002), 41–63.

[4] M.D.E. Conder, P. Lorimer, *Automorphism groups of symmetric graphs of valency 3*, J. Combin. Theory, Ser. B **47** (1989), 60–72.

[5] M.D.E. Conder, C.E. Praeger, *Remarks on path-transitivity on finite graphs*, European J. Combin. **17** (1996), 371–378.

[6] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, R.A. Wilson, *Atlas of Finite Groups*, Clarendon Press, Oxford, 1985.

[7] D.Ž. Djoković, G.L. Miller, *Regular groups of automorphisms of cubic graphs*, J. Combin. Theory, Ser. B **29** (1980), 195–230.

[8] Y.Q. Feng, J.H. Kwak, *Cubic symmetric graphs of order twice an odd prime-power*, J. Aust. Math. Soc. **81** (2006), 153–164.

[**9**] Y.Q. Feng, J.H. Kwak, *Cubic symmetric graphs of order a small number times a prime or a prime square*, J. Combin. Theory, Ser. B **97** (2007), 627–646.

[**10**] Y.Q. Feng, J.H. Kwak, M.Y. Xu, *Cubic s-regular graphs of order* $2p^3$, J. Graph Theory **52** (2006), 341–352.

[**11**] D. Gorenstein, *Finite Simple Groups*, Plenum Press, New York, 1982.

[**12**] J.L. Gross, T.W. Tucker, *Generating all graph coverings by permutation voltages assignment*, Discrete Math. **18** (1977), 273–283.

[**13**] P. Lorimer, *Vertex-transitive graphs: Symmetric graphs of prime valency*, J. Graph Theory **8** (1984), 55–68.

[**14**] A. Malnič, *Group actions, coverings and lifts of automorphisms*, Discrete Math. **182** (1998), 203–218.

[**15**] A. Malnič, D. Marušić, C.Q. Wang, *Cubic edge-transitive graphs of order* $2p^3$, Discrete Math. **274** (2004), 187–198.

[**16**] D. Marušić, T. Pisanski, *Symmetries of hexagonal graphs on the torus*, Croat. Chemica Acta **73** (2000), 69–81.

[**17**] D. Marušić, M.Y. Xu, *A* $\frac{1}{2}$*-transitive graph of valency 4 with a nonsolvable group of automorphisms*, J. Graph Theory **25** (1997), 133–138.

[**18**] R.C. Miller, *The trivalent symmetric graphs of girth at most six*, J. Combin. Theory, Ser. B **10** (1971), 163–182.

[**19**] D.J. Robinson, *A Course in the Theory of Groups*, Springer-Verlag, New York, 1982.

[**20**] M. Škoviera, *A contribution to the theory of voltage groups*, Discrete Math. **61** (1986), 281–292.

[**21**] W.T. Tutte, *A family of cubical graphs*, Proc. Cambridge Philos. Soc. **43** (1947), 459–474.

[**22**] W.T. Tutte, *On the symmetry of cubic graphs*, Canad. J. Math. **11** (1959), 621–624.

[**23**] H. Wielandt, *Finite Permutation Groups*, Academic Press, New York, 1964.

[**24**] M.Y. Xu, *Half-transitive graphs of prime-cube order*, J. Algebraic Combin. **1** (1992), 275–282.

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