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TOWARDS CANTOR INTERSECTION THEOREM AND BAIRE CATEGORY THEOREM IN PARTIAL METRIC SPACES

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Abstract. In this paper we consider a suitable definition of convergence and introduce star closed sets that enable us to establish a variant of Cantor intersection theorem as well as Baire category theorem in partial metric spaces.

1. Introduction

As in [5], let $X = S^w$ be the set of all sequences over a nonempty set S. Then (X, d) is a metric space if for all $x, y \in X$, $d(x, y) = 2^{-k}$, where k is the largest nonnegative integer or ∞ such that $x_i = y_i$ for each $i < k$. In Computer Science, every sequence is approximated by a section of it however long it may be because it is not possible, by writing a computer program, to compute a sequence and print out its values in any finite amount of time. Suppose now that the above definition of 'd' is extended to Y, the set of all finite sequences over S. Then 'd' fails to be a metric on Y because $d(x, x) = 2^{-k}$ for some $k < \infty$, that is because nonzero self-distance is possible. Now, the question is, "is there a generalization of the metric space axioms in which nonzero self-distances are possible, such that most familiar metric and topological properties are retained?" Partial metric space appears as an affirmative answer to this question. In 1994, S. G. Matthews [7] introduced the concept of partial metric space to study denotational semantics of programming languages and brought the concept of nonzero self distances.

For a metric space (X, d) closed sets play an important role to characterize its completeness. In this paper we introduce star closed sets and show that Cantor intersection theorem and Baire category theorem can be achieved in partial metric spaces.

As in [7], a mapping $p : X \times X \to [0, \infty)$, where X is a nonempty set, is said to be a partial metric if whenever $x, y, z \in X$ the following conditions hold:

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(1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$;
(2) $p(x,y) = p(y,x)$;
(3) $p(x,y) \geq p(x,x)$;
(4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$

and the ordered pair (X,p) is called a partial metric space. A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x,y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}^+$.

Each partial metric p on X generates a T_0 topology τ_p on X having the family of open p-balls $B_p(x,\epsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}$ as a basis. According to [7], a sequence $\{x_n\}$ in (X,p) is said to converge to x if $\lim_{n\to\infty} p(x_n,x) = p(x,x)$.

Now let $X = [0,\infty)$ and $p(x,y) = \max\{x,y\}$, $\forall x, y \in X$. Then the sequence $\{1,2,1,2,\dots\}$ is convergent in the partial metric space (X,p), which is absurd. So, as in [5], we say that a sequence $\{x_n\}$ in a partial metric space (X,p) converges to $x \in X$ if $\lim_{n\to\infty} p(x_n,x) = p(x,x) = \lim_{n\to\infty} p(x_n,x_n)$. A sequence $\{x_n\}$ in (X,p) is a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and a partial metric space is said to be complete if every Cauchy sequence in (X,p) is convergent.

For a partial metric space (X,p), the functions $d_w, d_p : X \times X \to \mathbb{R}^+$ given by $d_w(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ and

$$\begin{aligned} d_p(x,y) &= \max\{p(x,y) - p(x,x), p(x,y) - p(y,y)\} \\ &= p(x,y) - \min\{p(x,x), p(y,y)\} \end{aligned}$$

are metrics on X.

Throughout this paper, by $\overline{A}^p$ we mean the closure of A in (X,p) and by $\overline{A}$ the closure of A in (X,d_p). We define the interior of A by $\operatorname{Int} A = \{x \in A : B_p(x,r) \subset A$ for some $r > 0\}$. The articles [2–4, 6, 8–11, 13, 14] are cited to provide more information about partial metric spaces achieved by several researchers working in this field.

2. Star closure, Cantor intersection theorem and Baire category theorem

Let us begin with the example stated above and consider the subset $A = (0,1)$ of X. Then $\overline{A}^p = [0,\infty)$. Now take $a \in [0,\infty) - [0,1]$. Then $a \in \overline{A}^p$, but there does not exist any sequence $x_n \in A$ such that $\lim_{n\to\infty} p(x_n,x_n) = p(a,a)$ and so $\{x_n\}$ does not converge to a. i.e., the sequence lemma does not hold in a partial metric space. Now we define the star closure of a subset and star closed sets in a partial metric space. We define $\overline{A}^*$ by

$$\overline{A}^* = A \cup \{\, l : l \text{ is a limit of a convergent sequence in } A\,\}.$$

A is said to be a star closed set in (X,p) if $A = \overline{A}^*$. In a partial metric space (X,p) we observe the following properties

(i) $\overline{\emptyset}^* = \emptyset$.
(ii) $\overline{X}^* = X$.

(iii) $A \subset B \Rightarrow \overline{A}^* \subset \overline{B}^*$.

(iv) $\overline{A}^*$ is the smallest star closed set containing A.

(v) $\overline{\bigcup_{n=1}^{m} A_n}^* = \bigcup_{n=1}^{m} \overline{A}_n^*$.

Now since a partial metric space (in the sense of [7]) is not even T_1 in general, there is a doubt about the uniqueness of limit of a convergent sequence therein. In the case of convergence defined as in [5] (which we assume in the sequel), there is no such doubt according to the following

Proposition 2.1. *In a partial metric space (X, p) the limit of a convergent sequence (in the sense of* [5]*) is unique.*

Proof. Let $\{x_n\}$ be a sequence in (X, p) that converges to x and y. Then

$$\lim_{n\to\infty} p(x_n, x) = p(x, x) = \lim_{n\to\infty} p(x_n, x_n), \text{ and} \tag{2.1}$$

$$\lim_{n\to\infty} p(x_n, y) = p(y, y) = \lim_{n\to\infty} p(x_n, x_n). \tag{2.2}$$

From (2.1) and (2.2) we get

$$p(x, x) = p(y, y). \tag{2.3}$$

Now $p(x, y) \leq p(x, x_n) + p(x_n, y) - p(x_n, x_n)$. Taking the limit we have $p(x, y) \leq p(x, x)$. Also we have $p(x, y) \geq p(x, x)$. Hence $p(x, y) = p(x, x)$. Using (2.3) we have $p(x, y) = p(x, x) = p(y, y)$, which implies $x = y$. ■

Proposition 2.2. *$\overline{A}^*$ is a star closed set.*

Proof. Let $x \in \overline{\overline{A}^*}^*$. Then there exists a sequence $x_n \in \overline{A}^*$ such that $\lim_{n\to\infty} p(x_n, x) = p(x, x) = \lim_{n\to\infty} p(x_n, x_n)$. Since $x_n \in \overline{A}^*$ there exists a sequence $u_m^n \in A$ such that $\lim_{m\to\infty} p(u_m^n, x_n) = p(x_n, x_n) = \lim_{m\to\infty} p(u_m^n, u_m^n)$. Hence

$$\lim_{n,m\to\infty} p(u_m^n, x_n) = p(x, x) = \lim_{m\to\infty} p(u_m^n, u_m^n). \tag{2.4}$$

Now $p(u_m^n, x) \leq p(u_m^n, x_n) + p(x_n, x) - p(x_n, x_n)$. Taking the limit as $n, m \to \infty$ we get $\lim_{m\to\infty} p(u_m^n, x) \leq p(x, x)$. Hence $\lim_{m\to\infty} p(u_m^n, x) = p(x, x)$. Using (2.4) we get $\lim_{m\to\infty} p(u_m^n, x) = p(x, x) = \lim_{m\to\infty} p(u_m^n, u_m^n)$. So $x \in \overline{A}^*$. Thus $\overline{\overline{A}^*}^* \subset \overline{A}^*$. Hence $\overline{\overline{A}^*}^* = \overline{A}^*$. ■

Note. Defining the star closure of a subset of a topological space in the same way it can be seen that the properties (i)–(v) hold but in this case the star closure may not be idempotent.

Proposition 2.3. *Any closed set in (X, p) is a star closed set in (X, p).*

Proof. Let A be a closed in (X, p). Then $A = \overline{A}^p$. Let $x \in \overline{A}^*$. Then there exists a sequence $x_n \in A$ such that $\lim_{n\to\infty} p(x_n, x) = p(x, x) = \lim_{n\to\infty} p(x_n, x_n)$. It follows that $x \in \overline{A}^p = A$. Thus $\overline{A}^* = A$. Hence A is star closed in (X, p). ■

Example 2.1. The subset $A = \{1, 2, 3, 4\}$ is a star closed set but not a closed set in the partial metric space $(\mathbb{R}^+, p)$ with $p(x, y) = \max\{x, y\}$ $\forall x, y \in \mathbb{R}^+$.

Lemma 2.1. [1] *Let $\{x_n\}$ be a sequence in a partial metric space (X, p) and $x \in X$. Then $\lim_{n\to\infty} d_p(x_n, x) = 0$ if and only if $\lim_{n\to\infty} p(x_n, x) = p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.*

Proposition 2.4. *Any star closed set in (X, p) is a closed set in (X, d_p).*

Proof. Let A be a star closed set in (X, p). i.e., $A = \overline{A}^*$. Let $x \in \overline{A}$. Then there exists a sequence $\{x_n\}$ in A that converges to x with respect to $\tau(d_p)$ and using Lemma 2.1 we have $x \in \overline{A}^* = A$. Thus $\overline{A} \subset A$. Hence $A = \overline{A}$. ∎

Example 2.2. Let us consider $(\mathbb{R}^+, p)$ with $p(x, y) = \max\{x, y\}$ $\forall x, y \in \mathbb{R}^+$. Let $C = \{a, b\}$. Then $\overline{C} = C$ but $\overline{C}^p \neq C$. Hence a closed subset of (X, d_p) may not be closed set in (X, p).

Lemma 2.2. [12] *In a partial metric space, if $\{x_n\}$ converges to x and $\{y_n\}$ converges to y then $\lim_{n\to\infty} p(x_n, y_n) = p(x, y)$.*

Lemma 2.3. *In a partial metric space (X, p),*

(a) *$\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, d_p).*

(b) *(X, p) is complete if and only if (X, d_p) is complete.*

Proof. Part (a) is done in [1]. We prove part (b). Let (X, d_p) be complete. If $\{x_n\}$ is a Cauchy sequence in (X, p), then by (a) $\{x_n\}$ is a Cauchy sequence in (X, d_p). Since (X, d_p) is complete there exists $y \in X$ such that $\lim_{n\to\infty} d_p(x_n, y) = 0$. So, $\lim_{n\to\infty} p(x_n, y) = p(y, y) = \lim_{n\to\infty} p(x_n, x_n)$. Conversely, let (X, p) be complete and $\{x_n\}$ be a Cauchy sequence in (X, d_p). Hence $\{x_n\}$ is a Cauchy sequence in (X, p). Since (X, p) is complete there exists $y \in X$ such that $\lim_{n\to\infty} p(x_n, y) = p(y, y) = \lim_{n\to\infty} p(x_n, x_n)$. Then for any $\epsilon > 0$ there exists a natural number M such that $p(x_n, y) - p(y, y) < \epsilon$ $\forall n \geq M$. Then

$$d_p(y, x_n) = p(x_n, y) - \min\{p(x_n, x_n), p(y, y)\} < \epsilon \quad \forall\, n \geq M.$$

Hence (X, d_p) is complete. ∎

Remark 2.1. Since d_p and d_w are equivalent metrics, we can take d_w instead of d_p in the above lemma.

Lemma 2.4. *$p(A) = p(\overline{A}^*)$ where $p(A) = \sup\{p(x, y) - p(x, x) : x, y \in A\}$.*

Proof. For any subsets A, B of the underlying partial metric space (X, p), $A \subset B$ implies $A \times A \subset B \times B$ and so

$$\sup\{p(x, y) - p(x, x) : x, y \in A\} \leq \sup\{p(x, y) - p(x, x) : x, y \in B\}$$

which means $p(A) \leq p(B)$. Since $A \subset \overline{A}^*$, $p(A) \leq p(\overline{A}^*)$. Let $x, y \in \overline{A}^*$; then there exist $\{x_n\}, \{y_n\}$ in A such that $\{x_n\}$ converges to x and $\{y_n\}$ converges to y. Now

$p(x,y) \leq p(x,x_n) + p(x_n,y_n) + p(y_n,y) - p(x_n,x_n) - p(y_n,y_n)$, so after passing to the limit $p(x,y) - p(x,x) \leq p(A)$. Thus, $\sup\{p(x,y) - p(x,x) : x,y \in \overline{A}^*\} \leq p(A)$ that is, $p(\overline{A}^*) \leq p(A)$ and we are done. ■

THEOREM 2.1. *A partial metric space (X,p) is complete if and only if for every sequence $\{F_n\}$ of star closed sets in (X,p) satisfying:*

(a) $F_{n+1} \subset F_n$ $\forall n \in \mathbb{N}$ *and*

(b) $p(F_n) \to 0$ *as* $n \to \infty$,

the intersection $\bigcap_{n=1}^{\infty} F_n$ is a singleton.

Proof. Let $\{x_n\}$ be a Cauchy sequence in (X,p) and $\lim_{n\to\infty} p(x_{n+p},x_n) = \alpha \in \mathbb{R}$. So for any $\epsilon > 0$, there exists a natural number v such that $|p(x_{n+p},x_n) - \alpha| < \frac{\epsilon}{2}$, $\forall n \geq v$. Let $F_n = \{x_{n+p-1} : p \in \mathbb{N}\}$. Then $F_{n+1} \subset F_n$ implies $\overline{F_{n+1}}^* \subset \overline{F_n}^*$. Now

$$\begin{aligned} |p(x_{n+p},x_n) - p(x_n,x_n)| &\leq |p(x_{n+p},x_n) - \alpha| + |p(x_n,x_n) - \alpha| \\ &< \epsilon \quad \forall n \geq v. \end{aligned}$$

Hence $\lim_{n\to\infty}\{p(x_{n+p},x_n) - p(x_n,x_n) : x_{n+p},x_n \in F_n\} = 0$. So, $p(F_n) \to 0$ as $n \to \infty$ and as a result $p(\overline{F_n}^*) \to 0$ as $n \to \infty$. So we have $\bigcap_{n=1}^{\infty} \overline{F_n}^* \neq \emptyset$. Let $x \in \bigcap_{n=1}^{\infty} \overline{F_n}^*$. Since $x_n \in F_n$, $0 \leq p(x_n,x) - p(x,x) \leq p(\overline{F_n}^*)$. Taking the limit and using Sandwich Theorem we get $\lim_{n\to\infty} p(x_n,x) = p(x,x)$. Similarly one can show that $0 \leq p(x,x_n) - p(x_n,x_n) \leq p(\overline{F_n}^*)$. Using Sandwich Theorem we get $\lim_{n\to\infty} p(x_n,x_n) = p(x,x)$. So, $\lim_{n\to\infty} p(x_n,x) = p(x,x) = \lim_{n\to\infty} p(x_n,x_n)$. Hence (X,p) is complete.

Conversely, let (X,p) be a complete partial metric space. Let us choose $x_n \in F_n$ $\forall n \in \mathbb{N}$. Since $F_{n+1} \subset F_n$, it follows that $x_m \in F_n$ $\forall m \geq n$. Now $0 \leq p(x_n,x_m) - p(x_n,x_n) \leq p(F_n)$ and $0 \leq p(x_m,x_n) - p(x_m,x_m) \leq p(F_n)$. So, $0 \leq p(x_n,x_m) - \min\{p(x_n,x_n), p(x_m,x_m)\} \leq p(F_n)$. Thus $0 \leq d_p(x_n,x_m) \leq p(F_n)$. Using condition (b) and Sandwich Theorem we see that $\{x_n\}$ is a Cauchy sequence in (X,d_p). Since (X,p) is complete, by Lemma 2.3 we can say that (X,d_p) is complete. Hence there exists $x \in X$ such that $d_p(x_n,x) \to 0$ as $n \to \infty$. This implies $\{x_n\}$ converges to x in (X,p). Therefore $x \in F_n$ as F_n is star closed in (X,p). Thus $x \in F_n$ $\forall n \in \mathbb{N}$. Let $y \in \bigcap_{n=1}^{\infty} F_n$. This implies $0 \leq p(x,y) - p(x,x) \leq p(F_n)$. Taking the limit and using Sandwich Theorem we get $p(x,y) = p(x,x)$. Similarly one can get $p(x,y) = p(y,y)$. Hence $p(x,y) = p(x,x) = p(y,y)$ and so $x = y$. Thus $\bigcap_{n=1}^{\infty} F_n$ is a singleton. ■

DEFINITION 2.1. A subset B of a partial metric space (X,p) is said to be nowhere dense in (X,p) if every open set contains an open set V such that $V \cap B = \emptyset$.

DEFINITION 2.2. A partial metric space (X,p) is said to be of second category if it cannot be written as a countable union of nowhere dense sets. Otherwise it is of first category.

LEMMA 2.5. *A closed ball in a partial metric space (X,p) is a star closed set in (X,p).*

Proof. $B_p[x,\epsilon] = \{y \in X : p(x,y) \le p(x,x)+\epsilon\}$ is a closed ball in (X,p). Let $z \in \overline{B_p[x,\epsilon]}^*$. Then there exists a sequence $x_n \in B_p[x,\epsilon]$ such that $\lim_{n\to\infty} p(x_n,z) = p(z,z) = \lim_{n\to\infty} p(x_n,x_n)$. Also $p(x_n,x) \le p(x,x)+\epsilon$. Now $p(x,z) \le p(x,x_n) + p(x_n,z) - p(x_n,x_n)$. Taking the limit we have $p(x,z) \le p(x,x)+\epsilon$ implying that $z \in B_p[x,\epsilon]$. Hence $\overline{B_p[x,\epsilon]}^* \subset B_p[x,\epsilon]$. ■

EXAMPLE 2.3. A closed ball in a partial metric space (X,p) may fail to be closed in (X,p). For example, let us consider the partial metric space $([0,\infty),p)$, where $p(x,y) = \max\{x,y\}$. Here every open ball is of the form $B_p(x,\epsilon) = [0,x+\epsilon)$ and every closed ball is of the form $B_p[x,\epsilon] = [0,x+\epsilon]$. So, no closed ball in this partial metric space can be closed.

THEOREM 2.2. *Every complete partial metric space is of second category.*

Proof. Let (X,p) be a complete partial metric space. Let $X = \bigcup_{n=1}^{\infty} A_n$, where A_n is nowhere dense in (X,p) for all $n \in \mathbb{N}$. Since A_1 is nowhere dense in (X,p), the open set X contains an open set $B_p(x_1,r_1)$ such that $B_p(x_1,r_1) \cap A_1 = \emptyset$, where $0 < r_1 < 1$. Let $F_1 = B_p[x_1,\frac{r_1}{2}]$ and $x_2 \in \operatorname{Int} F_1$. Since A_2 is nowhere dense in (X,p), $\operatorname{Int} F_1$ contains an open set $B_p(x_2,r_2)$ such that $B_p(x_2,r_2) \cap A_2 = \emptyset$, where $0 < r_2 < \frac{1}{2}$. Let $F_2 = B_p[x_2,\frac{r_2}{2}]$. Continuing in this way we get a decreasing sequence $\{F_n\}$ of star closed sets in X, where $F_n = B_p[x_n,\frac{r_n}{2}]$ and $0 < r_n < \frac{1}{2^{n-1}}$ with the properties

(a) $F_{n+1} \subset F_n$ $\forall n \in \mathbb{N}$ and

(b) $p(F_n) \to 0$ as $n \to \infty$.

Since (X,p) is complete, using Theorem 2.1 we have $\bigcap_{n=1}^{\infty} F_n = \{x\}$ for some $x \in X$. So, $x \in B_p(x_n,r_n)$ and $B_p(x_n,r_n) \cap A_n = \emptyset$ $\forall n \in \mathbb{N}$ which implies that $x \notin A_n$ $\forall n \in \mathbb{N}$. This contradicts $x \in X$. Hence (X,p) is of second category. ■

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