MATEMATIČKI VESNIK
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# ON GENERALIZATIONS OF BOEHMIAN SPACE AND HARTLEY TRANSFORM

**C. Ganesan and R. Roopkumar**

**Abstract.** Boehmians are quotients of sequences which are constructed by using a set of axioms. In particular, one of these axioms states that the set $S$ from which the *denominator* sequences are formed should be a commutative semigroup with respect to a binary operation. In this paper, we introduce a generalization of abstract Boehmian space, called generalized Boehmian space or $G$-Boehmian space, in which $S$ is not necessarily a commutative semigroup. Next, we provide an example of a $G$-Boehmian space and we discuss an extension of the Hartley transform on it.

## 1. Introduction

Motivated by the Boehme's regular operators [1], a generalized function space called Boehmian space is introduced by J. Mikusiński and P. Mikusiński [6] and two notions of convergence called $\delta$-convergence and $\Delta$-convergence on a Boehmian space are introduced in [7]. In general, an abstract Boehmian space is constructed by using a suitable topological vector space $\Gamma$, a subset $S$ of $\Gamma$, $\star : \Gamma \times S \to \Gamma$ and a collection $\Delta$ of sequences satisfying some axioms. In [9], the abstract Boehmian space is generalized by replacing $S$ with a commutative semi-group in such a way that $S$ is not even comparable with $\Gamma$ and the binary operation on $S$ need not be the same as $\star$. Using this generalization of Boehmians, a lot of Boehmian spaces have been constructed for extending various integral transforms. To mention a few recent works on Boehmians, we refer to [12–16, 18]. There is yet another generalization of Boehmians called generalized quotients or pseudoquotients [3, 10, 11].

According to the earlier constructions, we note that $S$ is assumed to be a commutative semi-group either with respect to the restriction of $\star$ or with respect to the binary operation defined on $S$. In this paper, we provide another generalization of an abstract Boehmian space, in which $S$ is not necessarily a commutative semi-group. We shall call such Boehmian space a generalized Boehmian space or simply a $G$-Boehmian space and we also provide a concrete example of a $G$-Boehmian

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space and study the Hartley transform on it. At this juncture, we point out that in a recent interesting paper on pseudoquotients [5], the commutativity of $S$ is relaxed by Ore type condition, which is entirely different from the generalization discussed in this paper.

## 2. Preliminaries

**2.1. Boehmians.** From [7], we briefly recall the construction of a Boehmian space $\mathcal{B} = \mathcal{B}(\Gamma, S, \star, \Delta)$, where $\Gamma$ is a topological vector space over $\mathbb{C}$, $S \subseteq \Gamma$, $\star : \Gamma \times S \to \Gamma$ satisfies the following conditions:

$(A_1)$ $(g_1 + g_2) \star s = g_1 \star s + g_2 \star s$, $\forall g_1, g_2 \in \Gamma$ and $\forall s \in S$,

$(A_2)$ $(cg) \star s = c(g \star s)$, $\forall c \in \mathbb{C}$, $\forall g \in \Gamma$ and $\forall s \in S$,

$(A_3)$ $g \star (s \star t) = (g \star s) \star t$, $\forall g \in \Gamma$ and $\forall s, t \in S$,

$(A_4)$ $s \star t = t \star s$, $\forall s, t \in S$,

$(A_c)$ If $g_n \to g$ as $n \to \infty$ in $\Gamma$ and $s \in S$, then $g_n \star s \to g \star s$ as $n \to \infty$ in $\Gamma$,

and $\Delta$ is a collection of sequences from $S$ with the following properties:

$(\Delta_1)$ If $(s_n), (t_n) \in \Delta$, then $(s_n \star t_n) \in \Delta$,

$(\Delta_2)$ If $g \in \Gamma$ and $(s_n) \in \Delta$, then $g \star s_n \to g$ as $n \to \infty$ in $\Gamma$.

We call a pair $((g_n), (s_n))$ of sequences satisfying the conditions $g_n \in \Gamma$, $\forall n \in \mathbb{N}$, $(s_n) \in \Delta$ and

$$g_n \star s_m = g_m \star s_n, \ \forall m, n \in \mathbb{N},$$

a quotient and is denoted by $\frac{g_n}{s_n}$. The equivalence class $\left[\frac{g_n}{s_n}\right]$ containing $\frac{g_n}{s_n}$ induced by the equivalence relation $\sim$ defined on the collection of all quotients by

$$\frac{g_n}{s_n} \sim \frac{h_n}{t_n} \text{ if } g_n \star t_m = h_m \star s_n, \ \forall m, n \in \mathbb{N} \tag{1}$$

is called a Boehmian and the collection $\mathcal{B}$ of all Boehmians is a vector space with respect to the addition and scalar multiplication defined as follows.

$$\left[\frac{g_n}{s_n}\right] + \left[\frac{h_n}{t_n}\right] = \left[\frac{g_n \star t_n + h_n \star s_n}{s_n \star t_n}\right], \ c\left[\frac{g_n}{s_n}\right] = \left[\frac{cg_n}{s_n}\right].$$

Every member $g \in \Gamma$ can be uniquely identified as a member of $\mathcal{B}$ by $\left[\frac{g \star s_n}{s_n}\right]$, where $(s_n) \in \Delta$ is arbitrary and the operation $\star$ is also extended to $\mathcal{B} \times S$ by $\left[\frac{g_n}{\phi_n}\right] \star t = \left[\frac{g_n \star t}{\phi_n}\right]$. There are two notions of convergence on $\mathcal{B}$ namely $\delta$-convergence and $\Delta$-convergence which are defined as follows.

Definition 2.1. We say that $X_m \xrightarrow{\delta} X$ as $m \to \infty$ in $\mathcal{B}$, if there exists $(s_n) \in \Delta$ such that $X_m \star \delta_n, X \star \delta_n \in \Gamma$, $\forall m, n \in \mathbb{N}$ and for each $n \in \mathbb{N}$, $X_m \star \delta_n \to X \star \delta_n$ as $m \to \infty$ in $\Gamma$.

Definition 2.2. We say that $X_m \xrightarrow{\Delta} X$ as $m \to \infty$ in $\mathcal{B}$, if there exists $(s_n) \in \Delta$ such that $(X_m - X) \star \delta_m \in \Gamma$, $\forall m \in \mathbb{N}$ and $(X_m - X) \star \delta_m \to 0$ as $m \to \infty$ in $\Gamma$.

**2.2. Hartley transform.** For an arbitrary integrable function $f$, the Hartley transform was defined by

$$[\mathcal{H}(f)](t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)[\cos xt + \sin xt]\, dx, \ \forall t \in \mathbb{R}$$

and its inverse is obtained from the formula $\mathcal{H}[\mathcal{H}(f)] = f$, whenever $\mathcal{H}(f) \in \mathcal{L}^1(\mathbb{R})$. For more details on the classical theory of Hartley transform, we refer to [2, 4].

The Hartley transform is one of the integral transforms which is closely related to Fourier transform in the following sense.

$$\mathcal{F}(f) = \frac{\mathcal{H}(f) + \mathcal{H}(-f)}{2} + i\frac{\mathcal{H}(f) - \mathcal{H}(-f)}{2} \text{ and } \mathcal{H}(f) = \frac{1+i}{2}\mathcal{F}(f) + i\frac{1-i}{2}\mathcal{F}(-f),$$

where $\mathcal{F}(f)$ is the Fourier transform of $f$, which is defined by

$$\mathcal{F}(f)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ixt}\, dx, \ \forall t \in \mathbb{R}.$$

However N. Sundararajan [19] pointed out that Hartley transform has some computational advantages over the Fourier transform and therefore it can be an ideal alternative of Fourier transform.

Furthermore, as $|[\mathcal{H}(f)](t)| \le 2|\mathcal{F}(f)(t)|$, $\forall t \in \mathbb{R}$, using the properties of Fourier transform, we have $\mathcal{H}(f) \in C_0(\mathbb{R})$, $\|\mathcal{H}(f)\|_\infty \le 2\|\mathcal{F}(f)\|_\infty \le \|f\|_1$ and hence the Hartley transform $\mathcal{H} : \mathcal{L}^1(\mathbb{R}) \to C_0(\mathbb{R})$ is continuous.

## 3. Generalized Boehmian spaces

We introduce a generalization of Boehmain space called $G$-Boehmian space $\mathcal{B}^\star(\Gamma, S, \star, \Delta)$, which is obtained by relaxing the Boehmian-axiom $(A_4)$ in Subsection 2.1 by

$$(A_4')\ f \star (s \star t) = (f \star t) \star s, \ \forall f \in \Gamma \text{ and } s, t \in S.$$

If we probe into know the necessity for introducing the axioms $(A_3)$ and $(A_4)$ for constructing Boehmians, we could see that these two axioms are used to prove the transitivity of the relation $\sim$ defined on the collection of all quotients in (1).

It is easy to see that the verification of reflexivity and symmetry for the relation $\sim$ are straightforward. So we now verify the transitivity of $\sim$ using $(A_3)$ and $(A_4')$.

Let $\frac{g_n}{s_n}$, $\frac{h_n}{t_n}$ and $\frac{p_n}{u_n}$ be quotients such that $\frac{g_n}{s_n} \sim \frac{h_n}{t_n}$ and $\frac{h_n}{t_n} \sim \frac{p_n}{u_n}$. Then, we have $g_n, h_n, p_n \in \Gamma$, $\forall n \in \mathbb{N}$, $(s_n)$, $(t_n)$, $(u_n) \in \Delta$ and

$$\begin{aligned} g_n \star s_m &= g_m \star s_n, \ \forall m, n \in \mathbb{N} \\ h_n \star t_m &= h_m \star t_n, \ \forall m, n \in \mathbb{N} \\ p_n \star u_m &= p_m \star u_n, \ \forall m, n \in \mathbb{N} \\ g_n \star t_m &= h_m \star s_n, \ \forall m, n \in \mathbb{N} \\ h_n \star u_m &= p_m \star t_n, \ \forall m, n \in \mathbb{N}. \end{aligned} \tag{2}$$

For arbitrary $m, n, j \in \mathbb{N}$, applying $(A_4')$, $(A_3)$ and (2), we get

$$\begin{aligned}(g_n \star u_m) \star t_j &= g_n \star (t_j \star u_m) = (g_n \star t_j) \star u_m \\ &= (h_j \star s_n) \star u_m = h_j \star (u_m \star s_n) \\ &= (h_j \star u_m) \star s_n = (p_m \star t_j) \star s_n \\ &= p_m \star (s_n \star t_j) = (p_m \star s_n) \star t_j.\end{aligned}$$

Next applying $(\Delta_2)$, we get $g_n \star u_m = p_m \star s_n$, $\forall m, n \in \mathbb{N}$, and hence $\frac{g_n}{s_n} \sim \frac{p_n}{u_n}$. Thus, the transitivity of $\sim$ follows.

We note that the axioms $(A_3)$ and $(A_4)$ are also used in the proof of the following statements:

- $\frac{g \star s_n}{s_n}$ is a quotient, $\forall g \in \Gamma$ and $(s_n) \in \Delta$,
- $\frac{g_n}{s_n} \sim \frac{g_n \star t_n}{s_n \star t_n}$, for each quotient $\frac{g_n}{s_n}$ and for each $(t_n) \in \Delta$,
- $\frac{g_n \star t}{s_n}$ is a quotient whenever $\frac{g_n}{s_n}$ is a quotient,
- $\frac{g_n \star t_n + h_n \star s_n}{s_n \star t_n}$ is a quotient whenever $\frac{g_n}{s_n}$ and $\frac{h_n}{t_n}$ are quotients,

and these statements can also be proved by using $(A_3)$ and $(A_4')$ as above.

Now we construct an example of a $G$-Boehmian space by proving the required auxiliary results. Let $\Gamma = S = \mathcal{L}^1(\mathbb{R})$, $\Delta$ be the usual collection of all sequences $(\delta_n)$ from $\mathcal{L}^1(\mathbb{R})$ satisfying the following properties.

$(P_1)$ $\int_{-\infty}^{\infty} \delta_n(t)\, dt = 1,\ \forall n \in \mathbb{N}$,

$(P_2)$ $\int_{-\infty}^{\infty} |\delta_n(t)|\, dt \le M,\ \forall n \in \mathbb{N}$, for some $M > 0$,

$(P_3)$ supp $\delta_n \to 0$ as $n \to \infty$, where supp $\delta_n$ is the support of $\delta_n$;

and # be the following convolution

$$(f\#g)(x) = \frac{1}{2}\int_{-\infty}^{\infty} [f(x+y) + f(x-y)]g(y)dy,\ \forall x \in \mathbb{R},$$

for all $f, g \in \mathcal{L}^1(\mathbb{R})$.

Lemma 3.1. *If $f, g \in \mathcal{L}^1(\mathbb{R})$, then $\|f\#g\|_1 \le \|f\|_1\|g\|_1$ and hence $f\#g \in \mathcal{L}^1(\mathbb{R})$.*

*Proof.* By using Fubini's theorem, we obtain

$$\begin{aligned}\|f\#g\|_1 &= \frac{1}{2}\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} [f(x+y) + f(x-y)]g(y)\, dy \right| dx \\ &\le \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |[f(x+y) + f(x-y)]g(y)|\, dy\ dx \\ &\le \frac{1}{2}\int_{-\infty}^{\infty} |g(y)| \int_{-\infty}^{\infty} |f(x+y) + f(x-y)|\, dx\, dy \\ &\le \|f\|_1\|g\|_1 < +\infty\end{aligned}$$

and hence $f\#g \in \mathcal{L}^1(\mathbb{R})$. ∎

Lemma 3.2. *If* $f, g$ *and* $h \in L^1(\mathbb{R})$ *then* $(f\#g)\#h = f\#(g\#h) = (f\#h)\#g$.

*Proof.* Let $f, g, h \in L^1(\mathbb{R})$ and let $x \in \mathbb{R}$. Repeatedly applying the Fubini's theorem, we get that

$$\begin{aligned}
[f\#(g\#h)](x) &= \int_{-\infty}^{\infty} [f(x+y) + f(x-y)](g\#h)(y)\,dy \\
&= \int_{-\infty}^{\infty} [f(x+y) + f(x-y)] \int_{-\infty}^{\infty} [g(y+z) + g(y-z)]h(z)\,dz\,dy \\
&= \int_{-\infty}^{\infty} h(z)\bigg( \int_{-\infty}^{\infty} [f(x+y) + f(x-y)]g(y+z)\,dy \\
&\quad + \int_{-\infty}^{\infty} [f(x+y) + f(x-y)]g(y-z)\,dy\bigg)\,dz \\
&= \int_{-\infty}^{\infty} h(z)\bigg( \int_{-\infty}^{\infty} [f(x+u-z) + f(x-u+z)]g(u)\,du \\
&\quad + \int_{-\infty}^{\infty} [f(x+u+z) + f(x-u-z)]g(u)\,du\bigg)\,dz,
\end{aligned}$$

(by using $y + z = u$ in the first term and $y - z = u$ in the second term)

$$\begin{aligned}
&= \int_{-\infty}^{\infty} h(z) \int_{-\infty}^{\infty} [f(x+u-z) + f(x-u+z) \\
&\quad + f(x+u+z) + f(x-u-z)]g(u)\,du\,dz \\
&= \int_{-\infty}^{\infty} h(z)\bigg( \int_{-\infty}^{\infty} [f(x+z+u) + f(x+z-u)]g(u)\,du \\
&\quad + \int_{-\infty}^{\infty} [f(x-z+u) + f(x-z-u)]g(u)\,du\bigg)\,dz \\
&= \int_{-\infty}^{\infty} h(z)[(f\#g)(x+z) + (f\#g)(x-z)]\,dx \\
&= [(f\#g)\#h](x). \qquad (3)
\end{aligned}$$

Using (3), we get

$$\begin{aligned}
[f\#(g\#h)](x) &= \int_{-\infty}^{\infty} h(z) \int_{-\infty}^{\infty} [f(x+z+u) + f(x+z-u) \\
&\quad + f(x-z+u) + f(x-z-u)]g(u)\,du\,dz \\
&= \int_{-\infty}^{\infty} g(u) \int_{-\infty}^{\infty} [f(x+z+u) + f(x+z-u) \\
&\quad + f(x-z+u) + f(x-z-u)]h(z)\,dz\,du \\
&= \int_{-\infty}^{\infty} g(u) \int_{-\infty}^{\infty} [f(x+u+z) + f(x+u-z) + f(x-u+z) \\
&\quad + f(x-u-z)]h(z)\,dz\,du
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} g(u)\left[\int_{-\infty}^{\infty}[f(x+u+z)+f(x+u-z)]h(z)\,dz\right.\\
&\quad \left. + \int_{-\infty}^{\infty}[f(x-u+z)+f(x-u-z)]h(z)\,dz\right] du\\
&= \int_{-\infty}^{\infty} g(u)[(f\#h)(x+u)+(f\#h)(x-u)]\,du\\
&= [(f\#h)\#g](x).
\end{aligned}
$$

Since $x \in \mathbb{R}$ is arbitrary, the proof follows. ∎

Lemma 3.3. *If $f_n \to f$ as $n \to \infty$ in $\mathcal{L}^1(\mathbb{R})$ and if $g \in \mathcal{L}^1(\mathbb{R})$, then $f_n\#g \to f\#g$ as $n \to \infty$ in $\mathcal{L}^1(\mathbb{R})$.*

*Proof.* From the proof of Lemma 3.1, we have the estimate

$$
\|(f_n - f)\#g\|_1 \leq \|f_n - f\|_1 \|g\|_1. \tag{4}
$$

Since $f_n \to f$ as $n \to \infty$ in $\mathcal{L}^1(\mathbb{R})$, the right hand side of (4) tends to zero as $n \to \infty$. Hence the lemma follows. ∎

Lemma 3.4. *If $(\delta_n)$, $(\psi_n) \in \Delta$ then $(\delta_n\#\psi_n) \in \Delta$.*

*Proof.* By using Fubini's theorem, we get

$$
\begin{aligned}
\int_{-\infty}^{\infty}(\delta_n\#\psi_n)(x)\,dx &= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\delta_n(x+y)+\delta_n(x-y)]\,\psi_n(y)\,dy\,dx\\
&= \frac{1}{2}\int_{-\infty}^{\infty}\psi_n(y)\int_{-\infty}^{\infty}[\delta_n(x+y)+\delta_n(x-y)]\,dx\,dy\\
&= \frac{1}{2}\int_{-\infty}^{\infty}\psi_n(y)\left[\int_{-\infty}^{\infty}\delta_n(z)\,dz + \int_{-\infty}^{\infty}\delta_n(z)\,dz\right]dy\\
&= \frac{1}{2}\int_{-\infty}^{\infty}2\psi_n(y)\,dy = 1, \text{ for all } n \in \mathbb{N}.
\end{aligned}
$$

By a similar argument, it is easy to verify that $\int_{-\infty}^{\infty}|(\delta_n\#\psi_n)(x)|\,dx \leq M$ for some $M > 0$. Since supp $\delta_n\#\psi_n \subset [\text{supp } \delta_n + \text{supp } \psi_n] \cup [\text{supp } \delta_n - \text{supp } \psi_n]$, we get that supp $(\delta_n\#\psi_n) \to \{0\}$ as $n \to \infty$. Hence it follows that $(\delta_n\#\psi_n) \in \Delta$. ∎

Theorem 3.5. *Let $f \in \mathcal{L}^1(\mathbb{R})$ and let $(\delta_n) \in \Delta$, then $f\#\delta_n \to f$ as $n \to \infty$ in $\mathcal{L}^1(\mathbb{R})$.*

*Proof.* Let $\epsilon > 0$ be given. By the property $(P_2)$ of $(\delta_n)$, there exists $M > 0$ with $\int_{-\infty}^{\infty}|\delta_n(t)|dt \leq M$, $\forall\ n \in \mathbb{N}$. Using the continuity of the mapping $y \mapsto f_y$ from $\mathbb{R}$ in to $\mathcal{L}^1(\mathbb{R})$, (see [17, Theorem 9.5]), choose $\delta > 0$ such that

$$
\|f_y - f_0\|_1 < \frac{\epsilon}{M} \text{ whenever } |y| < \delta, \tag{5}
$$

where $f_y(x) = f(x-y)$, $\forall x \in \mathbb{R}$. By the property $(P_3)$ of $(\delta_n)$, there exists $N \in \mathbb{N}$ with supp $\delta_n \subset [-\delta, \delta]$, $\forall\, n \geq N$. By using the property $(P_1)$ of $(\delta_n)$ and Fubini's theorem, we obtain

$$\|f\#\delta_n - f\|_1 = \int_{-\infty}^{\infty}\left|\frac{1}{2}\int_{-\infty}^{\infty}[f(x+y)+f(x-y)]\ \delta_n(y)\,dy - f(x)\int_{-\infty}^{\infty}\delta_n(y)\,dy\right|dx$$

$$\leq \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(|f(x+y)-f(x)|+|f(x-y)-f(x)|)\ |\delta_n(y)|\,dx\,dy$$

$$\leq \frac{1}{2}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|f(x+y)-f(x)|\,dx + \int_{-\infty}^{\infty}|f(x-y)-f(x)|\,dx\right)|\delta_n(y)|\,dy$$

$$= \frac{1}{2}\int_{-\delta}^{\delta}(\|f_{-y}-f_0\|_1 + \|f_y - f_0\|_1)\ |\delta_n(y)|\,dy,\ \forall\ n \geq N$$

$$< \frac{1}{2}\int_{-\delta}^{\delta}(\frac{\epsilon}{M}+\frac{\epsilon}{M})\ |\delta_n(y)|\,dy, \quad \text{by (5)}$$

$$= \frac{\epsilon}{M}\int_{-\delta}^{\delta}|\delta_n(y)|\,dy \leq \epsilon,\ \forall\ n \geq N$$

and hence $f\#\delta_n \to f$ as $n \to \infty$ in $\mathcal{L}^1(\mathbb{R})$. ■

Lemma 3.6. *If $f_n \to f$ as $n \to \infty$ in $\mathcal{L}^1(\mathbb{R})$ and $(\delta_n) \in \Delta$, then $f_n\#\delta_n \to f$ as $n \to \infty$ in $\mathcal{L}^1(\mathbb{R})$.*

*Proof.* For any $n \in \mathbb{N}$ we have

$$\begin{aligned}\|f_n\#\delta_n - f\|_1 &= \|f_n\#\delta_n - f\#\delta_n + f\#\delta_n - f\|_1\\ &\leq \|(f_n - f)\#\delta_n\|_1 + \|f\#\delta_n - f\|_1\\ &\leq \|f_n - f\|_1\ \|\delta_n\|_1 + \|f\#\delta_n - f\|_1,\ \text{(by Lemma 3.1)}\\ &\leq M\|f_n - f\|_1 + \|f\#\delta_n - f\|_1\end{aligned}$$

Since $f_n \to f$ as $n \to \infty$ in $\mathcal{L}^1(\mathbb{R})$ and by Theorem 3.5, the right hand side of the last inequality tends to zero as $n \to \infty$. Hence the lemma follows. ■

Thus the $G$-Boehmian space $\mathcal{B}^\star_{\mathcal{L}^1} = \mathcal{B}^\star(\mathcal{L}^1(\mathbb{R}), \mathcal{L}^1(\mathbb{R}), \#, \Delta)$ has been constructed.

Finally, we justify that the convolution # introduced in this section is not commutative.

Example 3.7. If $f(x) = \begin{cases} e^{-x} & \text{if } x \geq 0\\ 0 & \text{if } x < 0\end{cases}$ and $g(x) = \begin{cases} 0 & \text{if } x > 0\\ e^{x} & \text{if } x \leq 0,\end{cases}$ then $f, g \in L^1(\mathbb{R})$ and $f\#g \neq g\#f$.

Indeed, for any $x \in \mathbb{R}$, we have

$$\begin{aligned}(f\#g)(x) &= \int_{-\infty}^{\infty}[f(x+y)+f(x-y)]g(y)\,dy = \int_{-\infty}^{0}[f(x+y)+f(x-y)]e^y\,dy\\ &= \int_{-\infty}^{0} f(x+y)e^y\,dy + \int_{-\infty}^{0} f(x-y)e^y\,dy\end{aligned}$$

$$= \begin{cases} \int_{-x}^{0} e^{-(x+y)} e^{y}\, dy + \int_{-\infty}^{0} e^{-(x-y)} e^{y}\, dy & \text{if } x \geq 0 \\ 0 + \int_{-\infty}^{x} e^{-(x-y)} e^{y}\, dy & \text{if } x < 0 \end{cases}$$

$$= \begin{cases} e^{-x}(\int_{-x}^{0} dy + \int_{-\infty}^{0} e^{2y}\, dy) & \text{if } x \geq 0 \\ e^{-x} \int_{-\infty}^{x} e^{2y}\, dy & \text{if } x < 0 \end{cases}$$

$$= \begin{cases} e^{-x}(x + \frac{1}{2}) & \text{if } x \geq 0 \\ \frac{e^{x}}{2} & \text{if } x < 0 \end{cases}$$

and

$$(g\#f)(x) = \int_{-\infty}^{\infty} [g(x+y) + g(x-y)] f(y)\, dy = \int_{0}^{\infty} [g(x+y) + g(x-y)] e^{-y}\, dy$$

$$= \int_{0}^{\infty} g(x+y) e^{-y}\, dy + \int_{0}^{\infty} g(x-y) e^{-y}\, dy$$

$$= \begin{cases} 0 + \int_{x}^{\infty} e^{x-y} e^{-y}\, dy & \text{if } x > 0 \\ \int_{0}^{-x} e^{x+y} e^{-y}\, dy + \int_{0}^{\infty} e^{x-y} e^{-y}\, dy & \text{if } x \leq 0 \end{cases}$$

$$= \begin{cases} e^{x} \int_{x}^{\infty} e^{-2y}\, dy & \text{if } x > 0 \\ e^{x}(\int_{0}^{-x} dy + \int_{0}^{\infty} e^{-2y}\, dy) & \text{if } x \leq 0 \end{cases}$$

$$= \begin{cases} \frac{1}{2} e^{-x} & \text{if } x > 0 \\ e^{x}(-x + \frac{1}{2}) & \text{if } x \leq 0. \end{cases}$$

From the above computations it is clear that $f\#g \neq g\#f$ and hence our claim holds.

## 4. Hartley transform on $G$-Boehmians

As in the general case of extending any integral transform to the context of Boehmians, we have to first obtain a suitable convolution theorem for Hartley transform. To obtain a compact version of a convolution theorem for Hartley transform, for $f \in \mathcal{L}^1(\mathbb{R})$, we define

$$[\mathcal{C}(f)](t) = \int_{-\infty}^{\infty} f(x) \cos xt\, dx, \; t \in \mathbb{R}.$$

We point out that $\mathcal{C}$ is not the usual Fourier cosine transform, as Fourier cosine transform is defined for integrable functions on non-negative real numbers.

THEOREM 4.1. *If $f, g \in \mathcal{L}^1(\mathbb{R})$, then $\mathcal{H}(f\#g) = \mathcal{H}(f) \cdot \mathcal{C}(g)$.*

*Proof.* Let $t \in \mathbb{R}$ be arbitrary. By using Fubini's theorem, we obtain that

$$[\mathcal{H}(f\#g)](t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f\#g)(x)[\cos xt + \sin xt]\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [f(x+y) + f(x-y)] g(y) dy [\cos xt + \sin xt]\, dx$$

$$
\begin{aligned}
&= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) \int_{-\infty}^{\infty} [f(x+y) + f(x-y)]\ [\cos xt + \sin xt]\, dx\, dy \\
&= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) \bigg( \int_{-\infty}^{\infty} f(x+y) \cos xt\, dx + \int_{-\infty}^{\infty} f(x+y) \sin xt\, dx \\
&\quad + \int_{-\infty}^{\infty} f(x-y) \cos xt\, dx + \int_{-\infty}^{\infty} f(x-y) \sin xt\, dx \bigg)\, dy \\
&= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) \bigg( \int_{-\infty}^{\infty} f(z) \cos(zt - yt)\, dz + \int_{-\infty}^{\infty} f(z) \sin(zt - yt)\, dz \\
&\quad + \int_{-\infty}^{\infty} f(z) \cos(zt + yt)', dz + \int_{-\infty}^{\infty} f(z) \sin(zt + yt)\, dz \bigg)\, dy \\
&= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) \int_{-\infty}^{\infty} f(z) [2 \cos zt \cos yt + 2 \sin zt \cos yt]\, dz\, dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) \cos yt \int_{-\infty}^{\infty} f(z) [\cos zt + \sin zt]\, dz\, dy \\
&= [\mathcal{H}(f)](t) \cdot [\mathcal{C}(g)](t).
\end{aligned}
$$

Thus we have $\mathcal{H}(f \# g) = \mathcal{H}(f)\ \cdot \mathcal{C}(g)$. ■

THEOREM 4.2. *If $f, g \in \mathcal{L}^1(\mathbb{R})$, then $\mathcal{C}(f \# g) = \mathcal{C}(f) \cdot \mathcal{C}(g)$.*

*Proof.* Let $t \in \mathbb{R}$ be arbitrary. By using Fubini's theorem, we obtain

$$
\begin{aligned}
[\mathcal{C}(f \# g)](t) &= \int_{-\infty}^{\infty} (f \# g)(x) \cos xt\, dx \\
&= \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [f(x+y) + f(x-y)] g(y)\, dy\ \cos xt\, dx \\
&= \frac{1}{2} \int_{-\infty}^{\infty} g(y) \bigg( \int_{-\infty}^{\infty} f(x+y) \cos xt\, dx + \int_{-\infty}^{\infty} f(x-y) \cos xt\, dx \bigg)\, dy \\
&= \frac{1}{2} \int_{-\infty}^{\infty} g(y) \bigg( \int_{-\infty}^{\infty} f(z) \cos(zt - yt)\, dz + \int_{-\infty}^{\infty} f(z) \cos(zt + yt)\, dz \bigg)\, dy \\
&= \int_{-\infty}^{\infty} g(y) \cos yt \int_{-\infty}^{\infty} f(z) \cos zt\, dz\, dy \\
&= [\mathcal{C}(f)](t) \cdot [\mathcal{C}(g)](t).
\end{aligned}
$$

Since $t \in \mathbb{R}$ is arbitrary, we have $\mathcal{C}(f \# g) = \mathcal{C}(f) \cdot \mathcal{C}(g)$. ■

LEMMA 4.3. *If $(\delta_n) \in \Delta$ then $\mathcal{C}(\delta_n) \to 1$ as $n \to \infty$ uniformly on compact subset of $\mathbb{R}$.*

*Proof.* Let $K$ be a compact subset of $\mathbb{R}$. Let $\epsilon > 0$ be given. Choose $M_1 > 0$, $M_2 > 0$ and a positive integer $N$ such that $\int_{-\infty}^{\infty} |\delta_n(t)|\, dt \leq M_1$, $\forall n \in \mathbb{N}$, $K \subset [-M_2, M_2]$ and $supp\, \delta_n \subset [-\epsilon, \epsilon]$ for all $n \geq N$. Then for $t \in K$ and $n \geq N$, we have

$$
|[\mathcal{C}(\delta_n)](t) - 1| = \left| \int_{-\infty}^{\infty} \delta_n(s) \cos ts\, ds - \int_{-\infty}^{\infty} \delta_n(s)\, ds \right|
$$

$$\leq \int_{-\infty}^{\infty} |\delta_n(s)|\,|\cos ts - 1|\,ds = \int_{-\epsilon}^{\epsilon} |\delta_n(s)|\,|\cos ts - 1|\,ds,\ \forall\, n \geq N$$
$$\leq \int_{-\epsilon}^{\epsilon} |\delta_n(s)|\,|ts|\,ds,$$

(by using mean-value theorem, and $|\sin x| \leq 1$, $\forall x \in \mathbb{R}$)

$$\leq M_2\epsilon \int_{-\epsilon}^{\epsilon} |\delta_n(s)|\,ds \leq M_2 M_1 \epsilon.$$

This completes the proof. ■

DEFINITION 4.1. For $\beta = \left[\frac{f_n}{\delta_n}\right] \in \mathcal{B}^{\star}_{\mathcal{L}^1}$, we define the extended Hartley transform of $\beta$ by $[\mathcal{H}(\beta)](t) = \lim_{n\to\infty}[\mathcal{H}(f_n)](t)$, $(t \in \mathbb{R})$.

The above limit exists and is independent of the representative $\frac{f_n}{\delta_n}$ of $\beta$. Indeed, for $t \in \mathbb{R}$, choose $k$ such that $[\mathcal{C}(\delta_k)](t) \neq 0$. Then, applying Theorem 4.1, we obtain that $[\mathcal{H}(f_n)](t) = \frac{[\mathcal{H}(f_n\#\delta_k)](t)}{[\mathcal{C}(\delta_k)](t)} = \frac{[\mathcal{H}(f_k\#\delta_n)](t)}{[\mathcal{C}(\delta_k)](t)} = \frac{[\mathcal{H}(f_k)](t)}{[\mathcal{C}(\delta_k)](t)}[\mathcal{C}(\delta_n)](t)$. Therefore, using Lemma 4.3, we get $[\mathcal{H}(f_n)](t) \to \frac{[\mathcal{H}(f_k)](t)}{[\mathcal{C}(\delta_k)](t)}$, as $n \to \infty$ uniformly on each compact subset of $\mathbb{R}$. If $\frac{f_n}{\delta_n} \sim \frac{g_n}{\psi_n}$, then $f_n\#\psi_m = g_m\#\delta_n$ for all $m, n \in \mathbb{N}$. Again using Theorem 4.1, we get $\lim_{n\to\infty}[\mathcal{H}(f_n)](t) = \frac{[\mathcal{H}(f_k)](t)}{[\mathcal{C}(\delta_k)](t)} = \frac{[\mathcal{H}(g_k)](t)}{[\mathcal{C}(\psi_k)](t)} = \lim_{n\to\infty}[\mathcal{H}(g_n)](t)$.

If $f \in \mathcal{L}^1(\mathbb{R})$ and $\beta = \left[\frac{f\#\delta_n}{\delta_n}\right]$, then

$$[\mathcal{H}(\beta)](t) = \lim_{n\to\infty}[\mathcal{H}(f\#\delta_n)](t) = [\mathcal{H}(f)](t) \lim_{n\to\infty}[\mathcal{C}(\delta_n)](t) = [\mathcal{H}(f)](t),$$

as $[\mathcal{C}(\delta_n)](t) \to 1$ as $n \to \infty$ uniformly on each compact subset of $\mathbb{R}$. This shows that the extended Hartley transform is consistent with the Hartley transform on $\mathcal{L}^1(\mathbb{R})$.

THEOREM 4.4. *If $\beta \in \mathcal{B}^{\star}_{\mathcal{L}^1}$, then the extended Hartley transform $\mathcal{H}(\beta) \in C(\mathbb{R})$.*

*Proof.* As $\mathcal{H}(\beta)$ is the uniform limit of $\{H(f_n)\}$ on each compact subset of $\mathbb{R}$ and each $H(f_n)$ is a continuous function on $\mathbb{R}$, $\mathcal{H}(\beta)$ is a continuous function on $\mathbb{R}$. ■

As proving the following properties of the Hartley transform on Boehmians is a routine exercise, as in the case of Fourier transform on integrable Boehmians [8], we just state them without proofs.

THEOREM 4.5. *The Hartley transform $\mathcal{H} : \mathcal{B}^{\star}_{\mathcal{L}^1} \to C(\mathbb{R})$ is linear.*

THEOREM 4.6. *The Hartley transform $\mathcal{H} : \mathcal{B}^{\star}_{\mathcal{L}^1} \to C(\mathbb{R})$ is one-to-one.*

THEOREM 4.7. *The Hartley transform $\mathcal{H} : \mathcal{B}^{\star}_{\mathcal{L}^1} \to C(\mathbb{R})$ is continuous with respect to $\delta$-convergence and $\Delta$-convergence.*

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C.G.: Department of Mathematics, V. H. N. S. N. College, Virudhunagar - 626001, India

*E-mail*: c.ganesan28@yahoo.com

R.R.:Department of Mathematics, Central University of Tamil Nadu, Thiruvarur - 610101, India

*E-mail*: roopkumarr@rediffmail.com