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CURVATURES OF TANGENT BUNDLE OF FINSLER MANIFOLD WITH CHEEGER-GROMOLL METRIC

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Abstract. Let (M, F) be a Finsler manifold and G be the Cheeger-Gromoll metric induced by F on the slit tangent bundle $\widetilde{TM} = TM\backslash\{0\}$. In this paper, we will prove that the Finsler manifold (M, F) is of scalar flag curvature $K = \alpha$ if and only if the unit horizontal Liouville vector field $\xi = \frac{y^i}{F}\frac{\delta}{\delta x^i}$ is a Killing vector field on the indicatrix bundle IM where $\alpha : TM \to \mathbb{R}$ is defined by $\alpha(x, y) = 1 + g_x(y, y)$. Also, we will calculate the scalar curvature of a tangent bundle equipped with Cheeger-Gromoll metric and obtain some conditions for the scalar curvature to be a positively homogeneous function of degree zero with respect to the fiber coordinates of $\widetilde{TM}$.

1. Introduction

Tangent bundles of differentiable manifolds are of great importance in many areas of mathematics and physics. In the last decade, a large number of publications have been devoted to the study of their special differential geometric properties [6].

Some authors have extended the study from Riemannian manifolds to Finsler manifolds. The geometry of Finsler manifolds of constant curvature is one of the fundamental subjects in Finsler geometry. Akbar-Zadeh [1] proved that, under some conditions on growth of the Cartan torsion, a Finsler manifold of constant curvature K is locally Minkowskian if $K = 0$ and Riemannian if $K = -1$. Shen [12] has also investigated the geometric structure of Finsler manifolds of positive constant curvature via the Riemannian Y-metrics. Bejancu [3–5] has initiated a study of interrelations between the geometries of both the tangent bundle and indicatrix bundle of a Finsler manifold on one side, and the geometry of the manifold itself, on the other side. Also he found some conditions for scalar curvature to be a positively homogeneous function. Peyghan and Tayebi [9] introduced two vector fields of the horizontal Liouville type on a slit tangent bundle endowed with a Riemannian metric

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of Sasaki-Finsler type, and proved that these vector fields are Killing if and only if the base Finsler manifold is of positive constant curvature. In the special case of one of them, they showed that if it is Killing vector field then the base manifold is Einstein-Finsler manifold. Also in [8], the two perviously mentioned authors together with Zhong introduced a class of g-natural metrics $G_{a,b}$ on the tangent bundle of a Finsler manifold (M, F) which generalizes the associated Sasaki−Matsumoto metric and Miron metric. They obtain the Weitzenböck formula of the horizontal Laplacian associated to $G_{a,b}$, which is a second-order differential operator for general forms on tangent bundle. Using the horizontal Laplacian associated to $G_{a,b}$, they give some characterizations of certain objects which are geometric interest (e.g. scalar and vector fields which are horizontal covariant constant) on the tangent bundle. Furthermore, Killing vector fields associated to $G_{a,b}$ are investigated. Tayebi and his collaborators studied the interrelations between the geometries of (M, g) and (TM, G). For other progress, see [10].

2. Finsler manifolds

Let (M, F) be an n-dimensional Finsler manifold, where F is the fundamental function of (M, F) that is supposed to be of class C^∞ on the slit tangent bundle $\widetilde{TM} = TM\backslash\{0\}$. Denote by (x^i, y^i), the local coordinates on TM, where (x^i) are the local coordinates of a point $x \in M$ and (y^i) are the coordinates of a vector $y \in T_xM$. Then the functions

$$g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j},$$

define a Finsler tensor field of type $(0, 2)$ on $\widetilde{TM}$. The $n \times n$ matrix $[g_{ij}]$ is supposed to be positive definite and its inverse is denoted by $[g^{ij}]$.

Next we consider the kernel $\mathcal{V}\widetilde{TM}$ of the differential of the projection map $\pi : \widetilde{TM} \to M$, which is known as the vertical bundle on $\widetilde{TM}$. Denote by $\Gamma(\mathcal{V}\widetilde{TM})$ the $\mathcal{F}(\widetilde{TM})$-module of sections of $\mathcal{V}\widetilde{TM}$, where $\mathcal{F}(\widetilde{TM})$ is the algebra of smooth functions on $\widetilde{TM}$. Locally, $\Gamma(\mathcal{V}\widetilde{TM})$ is spanned by the natural vector fields $\{\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}\}$. Then by using the functions N_j^i we define the local vector fields

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}, \qquad i \in \{1, \ldots, n\},$$

which enable us to construct a complementary vector subbundle $\mathcal{H}\widetilde{TM}$ to $\mathcal{V}\widetilde{TM}$ in $T\widetilde{TM}$, that is locally spanned by $\{\frac{\delta}{\delta x^1}, \ldots, \frac{\delta}{\delta x^n}\}$. We call $\mathcal{H}\widetilde{TM}$ the horizontal distribution on $\widetilde{TM}$. Thus the tangent bundle of $\widetilde{TM}$ admits the decomposition

$$T\widetilde{TM} = \mathcal{H}\widetilde{TM} \oplus \mathcal{V}\widetilde{TM}.$$

For a vector field $u \in TM$, we shall denote by U its canonical vertical vector field on TM which in local coordinates is given by $U = (u^i o\pi)(\frac{\partial}{\partial y^i})(p, u)$, where $u = (u^1, \ldots, u^n)$. We define the function $r : TM \to \mathbb{R}$ by $r(p, u) = |u| = \sqrt{g_p(u, u)}$

where $g_p(u,u) = F^2(u,u)$ and put $\alpha = 1 + r^2$.

Then we can define the Cheeger-Gromoll metric G on $\widetilde{TM}$ induced by F as follows

$$\begin{cases} G(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = g_{ij}, \\ G(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}) = 0, \\ G(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = \frac{1}{\alpha}(g_{ij\,(p,u)} + g_{is\,(p,u)} g_{jt\,(p,u)} u^s u^t). \end{cases} \tag{1}$$

We define some geometric objects of Finsler type on $\widetilde{TM}$. First, the Lie brackets of the above vector fields are expressed as follows:

$$\begin{aligned} &[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}] = R^k_{ij} \frac{\partial}{\partial y^k}, \\ &[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}] = (\Gamma^k_{ij} + L^k_{ij}) \frac{\partial}{\partial y^k}, \\ &[\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}] = 0. \end{aligned} \tag{2}$$

We note that R^k_{ij} define a skew-symmetric Finsler tensor field of type $(1,2)$, while $(\Gamma^k_{ij} + L^k_{ij})$ are the local coefficients of Berwald connection. Some other Finsler tensor fields defined by R^k_{ij} will be useful in study of Finsler manifolds of constant flag curvature:

$$R_{hij} = g_{hk} R^k_{ij}, \qquad R_{hj} = R_{hij} y^i, \qquad R^k_j = g^{kh} R_{hj}.$$

From this we have

$$y^h R_{hij} = 0, \qquad y^h R_{hj} = 0, \qquad R_{ij} = R_{ji}, \tag{3}$$

$$R^k_{ij} = \frac{1}{3}\left(\frac{\partial R^k_j}{\partial y^i} - \frac{\partial R^k_i}{\partial y^j}\right). \tag{4}$$

We define a symmetric Finsler tensor field of type $(1,2)$ whose local components are given by

$$B^k_{ij} = -L^k_{ij}. \tag{5}$$

As a consequence we have $B^k_{ij} y^j = 0$.

Also the Cartan tensor field is given by its local components:

$$C^k_{ij} = \frac{1}{2} g^{kh} \frac{\partial g_{ij}}{\partial y^h}, \tag{6}$$

and by the homogeneity condition for F we obtain $C^k_{ij} y^j = 0$.

The function Γ^k_{ij} is given by

$$\Gamma^k_{ij} = \frac{1}{2} g^{kh} \left\{ \frac{\delta g_{ki}}{\delta x^j} + \frac{\delta g_{hj}}{\delta x^i} - \frac{\delta g_{ij}}{\delta x^h} \right\}$$

and by Euler theorem we infer $\Gamma^k_{ij} y^j = N^k_i$.

The angular metric of (M,F) has the local components

$$h_{ij} = g_{ij} - l_i l_j, \qquad l_i = g_{ij} l^j = \frac{\partial F}{\partial y^i}.$$

3. Killing vector fields on $(\widetilde{TM}, G)$

We define the following adapted tensor fields by using the above Finsler tensor fields R^k_{ij}, C^k_{ij} and B^k_{ij}:

$$R : \Gamma(\mathcal{H}\widetilde{TM}) \times \Gamma(\mathcal{H}\widetilde{TM}) \to \Gamma(\mathcal{V}\widetilde{TM}), \quad R(X^h, Y^h) = R^k_{ij} Y^i X^j \frac{\partial}{\partial y^k}, \tag{7}$$

$$C : \Gamma(\mathcal{H}\widetilde{TM}) \times \Gamma(\mathcal{H}\widetilde{TM}) \to \Gamma(\mathcal{V}\widetilde{TM}), \quad C(X^h, Y^h) = C^k_{ij} X^j Y^i \frac{\partial}{\partial y^k}, \tag{8}$$

$$B : \Gamma(\mathcal{V}\widetilde{TM}) \times \Gamma(\mathcal{V}\widetilde{TM}) \to \Gamma(\mathcal{H}\widetilde{TM}), \quad B(U^v, W^v) = B^k_{ij} U^j W^i \frac{\delta}{\delta x^k}, \tag{9}$$

where we set

$$X^h = X^j \frac{\delta}{\delta x^j}, \quad Y^h = Y^i \frac{\delta}{\delta x^i}, \quad U^v = U^j \frac{\partial}{\partial y^j}, \quad W^v = W^i \frac{\partial}{\partial y^i},$$

and $R(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}) = R^k_{ij} \frac{\partial}{\partial y^k}$ and $R(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}) = -[\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}]^v$.

Hence $\mathcal{H}\widetilde{TM}$ is an integrable distribution if and only if $R = 0$. On the other hand, (M, F) becomes a Landsberg (resp. Riemannian) manifold if and only if $B = 0$ (resp. $C = 0$).

We define for each of the adapted tensor fields R, C and B a twin denoted by the same symbol as follows:

$$R : \Gamma(\mathcal{H}\widetilde{TM}) \times \Gamma(\mathcal{V}\widetilde{TM}) \to \Gamma(\mathcal{H}\widetilde{TM}), \tag{10}$$
$$G(R(X^h, Y^v), Z^h) = \alpha G(R(X^h, Z^h), Y^v)$$

$$C : \Gamma(\mathcal{H}\widetilde{TM}) \times \Gamma(\mathcal{V}\widetilde{TM}) \to \Gamma(\mathcal{H}\widetilde{TM}), \tag{11}$$
$$G(C(X^h, Y^v), Z^h) = \alpha G(C(X^h, Z^h), Y^v)$$

$$B : \Gamma(\mathcal{H}\widetilde{TM}) \times \Gamma(\mathcal{V}\widetilde{TM}) \to \Gamma(\mathcal{V}\widetilde{TM}), \tag{12}$$
$$G(B(X^h, Y^v), Z^v) = \frac{1}{\alpha} G(B(Y^v, Z^v), X^h),$$

for each $X, Y, Z \in \Gamma(T\widetilde{TM})$.

We put locally the twins of R, C and B

$$R(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}) = \bar{R}^k_{ij} \frac{\delta}{\delta x^k}, \tag{13}$$

$$C(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}) = \bar{C}^k_{ij} \frac{\delta}{\delta x^k}, \tag{14}$$

$$B(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}) = \bar{B}^k_{ij} \frac{\partial}{\partial y^k}. \tag{15}$$

Taking into account that C_{ijk} and B_{ijk} are symmetric with respect to all indices and by using (19)–(21) and (3)–(6), we have

$$\bar{C}_{ijk} = \bar{C}^p_{ij} g_{pk} = \alpha C^s_{ik} G(\frac{\partial}{\partial y^s}, \frac{\partial}{\partial y^j}) = C^s_{ik} g_{si} = C_{ikj} = C_{ijk}, \tag{16}$$

$$\bar{B}^p_{ij} = \frac{1}{\alpha} g^{pk} B^t_{jk} g_{ti} = \frac{1}{\alpha} B^p_{ji} = \frac{1}{\alpha} B^p_{ij}, \tag{17}$$

$$\bar{R}_{kji} = \bar{R}^t_{ji} g_{kt} = \alpha R^p_{ki} G(\frac{\partial}{\partial y^p}, \frac{\partial}{\partial y^j}) = R_{jki}. \tag{18}$$

Since $y^j R_{jki} = 0$ we obtain

$$y^j \bar{R}_{kji} = 0. \tag{19}$$

The Vrãnceanu connection ∇ on $\widetilde{TM}$ is defined by

$$\nabla_X Y = (\tilde{\nabla}_{X^v} Y^v)^v + (\tilde{\nabla}_{X^h} Y^h)^h + [X^h, Y^v]^v + [X^v, Y^h]^h, \tag{20}$$

for any $X, Y \in \Gamma(\widetilde{TM})$. By using (8) and (2) we obtain

$$\nabla_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta x^i} = \Gamma^k_{ij} \frac{\delta}{\delta x^k}, \tag{21}$$

$$\nabla_{\frac{\partial}{\partial y^j}} \frac{\delta}{\delta x^i} = 0, \tag{22}$$

$$\nabla_{\frac{\delta}{\delta x^j}} \frac{\partial}{\partial y^i} = (\Gamma^k_{ij} + L^k_{ij}) \frac{\partial}{\partial y^k}, \tag{23}$$

$$\begin{aligned}\nabla_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial y^i} &= C^t_{ji} \frac{\partial}{\partial y^t} + \frac{1}{\alpha}\big(- G(\frac{\partial}{\partial y^i}, U) \frac{\partial}{\partial y^j} - G(\frac{\partial}{\partial y^j}, U) \frac{\partial}{\partial y^i} \\ &\quad + (\alpha+1) G(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) U - G(\frac{\partial}{\partial y^j}, U) G(\frac{\partial}{\partial y^i}, U) U\big).\end{aligned} \tag{24}$$

THEOREM 3.1. *[11] Let (M, F) be a Finsler manifold. Then the Levi-civita connection $\tilde{\nabla}$ in terms of Vrãnceanu connection ∇ on $(\widetilde{TM}, G)$ are as follows:*

i) $\tilde{\nabla}_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} = \nabla_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} - \alpha C(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) - \frac{1}{2} R(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})$

ii) $\tilde{\nabla}_{\frac{\delta}{\delta x^i}} \frac{\partial}{\partial y^j} = C(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}) + \frac{1}{2\alpha} R(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}) + B(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}) + \nabla_{\frac{\delta}{\delta x^i}} \frac{\partial}{\partial y^j}$

iii) $\tilde{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\delta}{\delta x^j} = \nabla_{\frac{\partial}{\partial y^i}} \frac{\delta}{\delta x^j} + C(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}) + \frac{1}{2\alpha} R(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}) + B(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i})$

iv) $\tilde{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = -\frac{1}{\alpha} B(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) + \nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j}$

for each $i, j \in \{1, \ldots, n\}$.

THEOREM 3.2. *[11] Let (M, F) be a Finsler manifold, then the curvature tensor field $\tilde{R}$ of Levi-Civita connection on $(\widetilde{TM}, G)$ are as follows:*

i)
$$\begin{aligned}\tilde{R}(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) \frac{\delta}{\delta x^k} &= R(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) \frac{\delta}{\delta x^k} + B\big(\frac{\delta}{\delta x^k}, R(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})\big) \\ &\quad + \frac{1}{2\alpha} R\big(\frac{\delta}{\delta x^k}, R(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})\big) + C\big(\frac{\delta}{\delta x^k}, R(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})\big) \\ &\quad - \mathcal{A}_{(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})} \Big(\frac{1}{2} B\big(\frac{\delta}{\delta x^i}, R(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k})\big) \\ &\quad + \frac{1}{4\alpha} R\big(\frac{\delta}{\delta x^i}, R(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k})\big) + \frac{1}{2} C\big(\frac{\delta}{\delta x^i}, R(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k})\big)\end{aligned}$$

$$
\begin{aligned}
&+\alpha B\big(\frac{\delta}{\delta x^i},C(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k})\big)+\frac{1}{2}R\big(\frac{\delta}{\delta xi},C(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k})\big)\\
&+\alpha C\big(\frac{\delta}{\delta x^i},C(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k})\big)\\
&+\frac{1}{2}(\nabla_{\frac{\delta}{\delta x^i}}R)(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k})+\alpha(\nabla_{\frac{\delta}{\delta x^i}}C)(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k})\Big)
\end{aligned}
$$

$$
\begin{aligned}
ii)\ \tilde{R}(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j})\frac{\partial}{\partial y^k}&=R(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j})\frac{\partial}{\partial y^k}-\frac{1}{\alpha}B\big(R(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}),\frac{\partial}{\partial y^k}\big)\\
&+\mathcal{A}_{(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j})}\Big(B\big(\frac{\delta}{\delta x^i},B(\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^k})\big)\\
&+(\nabla_{\frac{\delta}{\delta x^i}}B)(\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^k})+(\nabla_{\frac{\delta}{\delta x^i}}C)(\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^k})\\
&+C\big(\frac{\delta}{\delta x^i},B(\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^k})\big)-\alpha C\big(\frac{\delta}{\delta x^i},C(\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^k})\big)\\
&-\frac{1}{2}C\big(\frac{\delta}{\delta x^i},R(\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^k})\big)+\frac{1}{2\alpha}(\nabla_{\frac{\delta}{\delta x^i}}R)(\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^k})\\
&+\frac{1}{2\alpha}R\big(\frac{\delta}{\delta x^i},B(\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^k})\big)-\frac{1}{2}R\big(\frac{\delta}{\delta x^i},C(\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^k})\big)\\
&-\frac{1}{4\alpha}R\big(\frac{\delta}{\delta x^i},R(\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^k})\big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
iii)\ \tilde{R}(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j})\frac{\delta}{\delta x^k}&=R(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^i})\frac{\delta}{\delta x^k}+\mathcal{A}_{(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j})}\Big(-\frac{1}{\alpha}B\big(\frac{\partial}{\partial y^i},B(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j})\big)\\
&+B\big(C(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}),\frac{\partial}{\partial y^i}\big)+\frac{1}{2\alpha}B\big(R(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}),\frac{\partial}{\partial y^i}\big)\\
&+C\big(C(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}),\frac{\partial}{\partial y^i}\big)+\frac{1}{2\alpha}C\big(R(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}),\frac{\partial}{\partial y^i}\big)\\
&+\frac{1}{2\alpha}R\big(C(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}),\frac{\partial}{\partial y^i}\big)+\frac{1}{4\alpha^2}R\big(R(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}),\frac{\partial}{\partial y^i}\big)\\
&+(\nabla_{\frac{\partial}{\partial y^i}}B)(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j})+(\nabla_{\frac{\partial}{\partial y^i}}C)(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j})\\
&+\frac{1}{2\alpha}(\nabla_{\frac{\partial}{\partial y^i}}R)(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j})-\frac{1}{\alpha^2}G(U,\frac{\partial}{\partial y^i})R(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j})\Big)
\end{aligned}
$$

$$
\begin{aligned}
iv)\ \tilde{R}(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j})\frac{\partial}{\partial y^k}&=R(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j})\frac{\partial}{\partial y^k}\\
&-\mathcal{A}_{(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j})}\Big(-\frac{1}{\alpha^2}G(U,\frac{\partial}{\partial y^i})B(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k})\\
&+\frac{1}{\alpha}(\nabla_{\frac{\partial}{\partial y^i}}B)(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k})+\frac{1}{\alpha}C\big(B(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}),\frac{\partial}{\partial y^i}\big)\\
&+\frac{1}{\alpha}B\big(B(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}),\frac{\partial}{\partial y^i}\big)+\frac{1}{2\alpha^2}R\big(B(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}),\frac{\partial}{\partial y^i}\big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
v)\ \tilde{R}(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j})\frac{\delta}{\delta x^k} &= R(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j})\frac{\delta}{\delta x^k} + B\big(\frac{\delta}{\delta x^i}, B(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j})\big) \\
&- B\big(\frac{\partial}{\partial y^j}, C(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^k})\big) - \frac{1}{2\alpha}B\big(\frac{\partial}{\partial y^j}, R(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^k})\big) \\
&+ (\nabla_{\frac{\delta}{\delta x^i}}B)(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}) + C\big(\frac{\delta}{\delta x^i}, B(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j})\big) \\
&- \alpha C\big(\frac{\delta}{\delta x^i}, C(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j})\big) - \frac{1}{2}C\big(\frac{\delta}{\delta x^i}, R(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j})\big) \\
&- 2G(U,\frac{\partial}{\partial y^j})C(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^k}) + (\nabla_{\frac{\delta}{\delta x^i}}C)(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}) \\
&+ \alpha(\nabla_{\frac{\partial}{\partial y^j}}C)(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^k}) + \frac{1}{2\alpha}R\big(\frac{\delta}{\delta x^i}, B(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j})\big) \\
&- \frac{1}{2}R\big(\frac{\delta}{\delta x^i}, C(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j})\big) - \frac{1}{4\alpha}R\big(\frac{\delta}{\delta x^i}, R(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j})\big) \\
&+ \frac{1}{2\alpha}(\nabla_{\frac{\delta}{\delta x^i}}R)(\frac{\delta}{\delta x^k},\frac{\partial}{\partial y^j}) + \frac{1}{2}(\nabla_{\frac{\partial}{\partial y^j}}R)(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^k}) \\
vi)\ \tilde{R}(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j})\frac{\partial}{\partial y^k} &= R(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j})\frac{\partial}{\partial y^k} - \frac{1}{\alpha}(\nabla_{(\frac{\delta}{\delta x^i}}B)(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}) \\
&- (\nabla_{\frac{\partial}{\partial y^j}}B)(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k}) - (\nabla_{\frac{\partial}{\partial y^j}}C)(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k}) \\
&- \frac{1}{2\alpha}(\nabla_{\frac{\partial}{\partial y^j}}R)(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k}) + C\big(\frac{\delta}{\delta x^i}, B(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k})\big) \\
&- C\big(C(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k}),\frac{\partial}{\partial y^j}\big) - \frac{1}{2\alpha}C\big(R(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k}),\frac{\partial}{\partial y^j}\big) \\
&+ \frac{1}{2\alpha}R\big(\frac{\delta}{\delta x^i}, B(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k})\big) - \frac{1}{2\alpha}R\big(C(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k}),\frac{\partial}{\partial y^j}\big) \\
&- \frac{1}{4\alpha^2}R\big(R(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k}),\frac{\partial}{\partial y^j}\big) + \frac{1}{\alpha}B\big(\frac{\partial}{\partial y^j}, B(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k})\big) \\
&- B\big(C(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k}),\frac{\partial}{\partial y^j}\big) - \frac{1}{2\alpha}B\big(R(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k}),\frac{\partial}{\partial y^j}\big) \\
&+ \frac{1}{\alpha^2}G(U,\frac{\partial}{\partial y^j})R(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^k})
\end{aligned}
$$

Definition 3.3. [7] A vector field X on M is called an infinitesimal isometry (or, a Killing vector field) if the local 1-parameter group of local transformations generated by X in a neighborhood of each point of M consists of local isometries.

Proposition 3.4. *[7] For a vector field X on a Riemannian manifold M, the following conditions are mutually equivalent:*

- *X is a Killing vector field;*

- *The natural lift g of X to $L(M)$ is tangent to $O(M)$ at every point of $O(M)$;*
- *$L_X g = 0$, where g is the metric tensor field of M;*
- *The tensor field $A_X = L_X - \nabla_X$ of type $(1,1)$ is skew-symmetric with respect to g everywhere on M, that is, $g(A_X Y, Z) = -g(A_X Z, Y)$ for arbitrary vector fields Y and Z.*

THEOREM 3.5. *The unit horizontal Liouville vector field $\xi = \frac{y^i}{F}\frac{\delta}{\delta x^i}$ is a Killing vector field on the indicatrix bundle IM if and only if*

$$h_{ij}(x.y) = \frac{1}{\alpha}R_{ij}(x,y) \qquad \forall (x,y) \in IM$$

Proof. The Lie derivative of G with respect to ξ is given by

$$(L_\xi G)(X,Y) = G(\nabla_X \xi, Y) + G(X, \nabla_Y \xi). \tag{25}$$

First, by using Theorem 3.1 and (25) we obtain

$$\begin{aligned}
(L_\xi G)(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) &= G(\nabla_{\frac{\delta}{\delta x^i}}\xi, \frac{\delta}{\delta x^j}) + G(\frac{\delta}{\delta x^i}, \nabla_{\frac{\delta}{\delta x^j}}\xi) \\
&= G(\frac{\delta(\frac{y^s}{F})}{\delta x^i}\frac{\delta}{\delta x^s} + \frac{y^s}{F}\Gamma^t_{is}\frac{\delta}{\delta x^t}, \frac{\delta}{\delta x^j}) + G(\frac{\delta(\frac{y^s}{F})}{\delta x^j}\frac{\delta}{\delta x^s} + \frac{y^s}{F}\Gamma^t_{js}\frac{\delta}{\delta x^t}, \frac{\delta}{\delta x^i}) \\
&= G(-\frac{N^s_i}{F}\frac{\delta}{\delta x^s} + \frac{N^t_i}{F}\frac{\delta}{\delta x^t}, \frac{\delta}{\delta x^j}) + G(-\frac{N^s_j}{F}\frac{\delta}{\delta x^s} + \frac{N^t_j}{F}\frac{\delta}{\delta x^t}, \frac{\delta}{\delta x^i}) = 0.
\end{aligned}$$

Next, by using Theorem 3.1 and (25) we get

$$\begin{aligned}
(L_\xi G)(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) &= G(\nabla_{\frac{\partial}{\partial y^i}}\xi, \frac{\partial}{\partial y^j}) + G(\frac{\partial}{\partial y^i}, \nabla_{\frac{\partial}{\partial y^j}}\xi) \\
&= G(\frac{y^s}{F}\bar{B}^t_{is}\frac{\partial}{\partial y^t}, \frac{\partial}{\partial y^j}) + G(\frac{y^s}{F}\bar{B}^t_{js}\frac{\partial}{\partial y^t}, \frac{\partial}{\partial y^i}) = 0.
\end{aligned}$$

Finally we have

$$\begin{aligned}
(L_\xi G)(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}) &= G(\nabla_{\frac{\delta}{\delta x^i}}\xi, \frac{\partial}{\partial y^j}) + G(\frac{\delta}{\delta x^i}, \nabla_{\frac{\partial}{\partial y^j}}\xi) \\
&= G(-\frac{y^s}{F}(\alpha C^t_{si} + \frac{1}{2}R^t_{is})\frac{\partial}{\partial y^t}, \frac{\partial}{\partial y^j}) \\
&\quad + G(\frac{\delta}{\delta x^i}, \frac{\partial(\frac{y^s}{F})}{\partial y^j}\frac{\delta}{\delta x^s} + \frac{y^s}{F}(C^t_{sj} + \frac{1}{2\alpha}R^t_{sj})\frac{\delta}{\delta x^t}) \\
&= -\frac{1}{\alpha F}R_{ij} + \frac{1}{F}(g_{ij} - l_i l_j) \\
&= \frac{1}{F}(h_{ij} - \frac{1}{\alpha}R_{ij}).
\end{aligned}$$

Now by using Proposition 3.4 we deduce that the unit horizontal Liouville vector field ξ is a Killing vector field on the indicatrix bundle IM if and only if

$$h_{ij}(x.y) = \frac{1}{\alpha}R_{ij}(x,y) \qquad \forall (x,y) \in IM,$$

This completes the proof. $\square$

THEOREM 3.6. *The Finsler manifold (M, F) is of scalar curvature $K = \alpha$ if and only if the unit horizontal Liouville vector field is a Killing vector field on the indicatrix bundle IM.*

Proof. The flag curvature of (M, F) at the point x with respect to the flag $\Pi(X) = \operatorname{span}\{y^i \frac{\delta}{\delta x^i}, X\}$ is the function (see Bao-Chern-Shen [2])

$$K(x, y, X) = \frac{R_{ij}X^iX^j}{F^2 h_{ij}X^iX^j}, \tag{26}$$

If $K = \alpha$ we have $h_{ij}(x, y) = \frac{1}{\alpha}R_{ij}(x, y)$ on IM. So from Theorem 3.5 we deduce that the unit horizontal Liouville vector field is a Killing vector field on the indicatrix bundle IM.

Let ξ be a Killing vector field on IM. If $(x, y) \in IM$ from Theorem 3.5 we get $K(x, y) = \alpha(x, y)$. Now suppose $(x, y) \notin IM$. Then $F(x, y) = c \in (0, \infty) - 1$, hence $(x, \frac{1}{c}y) \in IM$ and we have $h_{ij}(x, \frac{1}{c}y) = \frac{1}{\alpha}R_{ij}(x, \frac{1}{c}y)$. Taking into account that h_{ij} and R_{ij} are positively homogeneous of degrees 0 and 2 respectively, we deduce that $c^2 h_{ij}(x, y) = \frac{1}{\alpha}R_{ij}(x, y)$, so we have $K(x, y) = \alpha(x, y)$. □

4. Scalar curvature (TM, G)

Let (M, F) be a Finsler manifold and $(\widetilde{TM}, G)$ be its slit tangent bundle equipped with Cheeger-Gromoll metric induced by F. Let R be the curvature tensor field of Vrãnceanu connection on $(\widetilde{TM}, G)$. We have

$$R(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^j})\frac{\delta}{\delta x^i} = \big(\frac{\delta \Gamma^p_{ij}}{\delta x^k} - \frac{\delta \Gamma^p_{ik}}{\delta x^j} + \Gamma^t_{ij}\Gamma^p_{tk} - \Gamma^t_{ik}\Gamma^p_{tj}\big)\frac{\delta}{\delta x^p} = K^p_{ijk}\frac{\delta}{\delta x^p} \tag{27}$$

$$\begin{aligned} R(\frac{\partial}{\partial y^k}, \frac{\partial}{\partial y^j})\frac{\partial}{\partial y^i} &= \big(\frac{\partial C^p_{ij}}{\partial y^k} - \frac{\partial C^p_{ki}}{\partial y^j} + C^t_{ij}C^p_{kt} - C^t_{ki}C^p_{tj}\big)\frac{\partial}{\partial y^p} \\ &+ \frac{1+\alpha}{\alpha}C^t_{ij}G(\frac{\partial}{\partial y^t}, \frac{\partial}{\partial y^k})U - \frac{1+\alpha}{\alpha}C^t_{ik}G(\frac{\partial}{\partial y^t}, \frac{\partial}{\partial y^j})U \\ &+ \big(\frac{\alpha^2+\alpha+1}{\alpha^3}g_{ij} - \frac{\alpha-1}{\alpha^3}y_iy_j\big)\frac{\partial}{\partial y^k} \\ &- \big(\frac{\alpha^2+\alpha+1}{\alpha^3}g_{ik} - \frac{\alpha-1}{\alpha^3}y_iy_k\big)\frac{\partial}{\partial y^j} \\ &+ \frac{\alpha+2}{\alpha^3}g_{ik}y_jU - \frac{\alpha+2}{\alpha^3}g_{ij}y_kU \end{aligned} \tag{28}$$

Now consider the local orthonormal fields of frames $\{H_a\}$ and $\{V_a\}$, such that $H_a \in \Gamma(\mathcal{H}\widetilde{TM})$ and $V_a \in \Gamma(\mathcal{V}\widetilde{TM})$ for any $a \in \{1, 2, \ldots, n\}$. We set

$$H_a = H^i_a\frac{\delta}{\delta x^i}, \qquad V = V^i_a\frac{\partial}{\partial y^i}. \tag{29}$$

We get

$$g^{ij} = \sum_{a=1}^{n} H_a^i H_a^j = \sum_{a=1}^{n} V_a^i V_a^j. \tag{30}$$

Let S be the scalar curvature of Riemannian manifold $(\widetilde{TM}, G)$. We have

$$S = \sum_{a,b=1}^{n} \{G(\tilde{R}(H_a, H_b)H_b, H_a) + 2G(\tilde{R}(H_a, V_b)V_b, H_a) + G(\tilde{R}(V_a, V_b)V_b, V_a)\} \tag{31}$$

Now by using (7), (8), (10) and (11) we have

$$G(C(H_a, C(H_b, H_b)), H_a) = \alpha G(C(H_a, H_a), C(H_b, H_b)) \tag{32}$$

$$G(C(H_b, C(H_a, H_b)), H_a) = \alpha ||C(H_a, H_b)||^2 \tag{33}$$

$$\begin{aligned} G(C(H_b, R(H_a, H_b)), H_a) &= -G(R(H_b, C(H_a, H_b)), H_a) \\ &= \alpha G(C(H_a, H_b), R(H_a, H_b)) \end{aligned} \tag{34}$$

$$G(R(H_a, C(H_b, H_b)), H_a) = 0 \tag{35}$$

$$G(R(H_b, R(H_a, H_b)), H_a) = -\alpha ||R(H_a, H_b)||^2 \tag{36}$$

$$\sum_{a,b=1}^{n} G(C(H_a, H_b), R(H_a, H_b)) = 0. \tag{37}$$

By using Theorem 3.1, (1), (30), (32)–(37) we get

$$\begin{aligned} \sum_{a,b=1}^{n} \{G(\tilde{R}(H_a, H_b)H_b, H_a)\} = \sum_{a,b=1}^{n} \{&G(R(H_a, H_b)H_b, H_a) \\ &- \alpha G(C(H_a, H_a), C(H_b, H_b)) \\ &- \frac{3}{4\alpha}\|R(H_a, H_b)\|^2 + \alpha \|C(H_a, H_b)\|^2\} \end{aligned} \tag{38}$$

Then by using (7)–(9) we obtain

$$G(B(V_b, B(H_a, V_b)), H_a) = \alpha \|B(H_a, V_b)\|^2 \tag{39}$$

$$G(C(C(H_a, V_b), V_b), H_a) = \|C(H_a, V_b)\|^2 \tag{40}$$

$$G(C(R(H_a, V_b), V_b), H_a) + G(R(C(H_a, V_b), V_b), H_a) = 0 \tag{41}$$

$$G(R(R(H_a, V_b), V_b), H_a) = -\|R(H_a, V_b)\|^2. \tag{42}$$

By using Theorem 3.1, (39)–(42) we obtain

$$\begin{aligned} \sum_{a,b=1}^{n} \{G(\tilde{R}(H_a, V_b)V_b, H_a)\} &= \sum_{a,b=1}^{n} \{\|B(H_a, V_b)\|^2 - \|C(H_a, V_b)\|^2 \\ &+ \frac{1}{4\alpha^2}\|R(H_a, V_b)\|^2 - G(\frac{1}{\alpha}(\nabla_{H_a} B)(V_b, V_b), H_a) - G((\nabla_{V_b} C)(H_a, V_b), H_a) \\ &- G(\frac{1}{2\alpha}(\nabla_{V_b} R)(H_a, V_b), H_a) + \frac{1}{\alpha^2} G(U, V_b) G(R(H_a, V_b), H_a)\}. \end{aligned} \tag{43}$$

By using (12) we have

$$G(B(B(V_b,V_b),V_a),V_a)=\frac{1}{\alpha}G(B(V_a,V_a),B(V_b,V_b)) \tag{44}$$

$$G(B(B(V_a,V_b),V_b),V_a)=\frac{1}{\alpha}\|B(V_a,V_b)\|^2. \tag{45}$$

Finally, by using Theorem 3.1, (44) and (45) we get

$$\begin{aligned}\sum_{a,b=1}^{n}\{G(\tilde{R}(V_a,V_b)V_b,V_a)\}=\sum_{a,b=1}^{n}\{&G(R(V_a,V_b)V_b,V_a)+\frac{1}{\alpha^2}\|B(V_a,V_b)\|^2\\ &-\frac{1}{\alpha^2}G(B(V_a,V_a),B(V_b,V_b))\}.\end{aligned} \tag{46}$$

Now we use (16)–(18), (29) and (30) and obtain

$$\sum_{a,b=1}^{n}\|R(H_a,V_b)\|^2=\alpha\sum_{a,b=1}^{n}\|R(H_a,H_b)\|^2=g_{sp}g^{it}g^{hj}R^{p}_{jt}R^{s}_{hi} \tag{47}$$

$$\sum_{a,b=1}^{n}\|C(H_a,V_b)\|^2=\alpha\sum_{a,b=1}^{n}\|C(H_a,H_b)\|^2=g_{ks}g^{jl}g^{ip}C^{k}_{ij}C^{s}_{pl} \tag{48}$$

$$\sum_{a,b=1}^{n}\|B(V_a,V_b)\|^2=\alpha^2\sum_{a,b=1}^{n}\|B(H_a,V_b)\|^2=g_{ts}g^{ik}g^{jl}B^{t}_{ij}B^{s}_{kl} \tag{49}$$

Now by using (31), (38), (43), (46) and (47)–(49) we deduce that

$$\begin{aligned}S=\sum_{a,b=1}^{n}\{&G(R(H_a,H_b)H_b,H_a)+G(R(V_a,V_b)V_b,V_a)\\ &-2G(\frac{1}{\alpha}(\nabla_{H_a}B)(V_b,V_b),H_a)-2G((\nabla_{V_b}C)(H_a,V_b),H_a)\\ &-2G(\frac{1}{2\alpha}(\nabla_{V_b}R)(H_a,V_b),H_a)+\frac{2}{\alpha^2}G(U,V_b)G(R(H_a,V_b),H_a)\\ &-\alpha G(C(H_a,H_a),C(H_b,H_b))-\frac{1}{\alpha^2}G(B(V_a,V_a),B(V_b,V_b))\\ &-\frac{1}{4\alpha}\|R(H_a,H_b)\|^2-\alpha\|C(H_a,H_b)\|^2+\frac{3}{\alpha^2}\|B(V_a,V_b)\|^2\}.\end{aligned} \tag{50}$$

Then from (13)–(15), (27) and (28) we deduce that

$$\sum_{a,b=1}^{n}\{G(R(H_a,H_b)H_b,Ha)\}=g^{is}g^{jk}K_{ijks} \tag{51}$$

$$\sum_{a,b=1}^{n}\{G(R(V_a,V_b)V_b,V_a)\}=g^{is}g^{jk}S_{ijks}, \tag{52}$$

where $S^p_{ijk} = \big(\frac{\partial C^p_{ij}}{\partial y^k} - \frac{\partial C^p_{ki}}{\partial y^j} + C^t_{ij}C^p_{kt} - C^t_{ki}C^p_{tj}\big)\frac{\partial}{\partial y^p}$.

$$\sum_{a,b=1}^{n}\{G(C(H_a,H_a),C(H_b,H_b))\} = \frac{1}{\alpha}g_{ps}g^{ij}g^{kl}C^p_{ij}C^s_{kl} \tag{53}$$

$$\sum_{a,b=1}^{n}\{G(B(V_a,V_a),B(V_b,V_b))\} = g_{ps}g^{ij}g^{kl}B^p_{ij}B^s_{kl}. \tag{54}$$

By using (21)–(24), (29) and (30) we have

$$\sum_{a,b=1}^{n}\{G((\nabla_{H_a}B)(V_b,V_b),H_a)\} = g^{ij}\big(B^t_{ij}\Gamma^l_{lt} + \frac{\delta B^l_{ij}}{\delta x^l} - B^l_{ip}(\Gamma^p_{jl} + L^p_{jl}) - B^l_{jp}(\Gamma^p_{il} + L^p_{il})\big) \tag{55}$$

$$\sum_{a,b=1}^{n}\{G((\nabla_{V_b}C)(H_a,V_b),H_a)\} = g^{ij}(\frac{\partial \bar{C}^k_{kj}}{\partial y^i} - C^s_{ij}\bar{C}^k_{ks}) \tag{56}$$

$$\sum_{a,b=1}^{n}\{G((\nabla_{V_b}R)(H_a,V_b),H_a)\} = g^{ij}(\frac{\partial \bar{R}^k_{kj}}{\partial y^i} - C^s_{ij}\bar{R}^k_{sk}) \tag{57}$$

Finally, from (50)–(57) we deduce that

$$\begin{aligned} S = -2g^{ij}\big(&\frac{\partial \bar{C}^k_{kj}}{\partial y^i} - C^s_{ij}\bar{C}^k_{ks} + \frac{\partial \bar{R}^k_{kj}}{\partial y^i} - C^s_{ij}\bar{R}^k_{sk} \\ &+ B^t_{ij}\Gamma^l_{lt} + \frac{\delta B^l_{ij}}{\delta x^l} - B^l_{ip}(\Gamma^p_{jl} + L^p_{jl}) - B^l_{jp}(\Gamma^p_{il} + L^p_{il})\big) \\ &+ g^{ik}g^{js}\big(K_{ijks} + S_{ijks} - C^p_{ik}C^h_{js}g_{ph} - C^p_{is}C^h_{jk}g_{ph} \\ &- \frac{1}{4\alpha^2}R^p_{si}R^h_{jk}g_{ph} + \frac{3}{\alpha^2}B^p_{ks}B^h_{ij}g_{ph}\big); \end{aligned}$$

we get

$$g^{ik}g^{js}\big(S_{ijks} - C^p_{ik}C^h_{js}g_{ph} - C^p_{is}C^h_{jk}g_{ph}\big) - 2g^{ij}\big(\frac{\partial \bar{C}^k_{kj}}{\partial y^i} - C^s_{ij}\bar{C}^k_{ks}\big) = -2g^{ij}\frac{\partial C^k_{kj}}{\partial y^i},$$

where $C_j = C^k_{kj} = g^{sk}C_{jsk}$ and $C^i = g^{ij}C_j$.

Also we can get

$$g^{ij}\frac{\partial C_j}{\partial y^i} = \frac{\partial C^i}{\partial y^i} + 2g_{jk}C^jC^k. \tag{58}$$

THEOREM 4.1. *Let (M,F) be a Finsler manifold. The scalar curvature S of the tangent bundle with Cheeger-Gromoll induced by F is given by*

$$\begin{aligned} S = -2g^{ij}\big(&\frac{\partial \bar{C}_j}{\partial y^i} + \frac{\partial \bar{R}^k_{kj}}{\partial y^i} - C^s_{ij}\bar{R}^k_{sk} + B^t_{ij}\Gamma^l_{lt} + \frac{\delta B^l_{ij}}{\delta x^l} \\ &- B^l_{ip}(\Gamma^p_{jl} + L^p_{jl}) - B^l_{jp}(\Gamma^p_{il} + L^p_{il})\big) \\ &+ g^{ik}g^{js}\big(K_{ijks} - \frac{1}{4\alpha^2}R^p_{si}R^h_{jk}g_{ph} + \frac{3}{\alpha^2}B^p_{ks}B^h_{ij}g_{ph}\big). \end{aligned}$$

Corollary 4.2. *Let (M,F) be a flat Landsberg manifold and the torsion vector field of (M,F) satisfies*

$$\frac{\partial C^i}{\partial y^i} + 2g_{jk}C^jC^k = 0.$$

Then the scalar curvature is zero.

Proof. This is a direct consequence of Theorem 4.1 and (58). □

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