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REMARKS ON WEAKLY STAR COUNTABLE SPACES

Yan-Kui Song and Wei-Feng Xuan

Abstract. A space X is *weakly star countable* if for each open cover $\mathcal{U}$ of X there exists a countable subset F of X such that $\overline{\bigcup_{x\in F} St(x,\mathcal{U})} = X$. In this paper, we investigate topological properties of weakly star countable spaces.

1. Introduction

By a space, we mean a topological space. In this section, we give definitions of terms which are used in this paper. Let X be a space and $\mathcal{U}$ a collection of subsets of X. For $A \subseteq X$, let $St(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$. As usual, we write $St(x,\mathcal{U})$ instead of $St(\{x\},\mathcal{U})$.

Definition 1.1. ([1–3, 8]) Let P be a topological property. A space X is said to be a *star* P if whenever $\mathcal{U}$ is an open cover of X, there exists a subspace $A \subseteq X$ with property P such that $X = St(A,\mathcal{U})$. The set A will be called a *star kernel* of the cover $\mathcal{U}$.

The term star P was coined in [1–3,8] but certain star properties, specifically those corresponding to “P=finite” and “P=countable” were first studied by van Douwen et al. in [5] and later by many other authors. A survey of star covering properties with a comprehensive bibliography can be found in [5,7]. The authors believe the terminology from [1–3, 8] and the terminology used in the paper to be simple and logical. But we must mention that authors of previous works have used many different notations to define properties of this sort. For example, in [7] and earlier [5], a star finite space is called starcompact and strongly 1-starcompact, a star countable space is called star Lindelöf and strongly 1-star Lindelöf.

As a generalization of star countable spaces, it is natural in this context to introduce the following definition:

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Definition 1.2. ([4]) A space X is said to be *weakly star P* (respectively, *almost star P*) if whenever $\mathcal{U}$ is an open cover of X, there exists a subspace $A \subseteq X$ with property P such that $X = \overline{St(A, \mathcal{U})}$ (respectively, $X = \bigcup\{\overline{St(x, \mathcal{U})} : x \in A\}$). The set A will be called a *weak star kernel* (respectively, almost star kernel) of the cover $\mathcal{U}$.

In [9], the first author studied the relationship between weakly star countable spaces and related to spaces, and topological properties of almost star countable spaces. The purpose of this paper is to investigate topological properties of weakly star countable spaces.

Throughout the paper, the cardinality of a set A is denoted by $|A|$. Let $\mathfrak{c}$ denote the cardinality of the continuum, ω_1 the first uncountable cardinal and ω the first infinite cardinal. For a pair of ordinals α, β with $\alpha < \beta$, we write $(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}$, $(\alpha, \beta] = \{\gamma : \alpha < \gamma \leq \beta\}$ and $[\alpha, \beta] = \{\gamma : \alpha \leq \gamma \leq \beta\}$. As usual, a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. Every cardinal is often viewed as a space with the usual order topology. All spaces are assumed to be at least T_1. Other terms and symbols that we do not define will be used as in [6].

2. Properties of weakly star countable spaces

It is well-known that star finiteness is equivalent to countable compactness for Hausdorff spaces (see [5, 7] under different names). For the next example, we need the following lemma.

Lemma 2.1. *A space X with a dense star countable subspace is weakly star countable.*

Proof. Let D be a dense star countable subspace X. We show that X is weakly star countable. Let $\mathcal{U}$ be an open cover of X. Since D is a star countable subset of X, then there exists a countable subset F of D such that $D \subseteq \bigcup_{x \in F} St(x, \mathcal{U})$. Hence $X = \overline{D} \subseteq \overline{\cup_{x \in F} St(x, \mathcal{U})}$, which completes the proof. □

Since every σ-countably compact space is star countable, thus we have the following corollary.

Corollary 2.2. *([4]) If a space X has a σ-countably compact dense subset, then X is weakly star countable.*

Since every σ-compact space is σ-countably compact, the following holds.

Corollary 2.3. *If a space X has a σ-compact dense subset, then X is weakly star countable.*

In the following, we give an example showing that a closed subset of a Tychonoff weakly star countable space need not be weakly star countable. For a Tychonoff space X, let βX denote the Čech-Stone compactification of X.

Example 2.4. There exists a Tychonoff weakly star countable space X having a closed subset which is not weakly star countable.

Let D be a discrete space of cardinality ω_1, let $X = (\beta D \times (\omega+1)) \setminus ((\beta D \setminus D) \times \{\omega\})$ be the subspace of the product of βD and $\omega + 1$. Since $\beta D \times \omega$ is a dense σ-compact subset of X, X is weakly star countable by Corollary 2.3. However, $D \times \{\omega\}$ is not weakly star countable, since it is a discrete closed subset of cardinality ω_1, which completes the proof.

In the following, we give a positive result, which can be easily proved.

THEOREM 2.5. *If X is a weakly star countable space, then every open and closed subset of X is weakly star countable.*

Recall that a space X has *the discrete countable chain condition (DCCC)* if every discrete collection of open sets is countable.

THEOREM 2.6. *If X is a normal DCCC space, then X is weakly star countable.*

Proof. Suppose that there exists an open cover $\mathcal{U}$ of X such that $\overline{\bigcup_{x\in F} St(x,\mathcal{U})} \neq X$ for any countable subset F of X. By transfinite induction, we can define a sequence $\{x_\alpha : \alpha < \omega_1\}$ such that for every $\beta < \omega_1$, $x_\beta \in X \setminus \overline{\bigcup_{\alpha<\beta} St(x_\alpha,\mathcal{U})}$. For $\alpha < \omega_1$, let

$$V_\alpha = St(x_\alpha,\mathcal{U}) \setminus \overline{\bigcup_{\beta<\alpha} St(x_\beta,\mathcal{U})}.$$

Then V_α is open, $x_\alpha \in V_\alpha$ and $V_\alpha \cap V_{\alpha'} = \emptyset$ for any $\alpha \neq \alpha'$. Let $G = \{x_\alpha : \alpha < \omega_1\}$. Then G is closed in X and U contains at most one element of G for any $U \in \mathcal{U}$. Since X is normal, there exists an open set U of X such that

$$G \subseteq U \subseteq \overline{U} \subseteq \bigcup_{\alpha<\omega_1} V_\alpha.$$

For any $\alpha < \omega_1$, let $U_\alpha = U \cap V_\alpha$. Then $\{U_\alpha : \alpha < \omega_1\}$ is an uncountable discrete collection of nonempty sets. Indeed, for each $x \in X$, we have the following two cases:

(1) if $x \notin \overline{U}$, then $x \in X \setminus \overline{U}$ and $(X \setminus \overline{U}) \cap U_\alpha = \emptyset$ for each $\alpha < \omega_1$;

(2) if $x \in \overline{U}$, then there exists an $\alpha < \omega_1$ such that $x \in V_\alpha$ and $V_\alpha \cap V_\beta = \emptyset$ for $\beta \neq \alpha$.

Thus we get a construction, which completes the proof. $\square$

Since a continuous image of a star countable space is star countable, similarly we have the following result.

THEOREM 2.7. *A continuous image of a weakly star countable space is weakly star countable.*

Proof. Let X be a weakly star countable space, $f : X \to Y$ be continuous and $\mathcal{U}$ be an open cover of Y. Let $\mathcal{V} = \{f^{-1}(U) : U \in \mathcal{U}\}$. Then $\mathcal{V}$ is an open cover of X. There exists a countable subset F of X such that $\overline{\bigcup_{x\in F} St(x,\mathcal{V})} = X$, since X is weakly star countable. Thus the countable subset $f(F)$ of Y witnesses for $\mathcal{U}$ that Y is weakly star countable. Indeed, let $y \in Y$, there exists $x' \in X$ such that $f(x') = y$. Since $\overline{\bigcup_{x\in F} St(x,\mathcal{V})} = X$, thus we have $x' \in \overline{\bigcup_{x\in F} St(x,\mathcal{V})}$. Hence $y = f(x') \in f(\overline{\bigcup_{x\in F} St(x,\mathcal{V})}) \subseteq \overline{f(\bigcup_{x\in F} St(x,\mathcal{V}))} \subseteq \overline{\bigcup_{x\in F} St(f(x),\mathcal{U})}$. $\square$

Next we turn to consider preimages. To show that the preimage of a weakly star countable space under a closed 2-to-1 continuous map need not be weakly star countable, we use the Alexandorff duplicate $A(X)$ of a space X. The underlying set of $A(X)$ is $X \times \{0,1\}$; each point of $X \times \{1\}$ is isolated and a basic neighborhood of a point $\langle x, 0\rangle \in X \times \{0\}$ is of the from $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x, 1\rangle\})$, where U is a neighborhood of x in X.

Example 2.8. There exists a closed 2-to-1 continuous map $f : A(X) \to X$ such that X is a weakly star countable space, but $A(X)$ is not weakly star countable.

Let X be the space X of Example 2.4. Then X is weakly star countable and has an uncountable discrete closed subset $A = \{\langle d_\alpha, \omega\rangle : \alpha < \omega_1\}$. Hence the Alexandroff duplicate $A(X)$ of X is not weakly star countable, since $A \times \{1\}$ is an uncountable discrete, open and closed set in $A(X)$ and every open and closed subset of a weakly star countable space is weakly star countable. Let $f : A(X) \to X$ be the projection. Then f is a closed 2-to-1 continuous map, which completes the proof.

Theorem 2.9. *If f is an open and closed, finite-to-one continuous mapping from a space X to a weakly star countable space Y, then X is weakly star countable.*

Proof. Let $\mathcal{U}$ be an open covers of X. Then for each $y \in Y$, there is a finite subfamily $\mathcal{U}_y$ of $\mathcal{U}$ such that $f^{-1}(y) \subseteq \bigcup \mathcal{U}_y$ and $U \cap f^{-1}(y) \neq \emptyset$ each $U \in \mathcal{U}_y$. Since f is closed, there exists an open neighborhood V_y of y in Y such that $f^{-1}(V_y) \subseteq \bigcup \mathcal{U}_y$. Since f is open, we assume that $V_y \subseteq \bigcap\{f(U) : U \in \mathcal{U}_y\}$. Let $\mathcal{V} = \{V_y : y \in Y\}$. Then $\mathcal{V}$ is an open cover of Y. There exists a countable subset F of Y such that $\overline{\bigcup_{y\in F} St(y, \mathcal{V})} = Y$, since Y is weakly star countable. Since f is finite-to-one, the set $f^{-1}(F)$ is a countable subset of X. We show that $\overline{\bigcup_{x\in f^{-1}(F)} St(x, \mathcal{U})} = X$. Let $x \in X$. Then

$$\begin{aligned} x \in f^{-1}(f(x)) &\subseteq f^{-1}(\overline{\bigcup_{y\in F} St(y, \mathcal{V})}) \subseteq \overline{f^{-1}(\bigcup_{f\in F} St(y, \mathcal{V}))} \\ &\subseteq \overline{\bigcup_{y\in F} St(f^{-1}(y), \{f^{-1}(V_y) : y \in Y\})} \\ &\subseteq \overline{\bigcup_{y\in F} St(f^{-1}(y), \mathcal{U})} = \overline{\bigcup_{x\in f^{-1}(F)} St(x, \mathcal{U})}. \end{aligned}$$ $\square$

The following well-known example shows that the product of two countably compact (and hence weakly star countable) spaces need not be almost star countable. Here we give the proof roughly for the sake of completeness.

Example 2.10. There exist two countably compact spaces X and Y such that $X \times Y$ is not weakly star countable.

Let D be a discrete space of cardinality $\mathfrak{c}$. We can define $X = \bigcup_{\alpha<\omega_1} E_\alpha$ and $Y = \bigcup_{\alpha<\omega_1} F_\alpha$, where E_α and F_α are the subsets of βD which are defined inductively so as to satisfy the following conditions (1), (2) and (3):

(1) $E_\alpha \cap F_\beta = D$ if $\alpha \neq \beta$;

(2) $|E_\alpha| \le \mathfrak{c}$ and $|F_\beta| \le \mathfrak{c}$;

(3) every infinite subset of E_α (resp., F_α) has an accumulation point in $E_{\alpha+1}$ (resp., $F_{\alpha+1}$).

These sets E_α and F_α are well-defined, since every infinite closed set in βD has cardinality $2^{\mathfrak{c}}$ (see [10]). Then $X \times Y$ is not weakly star countable, because the diagonal $\{\langle d, d\rangle : d \in D\}$ is a discrete open and closed subset of $X \times Y$ with cardinality $\mathfrak{c}$ and the open and closed subsets of weakly star countable spaces are weakly star countable.

In the following, we give an example showing that the product of a Tychonoff countably compact space and a Tychonoff Lindelöf space need not be weakly star countable.

Example 2.11. There exist a Tychonoff countably compact space X and a Tychonoff Lindelöf space Y such that $X \times Y$ is not weakly star countable.

Let $X = [0, \omega_1)$ with the usual order topology. Clearly, X is countably compact. Let $D = \{d_\alpha : \alpha < \omega_1\}$ be a discrete space of cardinality ω_1, and let $Y = D \cup \{d_{\omega_1}\}$ be the one-point Lindelöfication of D, where $d_{\omega_1} \notin D$. Then Y is Lindelöf.

Next we show that $X \times Y$ is not weakly star countable.

For each $\alpha < \omega_1$, let $U_\alpha = [0, \alpha) \times \{d_\beta : \beta > \alpha\}$ and $V_\alpha = [\alpha, \omega_1) \times \{d_\alpha\}$. Let $\mathcal{U} = \{U_\alpha : \alpha < \omega_1\} \cup \{V_\alpha : \alpha < \omega_1\}$. Then $\mathcal{U}$ is an open cover of $X \times Y$. Let us consider the open cover $\mathcal{U}$ of $X \times Y$. For each $\langle \alpha, d_\beta\rangle \in X \times Y$, we have the following cases by the construction of $\mathcal{U}$:

(1) If $\alpha \ge \beta$, then $St(\langle \alpha, d_\beta\rangle, \mathcal{U}) = V_\beta$;

(2) If $\alpha < \beta$, then $St(\langle \alpha, d_\beta\rangle, \mathcal{U}) = \bigcup_{\alpha<\gamma<\beta} U_\gamma$.

It remains to show that $\overline{St(F, \mathcal{U})} \ne X \times Y$ for any countable subset F of $X \times Y$. Let F be a countable subset of $X \times Y$. Let $\beta_1 = \sup\{\beta : d_\beta \in \pi(F)\}$, where $\pi : X \times Y \to Y$ is the projection. Then $\beta_1 < \omega_1$, since F is countable. If we pick a non-limit ordinal $\beta_2 > \beta_1$ and an ordinal $\alpha > \beta_2$, then $\langle \alpha, d_{\beta_2}\rangle \in V_{\beta_2}$. Hence $V_{\beta_2} \cap St(x, \mathcal{U}) = \emptyset$ for each $x \in F$ by the above cases. Thus $\langle \alpha, d_{\beta_2}\rangle \notin \overline{St(F, \mathcal{U})}$, which shows that $X \times Y$ is not weakly star countable.

We give a positive result.

Theorem 2.12. *If X is a star countable space and Y is a separable space, then $X \times Y$ is weakly star countable.*

Proof. Let D be a countable dense subset of Y. Then $X \times D$ is a dense star countable subspace of $X \times Y$. Thus $X \times Y$ is weakly star countable by Lemma 2.1. □

Remark 2.13. The authors do not know if there exist a weakly star countable space X and a compact space Y such that $X \times Y$ is not weakly star countable.

Now we give a condition under which weakly star countableness implies star countable. Recall that a space X is *paracompact* if every open cover $\mathcal{U}$ of X has a locally finite open refinement.

Theorem 2.14. *Every T_2 paracompact weakly star countable space is Lindelöf (hence star countable).*

Proof. Let X be a paracompact weakly star countable space and $\mathcal{U}$ be an open cover of X. For each $x \in X$, there exists an open neighborhood V_x of x such that $\overline{V_x} \subseteq U$ for some $U \in \mathcal{U}$. Since X is paracompact, there exists a locally finite open refinement $\mathcal{V}$ of $\{V_x : x \in X\}$, since every Hausdorff paracompact space is regular. Since X is weakly star countable, there exists a countable subset A of X such that $X = \overline{St(A, \mathcal{V})}$. Let $\mathcal{W} = \{V \in \mathcal{V} : V \cap A \neq \emptyset\}$. Then $\mathcal{W}$ is countable, since $\mathcal{V}$ is locally finite and A is countable. For each $W \in \mathcal{W}$, choose $U_W \in \mathcal{U}$ such that $\overline{W} \subseteq U_W$. Then $\{U_W : W \in \mathcal{W}\}$ is a countable subcover of X, since $X = \overline{St(A, \mathcal{V})} = \bigcup\{\overline{W} : W \in \mathcal{W}\} \subseteq \bigcup\{U_W : W \in \mathcal{W}\}$, which shows that X is Lindelöf. Thus we complete the proof. □

Corollary 2.15. *The following conditions are equivalent for a Hausdorff paracompact space X.*

(1) X is Lindelöf;

(2) X is star countable;

(3) X is weakly star countable.

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Institute of Mathematics, School of Mathematical Science, Nanjing Normal University, Nanjing 210023, P.R. China

E-mail: songyankui@njnu.edu.cn

School of Statistics and Mathematics, Nanjing Audit University, Nanjing 210093, P.R. China

E-mail: wfxuan@nau.edu.cn