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A SIMPLE METHOD FOR FINDING THE INVERSE MATRIX OF VANDERMONDE MATRIX

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Abstract. A simple method for computing the inverse of Vandermonde matrices is presented. The inverse is obtained by finding the cofactor matrix of Vandermonde matrices. Based on this, it is directly possible to evaluate the determinant and inverse for more general Vandermonde matrices.

1. Introduction

The Vandermonde matrices are an essential topic in applied mathematics, natural science and engineering. For example, they appear in the fields of numerical analysis, mathematical finance, statistics, geometry of curves and control theory (cf, e.g., [1,3–5,8,9] and references therein). Moreover, Vandermonde matrices have gained much interest in wireless communications due to their frequent appearance in numerous applications in signal reconstruction, cognitive radio, physical layer security, and MIMO channel modeling (cf, e.g., [6, 7, 11, 12] and references therein).

In particular, when Rawashdeh et al. [2] studied the numerical stability of collocation methods for Volteera higher order integro-differential equations, they computed the eigenvalues of a certain matrix. The key point for the evaluation of such eigenvalues is to find the inverse of a Vandermonde matrix. Recently, Vandermonde matrices and their inverses play an important role to determine logarithmic functions of the sub-system's density matrices [13].

In this paper, we present an explicit formula for finding the inverse of Vandermonde matrices. Then we compute the determinant as well as the inverse of more general Vandermonde matrices that are obtained by deleting one or two rows and columns of Vandermonde matrices.

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2. Main Results

It is known that the Vandermonde matrix is defined by

$$V = V(c_1, \dots, c_m) = \begin{pmatrix} 1 & c_1 & c_1^2 & \dots & c_1^{m-1} \\ 1 & c_2 & c_2^2 & \dots & c_2^{m-1} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & c_m & c_m^2 & \dots & c_m^{m-1} \end{pmatrix}$$

and its determinant is given by $\prod\limits_{1\le k<j\le m}^{m} (c_j - c_k)$.

Therefore, if the numbers $c_1, c_2, \dots, c_m$ are distinct, then V is invertible. Finding the inverse of the Vandermonde matrix has been investigated by many researchers, for example Yiu [14] used a technique based on partial fraction decomposition of a certain rational function to express the inverse of V as a product of two matrices, one of them being a lower triangular matrix. L. Richard [10] wrote the inverse of the Vandermonde matrix as a product of two triangular matrices. F. Soto and H. Moya [13] showed that $V^{-1} = DWL$, where D is a diagonal matrix, W is an upper triangular matrix and L is a lower triangular matrix. However, in all of these techniques V^{-1} is not determined explicitly. In this section we present a new, efficient and easy-to-use method for computing V^{-1}.

First we introduce the following notations.

$$S_k := S_k(c_1, ..., c_m) = \sum_{1\le i_1<\dots<i_k\le m}^{m} c_{i_1} c_{i_2} ... c_{i_k}, \quad \text{for} \quad 1 \le k \le m,$$

$$S_0 := S_0(c_1, ..., c_m) = 1, \quad \text{and} \quad S_k = 0 \quad \text{for} \quad k \notin \{0, \dots, m\}.$$

We also define

$$S_{k,j} := S_k(c_1, ..., c_{j-1}, c_{j+1}, ..., c_m), \quad \text{for} \quad 1 \le k \le m-1, \quad \text{and} \quad 1 \le j \le m.$$

It is clear that $\prod\limits_{k=1}^{m} (x - c_k) = \sum\limits_{k=0}^{m} (-1)^{m-k} S_{m-k} x^k$.

The following lemma can be used to compute the cofactor matrix of the Vandermonde matrices as we will see later.

LEMMA 2.1. *Let* $c_1, c_2, \dots, c_m$ *be real numbers and* $i \in \{0, 1, \dots, m\}$. *Then the determinant of the matrix*

$$V_i(c_1, c_2, \dots, c_m) = \begin{pmatrix} 1 & c_1 & c_1^2 & \dots & c_1^{i-1} & c_1^{i+1} & \dots & c_1^m \\ 1 & c_2 & c_2^2 & \dots & c_2^{i-1} & c_2^{i+1} & \dots & c_2^m \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ 1 & c_m & c_m^2 & \dots & c_m^{i-1} & c_m^{i+1} & \dots & c_m^m \end{pmatrix}$$

is given by $\det(V_i(c_1, c_2, \dots, c_m)) = S_{m-i} \prod\limits_{1\le k<j\le m}^{m} (c_j - c_k)$.

Proof. Define the polynomial

$$f(x) := \begin{vmatrix} 1 & c_1 & c_1^2 & \dots & c_1^{i-1} & c_1^i & c_1^{i+1} & \dots & c_1^m \\ 1 & c_2 & c_2^2 & \dots & c_2^{i-1} & c_2^i & c_2^{i+1} & \dots & c_2^m \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & c_m & c_m^2 & \dots & c_m^{i-1} & c_m^i & c_m^{i+1} & \dots & c_m^m \\ 1 & x & x^2 & \dots & x^{i-1} & x^i & x^{i+1} & \dots & x^m \end{vmatrix}.$$

Since $f(x)$ is the determinant of the Vandermonde matrix $V(c_1, c_2, \dots, c_m, x)$, we have

$$f(x) = \prod_{k<j}^{m}(c_j - c_k)\prod_{j=1}^{m}(x - c_j) = \prod_{k<j}^{m}(c_j - c_k)\sum_{k=0}^{m}(-1)^{m-k}S_{m-k}x^k.$$

The coefficient of x^i in $f(x)$ is $(-1)^{m+i+2}\det(V_i(c_1, c_2, \dots, c_m))$ which is equal to $(-1)^{m-i}S_{m-i}\prod_{k<j}^{m}(c_j - c_k)$. Hence, $\det(V_i(c_1, c_2, \dots, c_m)) = S_{m-i}\prod_{1\le k<j\le m}^{m}(c_j - c_k)$. □

The proof presented here is short and straightforward, unlike the alternative proof presented by Rawashdeh et al. [2] based on the mathematical induction on the size of matrices.

Now we are in the position to find a simple formula for computing the inverse of V.

Lemma 2.2. *Let $c_1, c_2, \dots, c_m$ be distinct real numbers and*

$$V = V(c_1, \dots, c_m) = \begin{pmatrix} 1 & c_1 & c_1^2 & \dots & c_1^{m-1} \\ 1 & c_2 & c_2^2 & \dots & c_2^{m-1} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & c_m & c_m^2 & \dots & c_{m-1}^m \end{pmatrix}$$

be the Vandermonde matrix. Then the inverse of V is the matrix whose elements are given by

$$(V^{-1})_{i,j} = (-1)^{i+j}\frac{S_{m-i,j}}{\prod_{l<k}^{m}(c_k - c_l)} \text{ with } l = j \text{ or } k = j, \text{ where } i, j = 1, \dots, m.$$

Proof. It is known that $V^{-1} = \frac{\mathrm{Adj}(V)}{\det(V)}$, where $\mathrm{Adj}(V)$ is the transpose of the cofactor matrix of V. From Lemma 2.1, the entries of $\mathrm{Adj}(V)$ are given by

$$\begin{aligned}(\mathrm{Adj}(V))_{i,j} =& (-1)^{i+j}\det(V_i(c_1, \dots, c_{j-1}, c_{j+1}, \dots, c_m)) \\ =& (-1)^{i+j}S_{m-i,j}\prod_{l<k(l,k\neq j)}^{m}(c_k - c_l),\end{aligned}$$

where $V_i(c_1, \dots, c_{j-1}, c_{j+1}, \dots, c_m)$ is the matrix obtained from the matrix V by erasing the i-th column and j-th row. Thus

$$(V^{-1})_{i,j} = (-1)^{i+j} \frac{\prod\limits_{l<k(l,k\neq j)}^{m} (c_k - c_l)}{\prod\limits_{l<k}^{m} (c_k - c_l)} S_{m-i,j} = (-1)^{i+j} \frac{S_{m-i,j}}{\prod\limits_{l<k}^{m} (c_k - c_l)} \text{ with } l = j \text{ or } k = j,$$

where $i = 1, \ldots, m$ and $j = 1, \ldots, m$. This completes the proof. $\square$

Example 2.3. If $V = \begin{pmatrix} 1 & c_1 & c_1^2 & c_1^3 \\ 1 & c_2 & c_2^2 & c_2^3 \\ 1 & c_3 & c_3^2 & c_3^3 \\ 1 & c_4 & c_4^2 & c_4^3 \end{pmatrix}$, then V^{-1} has the form:

$$\begin{pmatrix} \frac{c_2c_3c_4}{(c_4-c_1)(c_3-c_1)(c_2-c_1)} & \frac{-c_1c_3c_4}{(c_4-c_2)(c_3-c_2)(c_2-c_1)} & \frac{c_1c_2c_4}{(c_4-c_3)(c_3-c_2)(c_3-c_1)} & \frac{-c_1c_2c_3}{(c_4-c_3)(c_4-c_2)(c_4-c_1)} \\ \frac{-c_2c_3+c_2c_4+c_3c_4}{(c_4-c_1)(c_3-c_1)(c_2-c_1)} & \frac{c_1c_3+c_1c_4+c_3c_4}{(c_4-c_2)(c_3-c_2)(c_2-c_1)} & \frac{-c_1c_2+c_1c_4+c_2c_4}{(c_4-c_3)(c_3-c_2)(c_3-c_1)} & \frac{c_1c_2+c_1c_3+c_2c_3}{(c_4-c_3)(c_4-c_2)(c_4-c_1)} \\ \frac{c_2+c_3+c_4}{(c_4-c_1)(c_3-c_1)(c_2-c_1)} & \frac{-c_1-c_3-c_4}{(c_4-c_2)(c_3-c_2)(c_2-c_1)} & \frac{c_1+c_2+c_4}{(c_4-c_3)(c_3-c_2)(c_3-c_1)} & \frac{-c_1-c_2-c_3}{(c_4-c_3)(c_4-c_2)(c_4-c_1)} \\ \frac{-1}{(c_4-c_1)(c_3-c_1)(c_2-c_1)} & \frac{1}{(c_4-c_2)(c_3-c_2)(c_2-c_1)} & \frac{-1}{(c_4-c_3)(c_3-c_2)(c_3-c_1)} & \frac{1}{(c_4-c_3)(c_4-c_2)(c_4-c_1)} \end{pmatrix}.$$

In the next lemma, we find the determinant as well as the inverse of more general Vandermonde matrices of the form

$$V_{i_1,i_2,\ldots,i_m}(c_1, c_2, \ldots, c_m) = \begin{pmatrix} c_1^{i_1} & c_1^{i_2} & \ldots & c_1^{i_m} \\ c_2^{i_1} & c_2^{i_2} & \ldots & c_2^{i_m} \\ \vdots & \vdots & \ldots & \vdots \\ c_m^{i_1} & c_m^{i_2} & \ldots & c_m^{i_m} \end{pmatrix},$$

where $\{i_1, i_2, \ldots, i_m\}$ is an increasing sequence of non negative integers and satisfying $\{i_1, i_2, \ldots, i_m\} \subseteq \{0, 1, \ldots, m+1\}$ or $\{i_1, i_2, \ldots, i_m\} \subseteq \{0, 1, \ldots, m+2\}$.

Lemma 2.4. *Let $c_1, c_2, \ldots, c_m$ be real numbers.*

(I) The determinant of the matrix

$$V_{i,j}(c_1, c_2, \ldots, c_m) = \begin{pmatrix} 1 & c_1 & c_1^2 & \ldots & c_1^{i-1} & c_1^{i+1} & \ldots & c_1^{j-1} & c_1^{j+1} & \ldots & c_1^{m+1} \\ 1 & c_2 & c_2^2 & \ldots & c_2^{i-1} & c_2^{i+1} & \ldots & c_2^{j-1} & c_2^{j+1} & \ldots & c_2^{m+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & c_m & c_m^2 & \ldots & c_m^{i-1} & c_m^{i+1} & \ldots & c_m^{j-1} & c_m^{j+1} & \ldots & c_m^{m+1} \end{pmatrix}$$

is given by $\det(V_{i,j}(c_1, c_2, \ldots, c_m)) = \prod\limits_{k<j}^{m} (c_j - c_k)(S_{m-i}S_{m-j+1} - S_{m-i+1}S_{m-j})$ *where* $\{i, j\} \subseteq \{0, 1, \ldots, m+1\}$ *and* $i < j$.

(II) The determinant of the matrix

$$V_{i,j,r}(c_1, c_2, \ldots, c_m) =$$
$$\begin{pmatrix} 1 & c_1 & c_1^2 & \ldots & c_1^{i-1} & c_1^{i+1} & \ldots & c_1^{j-1} & c_1^{j+1} & \ldots & c_1^{r-1} & c_1^{r+1} & \ldots & c_1^{m+2} \\ 1 & c_2 & c_2^2 & \ldots & c_2^{i-1} & c_2^{i+1} & \ldots & c_2^{j-1} & c_2^{j+1} & \ldots & c_2^{r-1} & c_2^{r+1} & \ldots & c_2^{m+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & c_m & c_m^2 & \ldots & c_m^{i-1} & c_m^{i+1} & \ldots & c_m^{j-1} & c_m^{j+1} & \ldots & c_m^{r-1} & c_m^{r+1} & \ldots & c_m^{m+2} \end{pmatrix}$$

is given by

$$\det(V_{i,j,r}(c_1,c_2,\ldots,c_m)) = \prod_{k<j}^{m}(c_j-c_k)((S_{m-i}S_{m-j+1}-S_{m-i+1}S_{m-j})S_{m-r+2})$$
$$-(S_{m-i}S_{m-j+2}-S_{m-i+2}S_{m-j})S_{m-r+1}+(S_{m-i+1}S_{m-j+2}-S_{m-i+2}S_{m-j+1})S_{m-r})$$

where $\{i,j,r\}\subseteq\{0,1,\ldots,m+2\}$ *and* $i<j<r$.

Proof. (I) Define the polynomial

$$f(x) := \begin{vmatrix} 1 & c_1 & c_1^2 & \ldots & c_1^{i-1} & c_1^{i+1} & \ldots & c_1^{j-1} & c_1^j & c_1^{j+1} & \ldots & c_1^{m+1} \\ 1 & c_2 & c_2^2 & \ldots & c_2^{i-1} & c_2^{i+1} & \ldots & c_2^{j-1} & c_2^j & c_2^{j+1} & \ldots & c_2^{m+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & c_m & c_m^2 & \ldots & c_m^{i-1} & c_m^{i+1} & \ldots & c_m^{j-1} & c_m^j & c_m^{j+1} & \ldots & c_m^{m+1} \\ 1 & x & x^2 & \ldots & x^{i-1} & x^{i+1} & \ldots & x^{j-1} & x^j & x^{j+1} & \ldots & x^{m+1} \end{vmatrix}.$$

Then it is clear that $f(x)=(a_1x+a_0)\sum_{k=0}^{m}(-1)^{m-k}S_{m-k}x^k$. The leading coefficient of $f(x)$ is a_1 and from Lemma 2.1, we have

$$a_1 = \det(V_i(c_1,c_2,\ldots,c_m)) = S_{m-i}\prod_{k<j}^{m}(c_j-c_k).$$

The constant term of $f(x)$ is $(-1)^mS_ma_0$ which is equal to $(-1)^{m+2}\det(B)$, where

$$B = \begin{pmatrix} c_1 & c_1^2 & \ldots & c_1^{i-1} & c_1^{i+1} & \ldots & c_1^{j-1} & c_1^j & c_1^{j+1} & \ldots & c_1^{m+1} \\ c_2 & c_2^2 & \ldots & c_2^{i-1} & c_2^{i+1} & \ldots & c_2^{j-1} & c_2^j & c_1^{j+1} & \ldots & c_2^{m+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_m & c_m^2 & \ldots & c_m^{i-1} & c_m^{i+1} & \ldots & c_m^{j-1} & c_m^j & c_m^{j+1} & \ldots & c_m^{m+1} \end{pmatrix}.$$

Again from Lemma 2.1, we have $\det(B) = c_1c_2\ldots c_m\det(V_{i-1}(c_1,c_2,\ldots,c_m)) = S_mS_{m-i+1}\prod_{k<j}^{m}(c_j-c_k)$, thus $a_0 = S_{m-i+1}\prod_{k<j}^{m}(c_j-c_k)$.

Hence, we have

$$f(x) = \prod_{k<j}^{m}(c_j-c_k)\sum_{k=0}^{m}(-1)^{m-k}S_{m-k}x^k(S_{m-i}x+S_{m-i+1}).$$

Since $(-1)^{m+1+j}\det(V_{i,j}(c_1,c_2,\ldots,c_m))$ is the coefficient of x^j in $f(x)$ which is

$$\prod_{k<j}^{m}(c_j-c_k)((-1)^{m-j}S_{m-i+1}S_{m-j}+(-1)^{m-j+1}S_{m-i}S_{m-j+1}),$$

we obtain $\quad \det(V_{i,j}(c_1,c_2,\ldots,c_m)) = \prod_{k<j}^{m}(c_j-c_k)(S_{m-i}S_{m-j+1}-S_{m-i+1}S_{m-j}).$

(II) Define the polynomial $f(x)=\det(V_{i,j}(c_1,c_2,\ldots,c_m,x))$. Then $f(x)$ is a polynomial of degree less than or equal to $m+2$ and it is clear that

$$f(x)=(a_2x^2+a_1x+a_0)\sum_{k=0}^{m}(-1)^{m-k}S_{m-k}x^k.$$

The leading coefficient of $f(x)$ is a_2 and from part (I), we have

$$a_2=\det(V_{i,j}(c_1,c_2,\dots,c_m))=(S_{m-i}S_{m-j+1}-S_{m-i+1}S_{m-j})\prod_{k<j}^{m}(c_j-c_k).$$

The constant term of $f(x)$ is $(-1)^mS_ma_0$ which is equal to

$$(-1)^{m+2}S_m\det(V_{i-1,j-1}(c_1,c_2,\dots,c_m)),$$

so from part (I), $a_0=\prod\limits_{k<j}^{m}(c_j-c_k)(S_{m-i+1}S_{m-j+2}-S_{m-i+2}S_{m-j+1})$.

Now the coefficient of x^j is zero, so

$$(-1)^{m-j+2}a_2S_{m-j+2}+(-1)^{m-j+1}a_1S_{m-j+1}+(-1)^{m-j}a_0S_{m-j}=0,$$

and substituting the values of a_2 and a_0, yields

$$\begin{aligned}\prod_{k<j}^{m}(c_j-c_k)\big((S_{m-i}S_{m-j+1}-S_{m-i+1}S_{m-j})S_{m-j+2}&\\+(S_{m-i+1}S_{m-j+2}-S_{m-i+2}S_{m-j+1})S_{m-j}\big)-a_1S_{m-j+1}&=0.\end{aligned}$$

Thus we have $a_1=\prod\limits_{k<j}^{m}(c_j-c_k)(S_{m-i}S_{m-j+2}-S_{m-i+2}S_{m-j})$. Hence,

$$\begin{aligned}f(x)=\prod_{k<j}^{m}(c_j-c_k)\sum_{k=0}^{m}(-1)^{m-k}S_{m-k}x^k\Big((S_{m-i}S_{m-j+1}-S_{m-i+1}S_{m-j})x^2&\\+(S_{m-i}S_{m-j+2}-S_{m-i+2}S_{m-j})x+(S_{m-i+1}S_{m-j+2}-S_{m-i+2}S_{m-j+1})\Big)&.\end{aligned}$$

Since $(-1)^{m+1+r-1}\det(V_{i,j,r}(c_1,c_2,\dots,c_m)$ is the coefficient of x^r in $f(x)$ which is

$$(-1)^{m-r+2}a_2S_{m-r+2}+(-1)^{m-r+1}a_1S_{m-r+1}+(-1)^{m-r}a_0S_{m-r},$$

we obtain

$$\begin{aligned}\det(V_{i,j,r}(c_1,c_2,\dots,c_m))=\prod_{k<j}^{m}(c_j-c_k)\Big(&(S_{m-i}S_{m-j+1}-S_{m-i+1}S_{m-j})S_{m-r+2}\\&-(S_{m-i}S_{m-j+2}-S_{m-i+2}S_{m-j})S_{m-r+1}\\&+(S_{m-i+1}S_{m-j+2}-S_{m-i+2}S_{m-j+1})S_{m-r}\Big).\end{aligned}$$

□

Remark 2.5. Lemmas 2.1 and 2.4 can be used to find the determinants and cofactor matrices of the matrices $V_i(c_1,\dots,c_m)$ and $V_{i,j}(c_1,\dots,c_m)$. Thus computing the inverses of these matrices is easy to determine, provided that $V_i(c_1,\dots,c_m)$ and $V_{i,j}(c_1,\dots,c_m)$ are invertible. However, finding the cofactor matrix as well as the inverse of the matrix $V_{i,j,r}(c_1,\dots,c_m)$ are still needed to be evaluated. In view of

Lemmas 2.1 and 2.4, it will be interesting to find the inverse of the matrix

$$V_{i_1,i_2,\ldots,i_m}(c_1,c_2,\ldots,c_m)=\begin{pmatrix} c_1^{i_1} & c_1^{i_2} & \ldots & c_1^{i_m} \\ c_2^{i_1} & c_2^{i_2} & \ldots & c_2^{i_m} \\ \vdots & \vdots & \ldots & \vdots \\ c_m^{i_1} & c_m^{i_2} & \ldots & c_m^{i_m} \end{pmatrix},$$

where $\{i_1, i_2, \ldots, i_m\}$ is an increasing sequence of non-negative integers, which we plan to discuss in a forthcoming paper.

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