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# AN EXISTENCE THEOREM OF TRIPLED FIXED POINT FOR A CLASS OF OPERATORS ON BANACH SPACE WITH APPLICATIONS

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**Abstract**. In this paper, using the technique of measure of noncompactness, we prove some theorems on tripled fixed points for a class of operators in a Banach space. Also as an application, we discuss the existence of solution for a class of systems of nonlinear functional integral equations. Finally a concrete example illustrating the mentioned applicability is also included.

## 1. Introduction

The study of nonlinear integral equations is a subject of interest for researchers in the nonlinear functional analysis. Integral equations occur in many applications, in applied mathematics, as well as in physics. In this context, several authors have presented papers on the existence of solution for such equations. On the other hand, measure of noncompactness and Darbo's fixed point theorem are two main tools for proving this result. So far different definitions of measure of noncompactness have been suggested by many authors [7, 8, 16]. In this paper, we accept the definition which is presented in [8] and is convenient in application. Up to now, several authors have presented papers on the existence of solution for nonlinear integral equations which involve the use of measure of noncompactness, as well as other techniques, see, for example [1, 2, 6–8, 11–17].

After introducing the concept of coupled fixed point by Bhaskar and Lakshmikantham [10], many authors used it to generalize the Banach contraction principle, for example see [3–5, 9, 10, 18, 19]. Recently Aghajani and Sabzali [2] presented some generalizations of Darbo's theorem by using the concept of coupled fixed points.

In this paper, by using the concept of tripled fixed point which was first introduced by [9], we try to present some generalizations of Darbo's fixed point theorem. The

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organization of this paper is as follows. In Section 2, we present notations, definitions, and basic theorems along with some examples. Section 3 is devoted to state and prove some existence theorems on tripled fixed points for a class of operators. In Section 4, using the obtained results in the previous sections, we survey the problem of existence of solution for the system of nonlinear integral equations (2) and at the end of this section, there are two examples to illustrate the obtained results.

## 2. Preliminaries

Let us recall some definitions, notations and preliminary results from the theory of measure of noncompactness in Banach spaces.

From now on, let $(E, \|\cdot\|)$ be a real Banach space with zero element 0 and $\overline{B}_r$ denote the closed ball in $E$ centered at 0 with radius $r$. Denote by $\mathfrak{M}_E$ the family of nonempty bounded subsets of $E$ and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact subsets of $E$. If $X$ is a subset, we assume that $\overline{X}$, $co(X)$ are the closure and closed convex hull of $X$ in $E$, respectively.

Definition 2.1 ([8]). A mapping $\mu : \mathfrak{M}_E \to [0, \infty)$ is said to be a measure of noncompactness in $E$ if it satisfies the following conditions:

(MNC1) The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subseteq \mathfrak{N}_E$.

(MNC2) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.

(MNC3) $\mu(\overline{X}) = \mu(X)$.

(MNC4) $\mu(coX) = \mu(X)$.

(MNC5) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1]$.

(MNC6) If $(X_n)$ is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$, $n = 1, 2, \ldots$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the intersection set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker \mu$ described in (MNC1) is said to be the kernel of the measure of noncompactness $\mu$. Observe that the intersection set $X_\infty$ from (MNC6) is a member of the family $\ker \mu$. In fact, since $\mu(X_\infty) \leq \mu(X_n)$ for any $n$, we infer that $\mu(X_\infty) = 0$. This yields that $X_\infty \in \ker \mu$.

Definition 2.2 ([9]). An element $(x, y, z) \in X \times X \times X$ is called a tripled fixed point of a mapping $T : X \times X \times X \to X$ if $T(x, y, z) = x$ and $T(y, x, y) = y$ and $T(z, y, x) = z$.

Theorem 2.3 ([8]). *Suppose that $\mu_1, \mu_2, \ldots, \mu_n$ are measures of noncompactness in $E_1, E_2, \ldots, E_n$, respectively. Moreover, assume that a function $F : [0, \infty)^n \to [0, \infty)$ is convex and $F(x_1, x_2, \ldots, x_n) = 0$ if and only if $x_i = 0$ for $i = 1, 2, \ldots, n$. Then*

$$\tilde{\mu}(X) = F(\mu_1(X_1), \mu_2(X_2), \ldots, \mu_n(X_n))$$

*defines a measure of noncompactness in $E_1 \times E_2 \times \ldots \times E_n$, where $X_i$ denotes the natural projection of $X$ into $E_i$ for $i = 1, 2, \ldots, n$.*

Now, as illustrations of Theorem 2.3, we present the following examples.

Example 2.4. Let $\mu$ be a measure of noncompactness in $E$. We define $F(x,y,z) = x+y+z$ for any $(x,y,z) \in [0,\infty)^3$. Then $F$ has the properties mentioned in Theorem 2.3. Hence $\widetilde{\mu}(X) = \mu(X_1) + \mu(X_2) + \mu(X_3)$ is a measure of noncompactness in the space $E \times E \times E$ where $X_i$ $(i = 1,2,3)$ denotes the natural projections of $X$ into $E$.

Example 2.5. Let $\mu$ be a measure of noncompactness in $E$. We define $F(x,y,z) = \max\{x,y,z\}$ for any $(x,y,z) \in [0,\infty)^3$. Then $F$ has the properties mentioned in Theorem 2.3. Hence $\widetilde{\mu}(X) = \max\{\mu(X_1),\mu(X_2),\mu(X_3)\}$ is a measure of noncompactness in the space $E \times E \times E$ where $X_i$ $(i = 1,2,3)$ denotes the natural projections of $X$ into $E$.

Theorem 2.6 (Darbo [6]). *Let $\Omega$ be a nonempty, bounded, closed, and convex subset of a Banach space $E$ and let $T : \Omega \to \Omega$ be a continuous mapping. Assume that there exists a constant $k \in [0,1)$ such that $\mu(T(X)) \le k\mu(X)$ for any nonempty $X \subset \Omega$. Then $T$ has a fixed point.*

In the sequel, let us recall basic notations which were introduced and discussed in [8]. Consider the space of bounded continuous functions $BC(\mathbb{R}_+)$ with the norm

$$\|x\| = \sup\{|x(t)| : \ t \ge 0\}$$

for any $x \in BC(\mathbb{R}_+)$. Moreover, choose a nonempty bounded subset $X$ of $BC(\mathbb{R}_+)$ and a positive number $L > 0$. For $x \in X$, $\varepsilon \ge 0$ and $t \in \mathbb{R}_+$. Let

$$\begin{aligned}
&\omega^L(x,\varepsilon) = \sup\{|x(t) - x(s)| : \ t,s \in [0,L], |t-s| \le \varepsilon\},\\
&\omega^L(X,\varepsilon) = \sup\left\{\omega^L(x,\varepsilon) : \ x \in X\right\},\\
&\omega_0^L(X) = \lim_{\varepsilon\to 0} \omega^L(X,\varepsilon), \qquad \omega_0(X) = \lim_{L\to\infty} \omega_0^L(X),\\
&X(t) = \{x(t) : x \in X\}, \qquad \operatorname{diam} X(t) = \sup\{|x(t) - y(t)| : \ x,y \in X\}
\end{aligned}$$

and consider the measure of noncompactness

$$\mu(X) = \omega_0(X) + \limsup_{t\to\infty} \operatorname{diam} X(t), \tag{1}$$

where $\operatorname{diam} X(t) = \sup\{|x(t) - y(t)| : \ x,y \in X\}$.

It can be shown (see [3,8]) that the function $\mu(X)$ defines a measure of noncompactness in the sense of the above accepted definition.

In this paper, we present and prove some existence theorems of tripled fixed point for a class of operators. Moreover, as an application, we study the problem of existence of solutions for the following system of nonlinear integral equations of the form

$$\begin{cases}
x(t)=A(t)+f(t,x(\xi(t)),y(\xi(t)),z(\xi(t)))\varphi(\int_0^{\beta(t)} g(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))ds),\\
y(t)=A(t)+f(t,y(\xi(t)),x(\xi(t)),y(\xi(t)))\varphi(\int_0^{\beta(t)} g(t,s,y(\eta(s)),x(\eta(s)),y(\eta(s)))ds),\\
z(t)=A(t)+f(t,z(\xi(t)),y(\xi(t)),x(\xi(t)))\varphi(\int_0^{\beta(t)} g(t,s,z(\eta(s)),y(\eta(s)),x(\eta(s)))ds),
\end{cases} \tag{2}$$

where $A, f, g, \varphi, \xi, \eta$ and $\beta$ satisfy certain conditions.

## 3. Main results

In this section, we present and prove some theorems for the existence of tripled fixed point for a special class of operators. This basic result will be used in Section 4.

THEOREM 3.1. *Let $\mu$ be an arbitrary measure of noncompactness in $E$ and $\Omega$ be a nonempty bounded subset of $E$. Moreover, assume that $T : \Omega \times \Omega \times \Omega \to \Omega$ is a continuous function such that there exist nonnegative constants $k_1, k_2, k_3$ with $k_1 + 2k_2 + k_3 < 1$ such that*

$$\mu(T(X_1 \times X_2 \times X_3) \leq k_1\mu(X_1) + k_2\mu(X_2) + k_3\mu(X_3) \tag{3}$$

*for all $X_1, X_2, X_3 \subset \Omega$. Then $T$ has a tripled fixed point.*

*Proof.* Define $\widetilde{T}$ on $\Omega \times \Omega \times \Omega$ by the formula $\widetilde{T}(x, y, z) = (T(x, y, z), T(y, x, y), T(z, y, x))$, for all $(x, y, z) \in \Omega \times \Omega \times \Omega$. It is easy to see that $\widetilde{T}$ is continuous on $\Omega \times \Omega \times \Omega$. Now we show that for any $X \subset \Omega \times \Omega \times \Omega$, we have $\widetilde{\mu}(\widetilde{T}(X)) \leq (k_1 + 2k_2 + k_3)\widetilde{\mu}(X)$ where $\widetilde{\mu}$ is defined by Example 2.4. To obtain our purpose, we take an arbitrary nonempty subset $X$ of $\Omega \times \Omega \times \Omega$. Then by (MNC2) and (3), we get

$$\begin{aligned}
\widetilde{\mu}(\widetilde{T}(X)) &\leq \widetilde{\mu}(T(X_1 \times X_2 \times X_3), T(X_2 \times X_1 \times X_2), T(X_3 \times X_2 \times X_1)) \\
&= \mu(T(X_1 \times X_2 \times X_3)) + \mu(T(X_2 \times X_1 \times X_2)) + \mu(T(X_3 \times X_2 \times X_1)) \\
&\leq k_1\mu(X_1) + k_2\mu(X_2) + k_3\mu(X_3) + k_1\mu(X_2) + k_2\mu(X_1) + k_3\mu(X_2) \\
&\qquad + k_1\mu(X_3) + k_2\mu(X_2) + k_3\mu(X_1) \\
&= (k_1 + k_2 + k_3)\mu(X_1) + (k_1 + 2k_2 + k_3)\mu(X_2) + (k_1 + k_3)\mu(X_3) \\
&\leq (k_1 + 2k_2 + k_3)\mu(X_1) + (k_1 + 2k_2 + k_3)\mu(X_2) + (k_1 + 2k_2 + k_3)\mu(X_3) \\
&= (k_1 + 2k_2 + k_3)(\mu(X_1) + \mu(X_2) + \mu(X_3)) = (k_1 + 2k_2 + k_3)\widetilde{\mu}(X).
\end{aligned}$$

Hence, by Theorem 2.6, $T$ has a tripled fixed point. □

By using the above result, we have the following corollary.

COROLLARY 3.2. *Assume that $T : \Omega \times \Omega \times \Omega \to \Omega$ is a continuous function such that $\mu(T(X_1 \times X_2 \times X_3)) \leq \frac{k}{3}(\mu(X_1) + \mu(X_2) + \mu(X_3))$ for each $X_1, X_2, X_3 \subset \Omega$ where $0 \leq k < 1$ is a constant. Then $T$ has a tripled fixed point.*

*Proof.* Taking $k_1 = k_3 = \frac{k}{3}$ and $k_2 = \frac{k}{6}$ in Theorem 3.1, we obtain the result. □

THEOREM 3.3. *Let $T : \Omega \times \Omega \times \Omega \to \Omega$ be a continuous function such that*

$$\mu(T(X_1 \times X_2 \times X_3) \leq k \max\{\mu(X_1), \mu(X_2), \mu(X_3\} \tag{4}$$

*for any $X_1, X_2, X_3 \subset \Omega$, where $\mu$ is an arbitrary measure of noncompactness and $k$ is a constant with $0 \leq k < 1$. Then $T$ has at least a tripled fixed point.*

*Proof.* First note that Example 2.5 implies that $\widetilde{\mu}(X) = \max\{\mu(X_1), \mu(X_2), \mu(X_3)\}$ is a measure of noncompactness in the space $E \times E \times E$ where $X_i$ $(i = 1, 2, 3)$ denotes the natural projections of $X$ into $E$. Also the map $\widetilde{T} : \Omega \times \Omega \times \Omega \to \Omega \times \Omega \times \Omega$, where $\widetilde{T}(x, y, z) = (T(x, y, z), T(y, x, y), T(z, y, x))$ is clearly continuous on $\Omega \times \Omega \times \Omega$ by its definition. Now we claim that $\widetilde{T}$ satisfies all the conditions of Theorem 2.6. To prove this, let $X \subset \Omega\times\Omega\times\Omega$ be a nonempty subset. Then by (MNC2) and (4), we get

$$\begin{aligned}\widetilde{\mu}(\widetilde{T}(X)) &\leq \widetilde{\mu}(T(X_1 \times X_2 \times X_3), T(X_2 \times X_1 \times X_2), T(X_3 \times X_2 \times X_1))\\ &= \max\{\mu(T(X_1 \times X_2 \times X_3)), \mu(T(X_2 \times X_1 \times X_2)), \mu(T(X_3 \times X_2 \times X_1))\}\\ &\leq \max\left\{\begin{array}{c} k\max\{\mu(X_1), \mu(X_2), \mu(X_3)\}, k\max\{\mu(X_2), \mu(X_1), \mu(X_2)\},\\ k\max\{\mu(X_3), \mu(X_2), \mu(X_1)\}\end{array}\right\}\\ &= k\max\{\mu(X_1), \mu(X_2), \mu(X_3)\}.\end{aligned}$$

Hence $\widetilde{\mu}(\widetilde{T}(X)) \leq k\widetilde{\mu}(X)$. Thus, our conclusion follows from Theorem 2.6. □

Corollary 3.4. *Let $T : \Omega \times \Omega \times \Omega \to \Omega$ be a continuous function such that*

$$\|T(x, y, z) - T(u, v, w)\| \leq k\max\{\|x - u\|, \|y - v\|, \|z - w\|\}$$

*for any $(x, y, z)$, $(u, v, w) \in \Omega \times \Omega \times \Omega$ where $0 \leq k < 1$ is a constant. Then $T$ has a tripled fixed point.*

*Proof.* It is easy to see that the map $\mu : \mathfrak{M}_E \to [0, \infty)$ defined by $\mu(X) = \operatorname{diam}(X)$ is a measure of noncompactness. Therefore, it is sufficient to prove that the inequality (4) is satisfied. To do this, let $X_1, X_2, X_3 \subset \Omega$ and $(x, y, z)$, $(u, v, w) \in X_1 \times X_2 \times X_3$. Then, we get

$$\begin{aligned}\|T(x, y, z) - T(u, v, w)\| \leq& k\max\{\|x - u\|, \|y - v\|, \|z - w\|\}\\ \leq& k\max\{\operatorname{diam}(X_1), \operatorname{diam}(X_2), \operatorname{diam}(X_3)\}.\end{aligned}$$

Thus $\operatorname{diam}(T(X_1 \times X_2 \times X_3)) \leq k\max\{\operatorname{diam}(X_1), \operatorname{diam}(X_2), \operatorname{diam}(X_3)\}$. So, by Theorem 3.3, $T$ has a tripled fixed point. □

## 4. Applications and examples

In this section, we present two applications of Theorem 3.1 which can be used to prove the existence of solution for systems of nonlinear integral equations (2).

Theorem 4.1. *Let the following hold.*

*(i) The function $A : \mathbb{R}_+ \to \mathbb{R}$ is continuous and bounded.*

*(ii) The function $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist $k_1, k_2, k_3 \in [0, 1)$ such that $|f(t, x, y, z) - f(t, u, v, w)| \leq k_1 |x - u| + k_2 |y - v| + k_3 |z - w|$ for any $t \geq 0$ and for all $x, y, z, u, v, w \in \mathbb{R}$.*

*(iii) The function defined by $t \mapsto |f(t, 0, 0, 0)|$ is bounded on $\mathbb{R}_+$.*

*(iv) The functions $\xi$, $\eta$, $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and $\xi(t) \to \infty$ as $t \to \infty$.*

*(v) The function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is continuous and there exist positive constants $\alpha, \delta$ such that $|\varphi(t_1) - \varphi(t_2)| \leq \delta |t_1 - t_2|^\alpha$ for any $t_1, t_2 \in \mathbb{R}_+$.*

*(vi) The function $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that*

$$\lim_{t\to\infty} \int_0^{\beta(t)} |g(t, s, x(\eta(s)), y(\eta(s)), z(\eta(s))) - g(t, s, u(\eta(s)), v(\eta(s)), w(\eta(s))| \, ds = 0 \quad (5)$$

*uniformly with respect to* $x, y, z, u, v, w \in BC(\mathbb{R}_+)$*. In addition,* $M_2(k_1+k_2+k_3) < 1$ *where*

$$M_2 = \sup\Big\{\big|\varphi\Big(\int_0^{\beta(t)} g(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s))\Big)ds\big| : t \in \mathbb{R}_+, x, y, z \in BC(\mathbb{R}_+)\Big\}. \tag{6}$$

*Then the equation* (2) *has at least one solution in* $BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)$*.*

*Proof.* First we define an operator $T : BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)\times BC(\mathbb{R}_+) \to BC(\mathbb{R}_+)$ by

$$\begin{aligned} &T(x,y,z)(t) = A(t) \\ &+ f(t,x(\xi(t)),y(\xi(t)),z(\xi(t)))\varphi\Big(\int_0^{\beta(t)} g(t,s,x(\eta(s),y(\eta(s)),z(\eta(s)))ds\Big). \end{aligned} \tag{7}$$

Moreover, the space $BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)$ is equipped with the norm $\|(x,y,z)\| = \|x\| + \|y\| + \|z\|$ for any $(x,y,z) \in BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)$. Notice that the continuity of $T(x,y,z)$ for any $(x,y,z) \in BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)$ is obvious. Moreover by (6), (7), (i), (ii) and the triangle inequality, we know that

$$\begin{aligned} |T(x,y,z)(t)| &\le |A(t)| + [|f(t,x(\xi(t)),y(\xi(t)),z(\xi(t))) - f(t,0,0,0)| + |f(t,0,0,0)|] \\ &\quad \times \varphi\Big(\int_0^{\beta(t)} |g(t,s,x(\eta(s),y(\eta(s)),z(\eta(s)))|\, ds\Big) \\ &\le M_0 + [k_1 |x(\xi(t))| + k_2 |y(\xi(t))| + k_3 |z(\xi(t))| + M_1]M_2 \le r \end{aligned} \tag{8}$$

where $M_0 = \sup_{t\in\mathbb{R}_+} |A(t)|$, $M_1 = \sup_{t\in\mathbb{R}_+} |f(t,0,0,0)|$ and $r = \frac{M_0+M_1M_2}{1-(k_1+k_2+k_3)M_2}$. Thus $T$ is well defined and the estimate (8) implies that $T$ maps $\overline{B}_r$ into itself. Now, we prove that $T$ is continuous on $\overline{B}_r$. For this, take $(x,y,z) \in \overline{B}_r\times\overline{B}_r\times\overline{B}_r$ and $\varepsilon > 0$ arbitrarily. Moreover, consider $(u,v,w) \in \overline{B}_r\times\overline{B}_r\times\overline{B}_r$ with

$$\|(x,y,z) - (u,v,w)\|_{BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)} \le \varepsilon.$$

Then we have

$$\begin{aligned} &|T(x,y,z)(t) - T(u,v,w)(t)| \\ &\le \Big[\left|\begin{matrix} f(t,x(\xi(t)),y(\xi(t)),z(\xi(t))) \\ -f(t,u(\xi(t)),v(\xi(t)),w(\xi(t))) \end{matrix}\right|\Big] \left|\varphi\Big(\int_0^{\beta(t)} g(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))ds\Big)\right| \\ &+ [|f(t,u(\xi(t)),v(\xi(t)),w(\xi(t))) - f(t,0,0,0)| \\ &+ |f(t,0,0,0)|] \left|\begin{matrix} \varphi(\int_0^{\beta(t)} g(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))ds) \\ -\varphi(\int_0^{\beta(t)} g(t,s,u(\eta(s),v(\eta(s)),w(\eta(s)))ds) \end{matrix}\right| \\ &\le [k_1 |x(\xi(t)) - u(\xi(t))| + k_2 |y(\xi(t)) - v(\xi(t))| + k_3 |z(\xi(t)) - w(\xi(t))|]M_2 \\ &+ [k_1 |u(\xi(t))| + k_2 |v(\xi(t))| + k_3 |w(\xi(t))| + M_1] \cdot \delta \left|\begin{matrix} \int_0^{\beta(t)} g(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s))) \\ -g(t,s,u(\eta(s)),v(\eta(s)),w(\eta(s)))ds \end{matrix}\right|^{\alpha} \\ &\le (k_1+k_2+k_3)\varepsilon M_2 \\ &+ [(k_1+k_2+k_3)r + M_1] \cdot \delta \left|\begin{matrix} \int_0^{\beta(t)} g(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s))) \\ -g(t,s,u(\eta(s)),v(\eta(s)),w(\eta(s)))ds \end{matrix}\right|^{\alpha}. \end{aligned} \tag{9}$$

In addition, from (5), there exists $L > 0$ such that, if $t > L$, then

$$\int_0^{\beta(t)} |g(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))-g(t,s,u(\eta(s)),v(\eta(s)),w(\eta(s)))|\,ds \le \left(\frac{\varepsilon}{\delta}\right)^{\alpha} \quad (10)$$

for any $x, y, z, u, v, w \in BC(\mathbb{R}_+)$. There are now two cases:

**Case (a)** If $t > L$, then from (9) and (10), we get

$$|T(x,y,z)(t) - T(u,v,w)(t)| \le [(k_1 + k_2 + k_3)(M_2 + r) + M_1]\varepsilon. \quad (11)$$

**Case (b)** If $t \in [0, L]$, then using the continuity of $g$ on $[0, L] \times [0, \beta_L] \times [-r, r] \times [-r, r] \times [-r, r]$ we have $\omega_r^L(g, \varepsilon) \to 0$, as $\varepsilon \to 0$.

Thus by the argument similar to those given in (9), we have

$$\begin{aligned}
&|T(x,y,z)(t) - T(u,v,w)(t)| \\
&\le (k_1 + k_2 + k_3)\,\varepsilon M_2 + [(k_1 + k_2 + k_3)\,r + M_1] \\
&\times\delta \left| \int_0^{\beta(t)} \left(g\left(t,s,x\left(\eta\left(s\right)\right),y\left(\eta\left(s\right)\right),z\left(\eta\left(s\right)\right)\right)-g\left(t,s,u\left(\eta\left(s\right)\right),v\left(\eta\left(s\right)\right),w\left(\eta\left(s\right)\right)\right)\right) ds\right|^{\alpha} \\
&< (k_1 + k_2 + k_3)\,\varepsilon M_2 + [(k_1 + k_2 + k_3)\,r + M_1] \cdot \delta \left(\beta_L \omega_r^L\left(g,\varepsilon\right)\right)^{\alpha} \\
&< [(k_1 + k_2 + k_3)\left(M_2 + r\right) + M_1]\,\varepsilon, \qquad (12)
\end{aligned}$$

where $\omega_r^L\left(g,\varepsilon\right) = \sup\left\{|g(t_1,s,x,y,z) - g(t_2,s,x,y,z)| : t_1,t_2 \in [0,L]\,, |t_1 - t_2| \le \varepsilon,\ s \in [0,\beta_L]\,, x,y,z \in [-r,r]\right\}$ and $\beta_L = \sup\{\beta(t) : t \in [0, L]\}$. Hence, the inequalities (11), (12), imply that $T$ is a continuous function from $\overline{B}_r \times \overline{B}_r \times \overline{B}_r$ into $\overline{B}_r$.

Now, we only need to show that $T$ satisfies the conditions of Theorem 2.6. To prove that, let $L, \varepsilon \in \mathbb{R}_+$ and $X_1, X_2\ , X_3$ are arbitrary nonempty subsets of $\overline{B}_r$ and take $t_1,\ t_2 \in [0, L]$, such that $|t_1 - t_2| < \varepsilon$.

Without loss of generality, we may assume that $\beta(t_1) < \beta(t_2)$. We also assume that $(x, y, z) \in X_1 \times X_2 \times X_3$. Then we get

$$\begin{aligned}
&|T(x,y,z)(t_1)-T(x,y,z)(t_2)| \\
&\le |A(t_1)-A(t_2)| + \left[\left| \begin{array}{c} f(t_1,x(\xi(t_1)),y(\xi(t_1)),),z(\xi(t_1))) \\ -f(t_2,x(\xi(t_1)),y(\xi(t_1)),z(\xi(t_1))) \end{array} \right| \right. \\
&\quad + \left.\left| \begin{array}{c} f(t_2,x(\xi(t_1)),y(\xi(t_1)),z(\xi(t_1))) \\ -f(t_2,x(\xi(t_2)),y(\xi(t_2)),z(\xi(t_2))) \end{array} \right|\right] \left|\varphi(\int_0^{\beta(t)} g(t,s,x(\eta(s),y(\eta(s)),z(\eta(s)))ds)\right| \\
&\quad + [|f(t_2,x(\xi(t_2)),y(\xi(t_2)),z(\xi(t_2)))-f(t_2,0,0,0)| + |f(t_2,0,0,0)|] \\
&\quad \times \left| \begin{array}{c} \varphi(\int_0^{\beta(t_1)} g(t_1,s,x(\eta(s)),y(\eta(s)), \\ z(\eta(s)))ds) \end{array} -\varphi(\int_0^{\beta(t_2)} g(t_2,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))ds) \right| \\
&\le |A(t_1)-A(t_2)| + [\omega_r^L(f,\varepsilon) + k_1\,|x(\xi(t_1))-x(\xi(t_2))| \\
&\quad + k_2\,|y(\xi(t_1))-y(\xi(t_2))| + k_3\,|z(\xi(t_1))-z(\xi(t_2))|]M_2 \\
&\quad + [k_1\,|x(\xi(t_2)| + k_2\,|y(\xi(t_2)| + k_3\,|z(\xi(t_2))| + M_1] \\
&\quad \times \delta \left| \begin{array}{c} \int_0^{\beta(t_1)} g(t_1,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))ds \\ -\int_0^{\beta(t_2)} g(t_2,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))ds \end{array} \right|^{\alpha}
\end{aligned}$$

$$\leq\omega^L(A,\varepsilon))+[\omega_r^L(f,\varepsilon)+k_1(\omega^L(x,\omega^L(\xi,\varepsilon)))+k_2(\omega^L(y,\omega^L(\xi,\varepsilon))+k_3(\omega^L(z,\omega^L(\xi,\varepsilon))]M_2$$
$$+[(k_1+k_2+k_3)r+M_1]\cdot\delta(\int_0^{\beta(t_1)}\left|\begin{array}{c} g(t_1,s,x(\eta(s)),y(\eta(s)),z(\eta(s))) \\ -g(t_2,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))\end{array}\right|ds$$
$$+\int_{\beta(t_1)}^{\beta(t_2)}|g(t_2,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))|\,ds)^\alpha$$
$$\leq\omega^L(A,\varepsilon)+[\omega_r^L(f,\varepsilon)+k_1(\omega^L(x,\omega^L(\xi,\varepsilon))+k_2(\omega^L(y,\omega^L(\xi,\varepsilon))+k_3(\omega^L(z,\omega^L(\xi,\varepsilon))]M_2$$
$$+[(k_1+k_2+k_3)r+M_1]\cdot\delta(\beta_L\omega_r^L(g,\varepsilon)+G_r^L\omega^L(\beta,\varepsilon))^\alpha \tag{13}$$
where
$$\omega^L(\xi,\varepsilon)=\sup\{|\xi(t_1)-\xi(t_2)|:t_1,t_2\in[0,L],|t_1-t_2|\leq\varepsilon\},$$
$$\omega^L(\beta,\varepsilon)=\sup\{|\beta(t_1)-\beta(t_2)|:t_1,t_2\in[0,L],|t_1-t_2|\leq\varepsilon\},$$
$$\omega^L(x,\omega^L(\xi,\varepsilon))=\sup\{|x(t_1)-x(t_2)|:t_1,t_2\in[0,L],|t_1-t_2|\leq\omega^L(\xi,\varepsilon)\},$$
$$\omega_r^L(f,\varepsilon)=\sup\left\{\begin{array}{c}|f(t_1,x,y)-f(t_2,x,y)|:t_1,t_2\in[0,L],|t_1-t_2|\leq\varepsilon,\\ x,y,z\in[-r,r]\end{array}\right\},$$
$$\omega_r^{\beta_L}(g,\varepsilon)=\sup\left\{\begin{array}{c}|g(t_1,s,x,y,z)-g(t_2,s,x,y,z)|:t_1,t_2\in[0,L],\\ |t_1-t_2|\leq\varepsilon, s\in[0,\beta_L],x,y,z\in[-r,r]\end{array}\right\}$$
and $$G_r^L=\sup\{|g(t,s,x,y,z)|:t\in[0,L],s\in[0,\beta_L]\text{ and }x,y,z\in[-r,r]\}.$$
Since $(x,y,z)$ is an arbitrary element of $X_1\times X_2\times X_3$ in (13), we obtain
$$\omega^L(T(X_1\times X_2\times X_3),\varepsilon)\leq\omega^L(A,\varepsilon))+\left[\begin{array}{c}\omega_r^L(f,\epsilon)\\ k_1\omega^L\left(X_1,\omega^L(\xi,\epsilon)\right)+k_2\omega^L\left(X_2,\omega^L(\xi,\epsilon)\right)\\ k_3\omega^L\left(X_3,\omega^L(\xi,\epsilon)\right)\end{array}\right]M_2$$
$$+[(k_1+k_2+k_3)r+M_1]\cdot\delta(\beta_L\omega_r^L(g,\varepsilon)+G_r^L\omega^L(\beta,\varepsilon))^\alpha. \tag{14}$$
Otherwise, by the uniform continuity of $f$, $g$ on $[0,L]\times[0,\beta_L]\times[-r,r]\times[-r,r]$, $[0,L]\times[0,\beta_L]\times[-r,r]\times[-r,r]\times[-r,r]$, respectively, we have $\omega_r^L(f,\varepsilon)\to0$, $\omega_r^{\beta_L}(g,\varepsilon)\to0$. Also because of the uniform continuity of $\xi,\beta$ and $A$ on $[0,L]$, we derive that $\omega^L(\xi,\varepsilon)\to0$, $\omega_r^{\beta_L}(\beta,\varepsilon)$ and $\omega_r^L(A,\varepsilon)\to0$ as $\varepsilon\to0$. But $G_r^L$ is finite, so taking the limit from (14) as $\varepsilon\to0$, we get
$$\omega_0^L(T(X_1\times X_2\times X_3))\leq(k_1\omega_0^L(X_1)+k_2\omega_0^L(X_2)+k_3\omega_0^L(X_3))M_2. \tag{15}$$
By letting $L\to\infty$ in (15), we obtain
$$\omega_0(T(X_1\times X_2\times X_3))\leq k_1M_2\omega_0(X_1)+k_2M_2\omega_0(X_2)+k_3M_2\omega_0(X_3). \tag{16}$$
In addition, for arbitrary $(x,y,z),(u,\ v,w)\in X_1\times X_2\times X_3$ and $t\in\mathbb{R}_+$ we have
$$|T(x,y,z)(t)-T(u,v,w)(t)|$$
$$\leq\Big(\left|\begin{array}{c}f(t,x(\xi(t)),y(\xi(t)),z(\eta(s)))\\ -f(t,u(\xi(t)),v(\xi(t)),w(\eta(s)))\end{array}\right|\Big)\left|\varphi(\int_0^{\beta(t)}g(t,s,x(\eta(s),y(\eta(s)),z(\eta(s)))ds)\right|$$
$$+[|f(t,u(\xi(t)),v(\xi(t)),w(\xi(t)))-f(t,0,0,0)|+|f(t,0,0,0)|]$$
$$\times\left|\begin{array}{c}\varphi(\int_0^{\beta(t)}g(t,s,x(\eta(s),y(\eta(s)),z(\eta(s)))ds)\\ -\varphi(\int_0^{\beta(t)}g(t,s,u(\eta(s),v(\eta(s)),w(\eta(s)))ds)\end{array}\right|$$

$$
\begin{aligned}
&\leq (k_1 \left|x(\xi(t)) - u(\xi(t))\right| + k_2 \left|y(\xi(t)) - v(\xi(t))\right| + k_3 \left|z(\xi(t) - w(\xi(t))\right|)M_2 \\
&\quad + [k_1 \left|u(\xi(t))\right| + k_2 \left|v(\xi(t))\right| + k_3 \left|w(\xi(t))\right| + M_1] \\
&\quad \times \delta \left| \begin{matrix} \int_0^{\beta(t)} g(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s))) \\ -g(t,s,u(\eta(s)),v(\eta(s)),w(\eta(s)))ds \end{matrix} \right|^{\alpha} \\
&\leq (k_1 \operatorname{diam} X_1(\xi(t) + k_2 \operatorname{diam} X_2(\xi(t)) + k_3 \operatorname{diam} X_3(\xi(t)))M_2 \\
&+[((k_1+k_2+k_3)r + M_1)] \cdot \delta(\int_0^{\beta(t)} \left| \begin{matrix} g(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s)) \\ -g(t,s,u(\eta(s)),v(\eta(s)),w(\eta(s))) \end{matrix} \right| ds)^{\alpha} \qquad (17)
\end{aligned}
$$

Since $(x,y,z),(u,\ v,w)$ and $t$ are arbitrary in (17), we conclude that

$$
\begin{aligned}
&\operatorname{diam} T(X_1 \times X_2 \times X_3)(t) \leq (k_1 \operatorname{diam} X_1(\xi(t)) + k_2 \operatorname{diam} X_2(\xi(t) + k_3 \operatorname{diam} X_3(\xi(t)))M_2 \\
&+[((k_1+k_2+k_3)r+M_1)] \cdot \delta \left| \int_0^{\beta(t)} \begin{pmatrix} g\left(t,s,x\left(\eta\left(s\right)\right),y\left(\eta\left(s\right)\right),z\left(\eta\left(s\right)\right)\right) \\ -g\left(t,s,u\left(\eta\left(s\right)\right),v\left(\eta\left(s\right)\right),w\left(\eta\left(s\right)\right)\right) \end{pmatrix} ds \right|^{\alpha} \qquad (18)
\end{aligned}
$$

Suppose that $t \to \infty$ in the inequality (18), then using (5) we deduce that

$$
\begin{aligned}
\limsup_{t\to\infty} \operatorname{diam} T(X_1 \times X_2 \times X_3)(t) &\leq (k_1 \limsup_{t\to\infty} \operatorname{diam} X_1(\xi(t)) \\
&+ k_2 \limsup_{t\to\infty} \operatorname{diam} X_2(\xi(t)) + k_3 \limsup_{t\to\infty} \operatorname{diam} X_3(\xi(t)))M_2. \qquad (19)
\end{aligned}
$$

Now, combining (16) and (19) we obtain

$$
\begin{aligned}
&\omega_0(T(X_1 \times X_2 \times X_3)) + \limsup_{t\to\infty} \operatorname{diam} T(X_1 \times X_2 \times X_3)(t) \\
&\leq (k_1 M_2 \omega_0(X_1) + k_2 M_2 \omega_0(X_2) + k_3 M_2 \omega_0(X_3)) + k_1 M_2 \limsup_{t\to\infty} \operatorname{diam} X_1(\xi(t)) \\
&\quad + k_2 M_2 \limsup_{t\to\infty} \operatorname{diam} X_2(\xi(t)) + k_3 M_2 \limsup_{t\to\infty} \operatorname{diam} X_3(\xi(t)) \\
&= k_1 M_2(\omega_0(X_1)) + \limsup_{t\to\infty} \operatorname{diam} X_1(\xi(t))) + k_2 M_2(\omega_0(X_2)) \\
&\quad + \limsup_{t\to\infty} \operatorname{diam} X_2(\xi(t))) + k_3 M_2(\omega_0(X_3) + \limsup_{t\to\infty} \operatorname{diam} X_3(\xi(t))). \qquad (20)
\end{aligned}
$$

Therefore, from (1) and (20), we derive that $\mu(T(X_1 \times X_2 \times X_3)) \leq k_1 M_2 \mu(X_1) + k_2 M_2 \mu(X_2) + k_3 M_2 \mu(X_3)$. Consequently $\mu(T(X_1 \times X_2 \times X_3)) \leq k_1^{'} \mu(X_1) + k_2^{'} \mu(X_2) + k_3^{'} \mu(X_3)$, where $k_1^{'} = k_1 M_1$, $k_2^{'} = k_2 M_2$, $k_3^{'} = k_3 M_2$ and $k_1^{'} + k_2^{'} + k_3^{'} < 1$. Thus by Theorem 3.1, $T$ has a tripled fixed point in $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$. This completes the proof. $\square$

The following corollary is a special case of Theorem 4.1.

Corollary 4.2. *Let the conditions (i)–(iv) of Theorem 4.1 be satisfied. Furthermore, suppose that there exists continuous functions $a, b : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$
|g(t,s,x,y,z)| \leq a(t)b(s) \qquad (21)
$$

*for $t, s \in \mathbb{R}_+$. Also, assume that*

$$
\lim_{t\to\infty} a(t) \int_0^{\beta(t)} b(s)\, ds = 0 \qquad (22)
$$

*and* $M_2^{'}(k_1+k_2+k_3)<1$ *where*

$$M_2^{'}=\sup\Big\{a(t)\int_0^{\beta(t)} b(s)\,ds : t\in\mathbb{R}_+\Big\}. \tag{23}$$

*Then the system of integral equations*

$$\begin{cases} x(t)=A(t)+f(t,x(\xi(t)),y(\xi(t)),z(\xi(t)))(\int_0^{\beta(t)} g(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))\,ds)\\ y(t)=A(t)+f(t,y(\xi(t)),x(\xi(t)),y(\xi(t)))(\int_0^{\beta(t)} g(t,s,y(\eta(s)),x(\eta(s)),y(\eta(s)))\,ds)\\ z(t)=A(t)+f(t,z(\xi(t)),y(\xi(t)),x(\xi(t)))(\int_0^{\beta(t)} g(t,s,z(\eta(s)),y(\eta(s)),x(\eta(s)))\,ds)\end{cases} \tag{24}$$

*has at least one solution in the space* $BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)$.

*Proof.* Let us consider $\varphi(x)=x$ in Theorem 4.1. It follows that $\varphi$ is a Lipschitz function with constant 1. Now from (23) and inequality (21), we get

$$\begin{aligned} M_2&=\sup\{\Big|\varphi(\int_0^{\beta(t)} g(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))\,ds)\Big| : t\in\mathbb{R}_+, x,y,z\in BC(\mathbb{R}_+)\}\\ &=\sup\{\Big|\int_0^{\beta(t)} g(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))\,ds\Big| : t\in\mathbb{R}_+, x,y,z\in BC(\mathbb{R}_+)\}\\ &\le \sup\{a(t)\int_0^{\beta(t)} b(s)\,ds : t\in\mathbb{R}_+\}=M_2^{'}.\end{aligned}$$

Hence, the above inequality implies that $M_2(k_1+k_2+k_3)<1$. On the other hand by (21), (22) and the triangle inequality we have

$$\begin{aligned} &\lim_{t\to\infty}\int_0^{\beta(t)} |g(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))-g(t,s,u(\eta(s)),v(\eta(s)),w(\eta(s)))|\ ds\\ &\qquad\le 2\lim_{t\to\infty} a(t)\int_0^{\beta(t)} b(s)\,ds=0\end{aligned}$$

uniformly respect to $x,y,z,u,v,w\in BC(\mathbb{R}_+)$. Thus all of the conditions of Theorem 3.1 are satisfied. Therefore, applying Theorem 3.1, we conclude that the equation (24) has at least one solution in the space $BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)$. □

We finish this section with two examples.

EXAMPLE 4.3. Let $m>2$ and $n>1$. We set

$$\begin{cases} x(t)= & \frac{t}{t+1}+[\frac{1}{4}\sin(x(3t))+\frac{2}{4}\cos(y(3t))+\frac{1}{4}e^{z(3t)}]\\ & \times\int_0^{t^2}\frac{\ln(1+s^{\frac{m-2}{2}}\left|\sin^n(x(\sqrt{s}))\right|)\left|y(\sqrt{s})\right|\left|z(\sqrt{s})\right|}{(1+\sin^{2n}(x(\sqrt{s})))(1+t^{2m})(1+y^2(\sqrt{s}))(1+z^2(\sqrt{s}))}\,ds\\ y(t)= & \frac{t}{t+1}+[\frac{1}{4}\sin(y(3t))+\frac{2}{4}\cos(x(3t))+\frac{1}{4}e^{y(3t)}]\\ & \times\int_0^{t^2}\frac{\ln(1+s^{\frac{m-2}{2}}\left|\sin^n(y(\sqrt{s}))\right|)\left|x(\sqrt{s})\right|\left|y(\sqrt{s})\right|}{(1+\sin^{2n}(y(\sqrt{s})))(1+t^{2m})(1+x^2(\sqrt{s})(1+y^2(\sqrt{s})))}\,ds\\ z(t)= & \frac{t}{t+1}+[\frac{1}{4}\sin(z(3t))+\frac{2}{4}\cos(y(3t))+\frac{1}{4}e^{x(3t)}]\\ & \times\int_0^{t^2}\frac{\ln(1+s^{\frac{m-2}{2}}\left|\sin^n(z(\sqrt{s}))\right|)\left|y(\sqrt{s})\right|\left|x(\sqrt{s})\right|}{(1+\sin^{2n}(z(\sqrt{s})))(1+t^{2m})(1+y^2(\sqrt{s}))(1+x^2(\sqrt{s}))}\,ds\end{cases} \tag{25}$$

Then, we have

$$f(t,x,y,z)=\frac{t}{t+1}+[\frac{1}{4}\sin(x)+\frac{2}{4}\cos(y)+\frac{1}{4}\mathrm{e}^z],$$

$$g(t,s,x,y,z) = \frac{\ln(1 + s^{\frac{m-2}{2}} \left|\sin^n(x)\right|) \left|y\right| \left|z\right|}{(1+\sin^{2n}(x))(1+t^{2m})(1+y^2)(1+z^2)},$$
$$A(t) = \frac{t}{t+1}, \xi(t) = 3t, \eta(s) = \sqrt{s}, \beta(t) = t^2$$

comparing (25) with (24). Now we survey the conditions of Corollary 4.2:

(i) is clear.

(ii) Assume that $t \geq 0$ and $x, y, z, u, v, w \in \mathbb{R}$. Then we get

$$\begin{aligned} |f(t,x,y,z) - f(t,u,v,w)| &\leq \frac{1}{4}\left|\sin(x) - \sin(u)\right| + \frac{2}{4}\left|\cos(y) - \cos(v)\right| + \frac{1}{4}\left|\mathrm{e}^z - \mathrm{e}^w\right| \\ &\leq \frac{1}{4}|x-u| + \frac{2}{4}|y-v| + \frac{1}{4}|z-w| \end{aligned}$$

for all $t > 0$. Then the condition (ii) is satisfied with $k_1 = \frac{1}{4}$, $k_2 = \frac{2}{4}$, $k_3 = \frac{1}{4}$.

(iii) Clearly, the function $|f(t,0,0,0)| = \frac{t}{t+1} + \frac{3}{4}$ is bounded.

(iv) Obviously, $\xi, \eta, q$ are continuous and $\xi(t) \to \infty$ as $t \to \infty$.

Now suppose that $t, s \in \mathbb{R}_+$ and $x, y, z \in \mathbb{R}$; then

$$\begin{aligned} |g(t,s,x,y,z)| &= \left|\frac{\ln(1 + s^{\frac{m-2}{2}} \left|\sin^n(x)\right|) |y| |z|}{(1+\sin^{2n}(x))(1+t^{2m})(1+y^2)(1+z^2)}\right| \\ &\leq \frac{s^{\frac{m-2}{2}} \left|\sin^n(x)\right|) |y| |z|}{(1+\sin^{2n}(x))(1+t^{2m})(1+y^2)(1+z^2)} \leq \frac{1}{4}\frac{s^{\frac{m-2}{2}}}{(1+t^{2m})} = a(t) \cdot b(s) \end{aligned}$$

for any $t, s \in \mathbb{R}_+$, and $x, y, z \in \mathbb{R}$. Furthermore

$$\lim_{t\to\infty} a(t) \int_0^{t^2} b(s)\, ds = \lim_{t\to\infty} \frac{1}{4(1+t^{2m})} \int_0^{t^2} s^{\frac{m-2}{2}}\, ds = \lim_{t\to\infty} \frac{t^m}{2m(1+t^{2m})} = 0$$

and $M_2' = \sup\limits_{t\in\mathbb{R}_+} a(t) \int_0^{t^2} b(s)\, ds \leq \frac{1}{4m}$. Hence $(k_1 + k_2 + k_3)M_2' \leq (\frac{1}{4} + \frac{2}{4} + \frac{1}{4})\frac{1}{4m} < 1$.

These imply that the assumptions of Corollary 4.2 are satisfied. Therefore, as a result of Corollary 4.2, we conclude that the system of integral equations (25) has at least one solution in $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$.

Example 4.4. Let $n > 1$. Consider the equation

$$\begin{cases} x(t) = e^{-t^2} + [\frac{t}{2\pi(t^2+1)} \cos x(2t) + \frac{2}{4\pi}\frac{t}{t+1} \ln(1 + |y(2t)|) + \frac{t}{4\pi(t^4+1)} \sin z(2t)] \\ \cdot \operatorname{arctg}(\int_0^t \frac{4n \ln(1+s^{n-1}\left|x(\sqrt{s})\right|)\left|\cos y(\sqrt{s})\right|\left|\sin z(\sqrt{s})\right| + ns^{2n-1}(1+\left|x^2(\sqrt{s})\right|)(1+\cos^2 y(\sqrt{s}))(1+\sin^2 z(\sqrt{s}))}{(1+x^2(\sqrt{s}))(1+t^{2n})(1+\cos^2 y(\sqrt{s}))(1+\sin^2 z(\sqrt{s}))} ds) \\ y(t) = e^{-t^2} + [\frac{t}{2\pi(t^2+1)} \cos y(2t) + \frac{2}{4\pi} \ln(1 + |x(2t)|) + \frac{t}{4\pi(t^4+1)} \sin y(2t)] \\ \cdot \operatorname{arctg}(\int_0^t \frac{4n \ln(1+s^{n-1}\left|y(\sqrt{s})\right|)\left|\cos x(\sqrt{s})\right|\left|\sin y(\sqrt{s})\right| + ns^{2n-1}(1+\left|y^2(\sqrt{s})\right|)(1+\cos^2 x(\sqrt{s}))(1+\sin^2 y(\sqrt{s}))}{(1+y^2(\sqrt{s}))(1+t^{2n})(1+\cos^2 x(\sqrt{s}))(1+\sin^2 y(\sqrt{s}))} ds) \\ z(t) = e^{-t^2} + [\frac{t}{2\pi(t^2+1)} \cos z(2t) + \frac{2}{4\pi} \ln(1 + |y(2t)|) + \frac{t}{4\pi(t^4+1)} \sin x(2t)] \\ \cdot \operatorname{arctg}(\int_0^t \frac{4n \ln(1+s^{n-1}\left|z(\sqrt{s})\right|)\left|\cos y(\sqrt{s})\right|\left|\sin x(\sqrt{s})\right| + ns^{2n-1}(1+\left|z^2(\sqrt{s})\right|)(1+\cos^2 y(\sqrt{s}))(1+\sin^2 x(\sqrt{s}))}{(1+z^2(\sqrt{s}))(1+t^{2n})(1+\cos^2 y(\sqrt{s}))(1+\sin^2 x(\sqrt{s}))} ds) \end{cases}$$

Since the previous equation is a special case of the equation (2), we obtain

$$f(t,x,y,z) = e^{-t^2} + [\frac{t}{2\pi(t^2+1)} \cos(x) + \frac{2}{4\pi}\frac{t}{t+1} \ln(1 + |y|) + \frac{1}{4\pi}\frac{t}{t^4+1} \sin(z)]$$

$$g(t,s,x,y,z)=\frac{4n\ln(1+s^{n-1}|x|)\,|\cos y|\,|\sin z|+ns^{2n-1}(1+x^2)(1+\cos^2 y)(1+\sin^2 z)}{(1+x^2)(1+t^{2n})(1+\cos^2 y)(1+\sin^2 z)}$$

$$A(t)=e^{-t^2},\ \xi(t)=2t,\ \eta(s)=\sqrt{s},\ \beta(t)=t,\ \varphi(x)=\operatorname{arctg}(x).$$

We will prove that the conditions of Theorem 4.1 are satisfied for this equation. First, in the same way as stated in Example 4.4, (i), (iii) and (iv) are evident. Now, suppose that $t\in\mathbb{R}_+$ and $x,y,z\in\mathbb{R}$ with $|y|\ge|v|$. Then we get

$$|f(t,x,y,z)-f(t,u,v,w)|\le\frac{t}{2\pi(t^2+1)}|\cos(x)-\cos(u)|$$

$$+\frac{2}{4\pi}\frac{t}{t+1}|\ln(1+|y|)-\ln(1+|v|)|+\frac{t}{4\pi(t^4+1)}|\sin(z)-\sin(w)|$$

$$\le\frac{1}{4\pi}|x-u|+\frac{2}{4\pi}\left|\ln\Big(\frac{1+|y|}{1+|v|}\Big)\right|+\frac{1}{4\pi}|z-w|\le\frac{1}{4\pi}|x-u|+\frac{2}{4\pi}\left|\ln\Big(1+\frac{|y|-|v|}{1+|v|}\Big)\right|+\frac{1}{4\pi}|z-w|$$

$$\le\frac{1}{4\pi}|x-u|+\frac{2}{4\pi}\ln(1+|y-v|)+\frac{1}{4\pi}|z-w|\le\frac{1}{4\pi}|x-u|+\frac{2}{4\pi}|y-v|+\frac{1}{4\pi}|z-w|.$$

Therefore $f$ satisfies the condition (ii) of Theorem 4.1 with $k_1=\frac{1}{4\pi}$, $k_2=\frac{2}{4\pi}$and $k_3=\frac{1}{4\pi}$. On the other hand, according to the Mean Value Theorem, $\varphi$ is Lipschitz with constant 1. This means that the condition (v) of Theorem 4.1 is satisfied. Also, it is obvious that $g$ is continuous on $\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$. Since

$$\frac{ns^{n-1}|x|\,|\cos y|\,|\sin z|}{(1+x^2)(1+t^{2n})(1+\cos^2 y)(1+\sin^2 z)}\le\frac{1}{4}\frac{ns^{n-1}}{(1+t^{2n})}$$

for any $t,s\in\mathbb{R}_+$ and $x,y,z\in\mathbb{R}$, we have

$$\lim_{t\to\infty}\int_0^t|g(t,s,x,y,z)|\,ds=\lim_{t\to\infty}\int_0^t\frac{ns^{2n-1}}{(1+t^{2n})}\,ds=\frac{1}{2}.$$

Notice that this shows that the property (22) of Corollary 4.2 is not satisfied and this corollary cannot be used to conclude that the considered equation has a solution. But

$$\int_0^t|g(t,s,x,y,z)|\,ds\le\int_0^t\frac{4ns^{n-1}|x|\,|\cos y|\,|\sin z|}{(1+x^2)(1+t^{2n})(1+\cos^2 y)(1+\sin^2 z)}\,ds+\int_0^t\frac{ns^{2n-1}}{(1+t^{2n})}\,ds$$

$$\le\int_0^t\frac{ns^{n-1}}{(1+t^{2n})}\,ds+\int_0^t\frac{ns^{2n-1}}{(1+t^{2n})}\,ds\le\frac{t^n}{(1+t^{2n})}+\frac{t^{2n}}{2(1+t^{2n})}\le\frac{1}{2}+\frac{1}{2}=1$$

for any $t\in\mathbb{R}_+$ and $x,y,z\in\mathbb{R}$. Thus

$$M_2=\sup_{t\in\mathbb{R}_+}\left|\operatorname{arctg}\Big(\int_0^t g(t,s.x,y,z)\,ds\Big)\right|\le\sup_{x\in[-1,1]}|\operatorname{arctg}(x)|=\frac{\pi}{4}.$$

It follows that $(k_1+k_2+k_3)M_2=(\frac{1}{4\pi}+\frac{2}{4\pi}+\frac{1}{4\pi})\frac{\pi}{4}<1$. Moreover

$$\lim_{t\to\infty}\int_0^t|g(t,s,x,y,z)-g(t,s,u,v,w)|\,ds\le\lim_{t\to\infty}2\int_0^t\frac{ns^{n-1}}{(1+t^{2n})}\,ds=\lim_{t\to\infty}\frac{2t^n}{(1+t^{2n})}=0.$$

Hence the condition (vi) of Theorem 4.1 is satisfied too. Now, according to Theorem 4.1, we have a solution in $BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)$.

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